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A NOTE ON M-SUMMANDS IN DUAL SPACES

BY

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ABSTRACT. A theorem concerning M-summands in dual spaces is used to prove that certain known M-ideals are not M-summands. In some cases where this information was already known, our procedure greatly simplifies the earlier proofs. Finally, we give a condition to determine which M-ideals in dual spaces are M-summands and which are not.

1. **Preliminaries.** Let M be a closed subspace of a Banach space X. We say M is an M-summand of X provided there is a projection Q of X onto M satisfying $\|x\| = \max\{ \|Qx\|, \|(1 - Q)x\| \}$ for all $x \in X$. Such a projection is called an M-projection. The subspace M is said to be an M-ideal of X provided there is a projection P of X^* onto $M^{\perp}$ satisfying $\|f\| = \|Pf\| + \|(1 - P)f\|$ for all $f \in X^*$, where X^* is the dual space of X and $M^{\perp}$ is the annihilator of M in X^*. Such a projection is called an L-projection. It is a basic fact that every M-summand is an M-ideal though the converse is false. These concepts of M-summand and M-ideal were introduced in 1972 by Alfsen and Effros ([1]) and have been widely studied since. Of particular interest has been the application to approximation theory. Briefly, if M is an M-ideal of X then for each $x \in X$ there exists $m \in M$ such that $\|x + m\| = \inf_{y \in M} \|x + y\|$ (cf. [1, Cor. 5.6]). Furthermore, if $x \in X \setminus M$ then the set $\{m \in M: \|x + m\| = \inf_{y \in M} \|x + y\| \}$ algebraically spans M ([9]).

Below we consider three known examples of M-ideals and present simple arguments to show that they are not M-summands.

Throughout, H will denote a complex, separable, infinite dimensional Hilbert space. $\mathscr{L}(H)$ and $\mathscr{K}$ represent respectively the algebra of all bounded linear operators on H and the ideal of compact operators on H. If $\mathscr{P}$ is a linearly ordered set of (self-adjoint) projections in $\mathscr{L}(H)$ which is closed in the strong operator topology then the nest algebra associated with $\mathscr{P}$ is the collection $\operatorname{Alg} \mathscr{P} = \{A \in \mathscr{L}: AP = PAP, P \in \mathscr{P}\}$. The symbols L^{∞} and C denote respectively

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the spaces of bounded measurable and continuous functions on the unit circle in the complex plane. Via identification with boundary functions, the space H^∞ of bounded analytic functions on the open unit disk may be thought of as a subspace of L^∞. For any Banach space X, ball(X) will denote the closed unit ball of X. The dual space of X is denoted by X^*. If J is a subset of X then the annihilator of J is $J^\perp = \{f \in X^*: f(x) = 0 \text{ for all } x \in J\}$ while if M is a subset of X^* then the pre-annihilator of M is given by ${}^\perp M = \{x \in X: f(x) = 0$ for all $f \in M\}$. We shall also need the following basic facts concerning weak-* topologies.

(1.1) For every Banach space X, ball(X) is weak-* dense in ball(X^{**}). (This is sometimes called Goldstine's Theorem.)

(1.2) If M is a subspace of X^* for which ball(M) is weak-* dense in ball(X^*) and if J is a weak-* closed subset of X^* for which $J + M$ is norm-closed, then ball($(J + M)/J$) is weak-* dense in (X^*/J). This is a consequence of the identification $(X^*/J) \cong ({}^\perp J)^*$ and the separation principle.

(1.3) ball(C) is weak-* dense in ball(L^∞).

(1.4) ball($\mathscr{K}$) is weak-* dense in ball($\mathscr{L}(H)$). This is a consequence of (1.1) and the identification of $\mathscr{L}(H)$ with the second dual of $\mathscr{K}$ (cf. [11]).

The main tool used in discussing the examples below is the following theorem whose proof we present for completeness.

PROPOSITION A ([4]). *If X is a Banach space and M is an M-summand of X^*, then M is closed in the weak-* topology.*

PROOF. By the Krein-Smulyan Theorem (cf. [6, p. 430ff.]), M is weak-* closed if the closed unit ball of M is weak-* closed. For this, let $\{e_\lambda\}_{\lambda \in \Lambda} \subset M$ satisfy $\|e_\lambda\| \leqq 1$ for all $\lambda \in \Lambda$ and suppose the net $\{e_\lambda\}$ converges weak-* to f. We may write $X = M + M'$ and $f = f_0 + f'$ where $f_0 \in M$ and $f' \in M'$. Since the net $\{e_\lambda - f_0\}$ is a bounded net in M converging weak-* to f' we may assume that $f_0 = 0$ to begin with.

Assume that $f' \neq 0$. Since the closed unit ball of X^* is weak-* compact, it follows that $\|f'\| \leqq 1$. Choose a positive integer k such that $k \cdot \|f'\| \leqq 1$ but $(k + 1) \cdot \|f'\| > 1$ and note that the net $\{e_\lambda + kf'\}$ converges weak-* to $(k + 1)f'$. For each $\lambda \in \Lambda$, we have $\|e_\lambda + kf'\| = \max\{\|e_\lambda\|, k \cdot \|f'\|\} \leqq 1$ so that, by weak-* compactness of closed balls in X^*, $(k + 1)\|f'\| \leqq 1$. This contradicts the choice of k. Thus, $f' = 0$ as desired.

COROLLARY 1. *Let M be an M-summand of X^*. Then X^*/M is a dual space.*

PROOF. By Proposition A, M is weak-* closed. Thus $M = ({}^\perp M)^\perp$ and, by standard results, $({}^\perp M)^* \cong X^*/({}^\perp M)^\perp = X^*/M$.

2. **Examples.**

EXAMPLE 2.1. In [10] Luecking shows that $(H^\infty + C)/H^\infty$ is an M-ideal in L^∞/H^∞. In [2], it is shown that $L^\infty/(H^\infty + C)$ (which may be identified with $(L^\infty/H^\infty)/(H^\infty + C/H^\infty)$) is not a dual space. This is accomplished by demonstrating that the closed unit ball of $L^\infty/(H^\infty + C)$ has no extreme points. From this fact and Corollary 1 above we get the following result.

COROLLARY 2. *$(H^\infty + C)/H^\infty$ is not an M-summand in L^∞/H^∞.*

We are using the fact that L^∞/H^∞ is identified with the dual space of H^1, the space of analytic integrable functions. An alternate proof of Corollary 2 is obtained by noting that facts (1.2) and (1.3) imply that $(H^\infty + C)/H^\infty$ is weak-* dense in L^∞/H^∞. The corollary now follows from Proposition A.

We remark that several other examples exist of closed subalgebras B of L^∞ which contain H^∞ and for which B/H^∞ is an M-ideal in L^∞/H^∞ (cf. [12], [13]). Since every such B must contain $H^\infty + C$, the second proof of Corollary 2 implies that B/H^∞ is not an M-summand.

EXAMPLE 2.2. Though he did not know it at the time, Dixmier showed in [5] that $\mathscr{K}$ is an M-ideal in $\mathscr{L}(H)$. In [9] it is shown that $\mathscr{K}$ is not an M-summand. The proof relies on an investigation of metric complements and uses the approximation properties of M-ideals. We now offer a simpler proof.

COROLLARY 3. *$\mathscr{K}$ is not an M-summand in $\mathscr{L}(H)$.*

PROOF. It follows from (1.4) above that $\mathscr{K}$ is not weak-* closed. Proposition A now gives the result.

EXAMPLE 2.3. Let $\mathscr{A} = \text{Alg}\,\mathscr{P}$ be a nest algebra. It is shown in [7] that $\mathscr{L}(H)/\mathscr{A}$ is a dual space. In [8], this author proved that $(\mathscr{A} + \mathscr{K})/\mathscr{A}$ is an M-ideal in $\mathscr{L}(H)/\mathscr{A}$ but not an M-summand. The proof of this last fact used metric complements, the approximation properties of M-ideals, and Arveson's distance formula for nest algebras. A simpler proof is the following.

COROLLARY 4. *$(\mathscr{A} + \mathscr{K})/\mathscr{A}$ is not an M-summand in $\mathscr{L}(H)/\mathscr{A}$.*

PROOF. It follows from (1.2) and (1.4) above that $(\mathscr{A} + \mathscr{K})/\mathscr{A}$ is weak-* dense in $\mathscr{L}(H)/\mathscr{A}$. Thus, by Proposition A, $(\mathscr{A} + \mathscr{K})/\mathscr{A}$ is not an M-summand.

3. ***M*-ideals in dual spaces.** Proposition A shows that if M is an M-summand of the dual space X^* then M is closed in the weak-* topology. We now show that the converse holds for M-ideals of X^*. The result and its proof are related to but simpler than a theorem due to Cunningham ([3; Thm. 5]).

PROPOSITION B. *Let X be a Banach space and let J be an M-ideal of X^*. If J is closed in the weak-* topology, then J is an M-summand of X^*.*

PROOF. Since J is an M-ideal of X^*, there is an L-projection P of X^{**} onto $J^{\perp}$ satisfying $\|\phi\| = \|P\phi\| + \|\phi - P\phi\|$ for all $\phi \in X^{**}$. For $x \in X$, let $\hat{x} \in X^{**}$ be defined by $\hat{x}(f) = f(x)$ for $f \in X^*$. For each $f \in X^*$, define Qf and $(1 - Q)f$ by $(Qf)(x) = (\hat{x} - P\hat{x})(f)$ and $((1 - Q)f)(x) = (P\hat{x})(f)$ for all $x \in X$. Clearly, $Qf \in X^*$, $(1 - Q)f \in X^*$, and $f = Qf + (1 - Q)f$.

If $x \in {}^{\perp}J$ then $\hat{x} \in J^{\perp}$ so that $P\hat{x} = \hat{x}$. For such x we thus have $(Qf)(x) = 0$ for all $f \in X^*$. That is, $Qf \in ({}^{\perp}J)^{\perp}$. Since J is weak-* closed, it follows that $J = ({}^{\perp}J)^{\perp}$ and, hence, that $Qf \in J$ for all $f \in X^*$. On the other hand, if $f \in J$ then $(1 - Q)f = 0$ since $P\hat{x} \in J^{\perp}$ for all $x \in X$. This implies that $Q:X^* \to J$ is a projection of X^* onto J. It remains to show that Q is an M-projection.

For this, note first that, for any $f \in X^*$,

$$\|Qf\| = \sup_{\|x\|=1} |(\hat{x} - P\hat{x})(f)| \leqq \sup_{\|x\|=1} (|(\hat{x} - P\hat{x})(f)| + |(P\hat{x})(f)|)$$

$$\leqq \sup_{\|x\|=1} (\|\hat{x} - P\hat{x}\| + \|P\hat{x}\|) \cdot \|f\| = \sup_{\|x\|=1} \|\hat{x}\| \cdot \|f\| = \|f\|.$$

Similarly, $\|(1 - Q)f\| \leqq \|f\|$. Hence,

$$\|f\| \geqq \max\{\|Qf\|, \|(1 - Q)f\|\} \text{ for all } f \in X^*.$$

To prove the reverse inequality, notice that $(P\hat{x})(Qf) = (\hat{x} - P\hat{x})((1 - Q)f) = 0$ for all $f \in X^*$, $x \in X$. Thus, $(P\hat{x})(f) = (P\hat{x})((1 - Q)f)$ and $(\hat{x} - P\hat{x})(f) = (\hat{x} - P\hat{x})(Qf)$.

For $f \in X^*$ and $x \in X$ we now have

$$\begin{aligned} |f(x)| &= |\hat{x}(f)| \leqq |(\hat{x} - P\hat{x})(f)| + |(P\hat{x})(f)| \\ &= |(\hat{x} - P\hat{x})(Qf)| + |(P\hat{x})((1 - Q)f)| \\ &\leqq \|\hat{x} - P\hat{x}\| \cdot \|Qf\| + \|P\hat{x}\| \cdot \|(1 - Q)f\| \\ &\leqq (\|\hat{x} - P\hat{x}\| + \|P\hat{x}\|) \cdot \max\{\|Qf\|, \|(1 - Q)f\|\} \\ &= \|x\| \cdot \max\{\|Qf\|, \|(1 - Q)f\|\}. \end{aligned}$$

Hence, $\|f\| \leqq \max\{\|Qf\|, \|(1 - Q)f\|\}$.

We conclude that $\|f\| = \max\{\|Qf\|, \|(1 - Q)f\|\}$ for all $f \in X^*$ which implies that Q is an M-projection and, therefore, that J is an M-summand of X^*. □

Propositions A and B taken together assert that an M-ideal of X^* is an M-summand of X^* if and only if it is closed in the weak-* topology. For the case of dual Banach spaces, this answers a problem posed by Holmes, Scranton, and Ward in [9].

We conclude with some comments concerning metric complements which, as was mentioned above, were used in some earlier theorems on M-summands. For a closed subspace M of the Banach space X, the metric complement of M is given by $M^0 = \{x \in X: \|x\| = d(x, M)\}$.

In [8], it is shown that the metric complement of an M-summand always has non-empty interior and that the metric complement of $(\mathscr{A} + \mathscr{K})/\mathscr{A}$ is nowhere dense in $(\mathscr{L}(H))/\mathscr{A}$ where $\mathscr{A}$ is a nest algebra. In [9], the authors show that the metric complement of $\mathscr{K}$ is nowhere dense in $\mathscr{L}(H)$ and they speculate that M-summands may be distinguishable from other M-ideals according to whether the metric complement has non-empty or empty interior. We now add some evidence in favor of this speculation with the following result.

PROPOSITION C. *Let X be a Banach space and let M be a subspace of X^* with the property that the closed unit ball of M is dense in the closed unit ball of X^* in the weak-* topology. Then the metric complement of M is nowhere dense.*

REMARK. Facts (1.1), (1.2), (1.3), (1.4) show that this proposition applies to the M-ideals of Examples 2.1, 2.2, and 2.3.

PROOF. The metric complement of M is defined by $M^0 = \{f \in X^*: \|f\| = \text{dist}(f, M)\}$. Clearly, M^0 is a norm closed subset of X^*. We will show that M^0 contains no open balls.

First, let $m \in M$ satisfy $\|m\| = 1$. Then for $0 < \lambda < 1$ we have $\|\lambda m\| = \lambda$ and $\lambda m \in M$ so that $\lambda m \notin M^0$. Hence, M^0 contains no open ball centered at the origin.

Next, suppose $f \in X^*$ and $\delta > 0$ satisfy $0 < \delta < \|f\|$. Choose $x \in X$ such that $\|x\| = 1$ and $|f(x)| > \|f\| - \delta/4$. Define $f_1 \in X^*$ by $f_1(y) = \delta/(2|f(x)|) \cdot f(y)$. We have

$$\|f_1\| = \frac{\delta\|f\|}{2|f(x)|} < \frac{\delta \cdot \|f\|}{2(\|f\| - \delta/4)} < \frac{\delta \cdot \|f\|}{2 \cdot \|f\| - \delta} \leqq \frac{\delta\|f\|}{(\|f\| + \delta - \delta)} = \delta$$

so that f_1 is in the ball of radius δ centered at the origin. The density hypothesis implies that there is a sequence $\{g_n\} \subseteq M$ satisfying $\|g_n\| < \delta$ for all n and $g_n \to f_1$ in the weak-* topology. Hence, there exists N such that $|g_N(x) - f_1(x)| < \delta/4$. Let $g_N(x) = f_1(x) + \xi$ where $|\xi| < \delta/4$. We now have $\|(f + g_N) - f\| = \|g_N\| < \delta$ so that $f + g_N$ lies in a ball of radius δ about f. Also,

$$\begin{aligned}\|f + g_N\| &\geqq |(f + g_N)(x)| \\ &= |f(x) + f_1(x) + \xi| = \left| f(x)\left(1 + \frac{\delta}{2|f(x)|}\right) + \xi \right| \\ &\geqq |f(x)|\left(1 + \frac{\delta}{2|f(x)|}\right) - |\xi| > |f(x)| + \frac{\delta}{2} - \frac{\delta}{4} > \|f\|\end{aligned}$$

$$\geqq \text{dist}(f, M) = \text{dist}(f + g_N, M).$$

Thus, $f + g_N \notin M^0$. We conclude that M^0 contains no open ball centered at f. This completes the proof of the proposition.

We would like to thank the referee for making several useful suggestions and for pointing out to us the following construction which shows that there do exist M-ideals which are not M-summands but whose metric complements have non-empty interior. Let M be an M-ideal in a Banach space Y and let W be any other Banach space. Let $X = Y \oplus W$ where $\|y \oplus w\| = \max\{ \|y\|, \|w\| \}$ for all $y \in Y$ and $w \in W$. Then $M \oplus \{0\}$ is an M-ideal in X and its metric complement, $(M \oplus \{0\})^0$, contains the open set $\{ y \oplus w \in X : \|y\| < \|w\| \}$. If Y and W are dual spaces, then so is X. Also, if M is not an M-summand in Y, then neither is $M \oplus \{0\}$ in $Y \oplus W$.

Thus, a necessary condition for an M-ideal M in a Banach space to have metric complement which is nowhere dense is that there does not exist an M-summand N satisfying $M \subseteq N \subseteq X$.

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