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THE OPTIMUM PROCESSING OF CLIPPED SIGNALS: AN APPROACH BASED ON A LIKELIHOOD RATIO STATISTIC

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Abstract

The likelihood ratio approach to the detection of small signals in the presence of noise is investigated in the case where the available data have been clipped. The statistic obtained is the ratio of orthant probabilities and appears intractable; accordingly an approximation to this statistic is developed by truncating an appropriate Taylor expansion. Approximations are obtained for the mean and variance of this modified statistic and compared with those obtained from computer simulations.

1. Introduction

For many years there has been interest in signal processors which operate on clipped inputs; the input, $I_i(t)$, from each receiver, R_i, being first transformed to $\operatorname{sgn}[I_i(t)]$ before any processing is carried out. Early work in this field was described by Faran and Hills [4] since when the performance of such processors has been discussed by many authors (see, for example, [2, 3, 8, 13, 15]). Recently this work has been generalised [10] and extended to cover the optimum processing of clipped inputs [9]. Three approaches to such optimisation were mentioned in [9] and one approach, namely minimum signal distortion, was discussed in some detail.

Another approach, mentioned in [9], is based on maximum signal to noise ratio. This approach has proved useful in the case of linear processing, but is less helpful when non-linear processing is used. One difficulty lies with the definition of signal to noise ratio; definitions which are equivalent in the linear case are, in

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general, no longer equivalent when the processing is non-linear. Various definitions of signal to noise ratio have been suggested; see, for example, Blachman [1]. An optimum processor based on a signal to noise criterion will, in general, depend on the definition adopted; accordingly that criterion is not likely to prove generally useful.

One firmly based criterion for optimum signal detection is the likelihood ratio arising from the Neyman-Pearson lemma [5]. At first glance the application of that lemma to the present problem appears intractable, but some progress has been made in the case where the signal power is much less than the noise power; this progress is reported below.

2. Notation and assumptions

The input to the ith of n receivers, R_i, consists of a signal S_i, plus noise, N_i. These inputs

$$I_i(t) = S_i(t) + N_i(t), \qquad i = 1,\dots,n,$$

are clipped to give

$$A_i(t) = \operatorname{sgn}[I_i(t)]$$

and then sampled at equal time intervals, $k = 1,\dots,m$. It is assumed that the signal and noise are both normal, ergodic random processes with zero mean and the signal is not correlated with the noise. Thus in the presence of signal the input to the ith receiver at time k is

$$I_{ik} = S_{ik} + N_{ik}$$

and the covariance is

$$E[I_{ik}I_{jl}] = \sigma_i^S\sigma_j^S\rho_{ij}^S(l-k) + \sigma_i^N\sigma_j^N\rho_{ij}^N(l-k),$$

where σ_i^S and σ_i^N are the standard deviations of S_i and N_i respectively, $\rho_{ij}^S(l-k)$ is the correlation coefficient of S_{ik} and S_{jl}, and $\rho_{ij}^N(l-k)$ is the correlation coefficient of N_{ik} and N_{jl}. The correlation coefficient corresponding to $E[I_{ik}I_{jl}]$ is

$$\rho_{ik,jl}^I = a_i a_j \rho_{ij}^S(l-k) + (1-a_i^2)^{1/2}(1-a_j^2)^{1/2}\rho_{ij}^N(l-k), \tag{1}$$

where

$$a_i = \begin{cases} \dfrac{\sigma_i^S}{\left(\sigma_i^{2^S} + \sigma_i^{2^N}\right)^{1/2}}, & \text{signal present,} \\ 0, & \text{signal absent.} \end{cases}$$

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We now define two random vectors **I** and **A** each having nm elements with

$$I_{ik} \text{ as the } [(i-1)m+k]\text{th element of } \mathbf{I}$$

and

$$A_{ik} \text{ as the } [(i-1)m+k]\text{th element of } \mathbf{A}.$$

A realisation of **A** will be denoted by $\boldsymbol{\alpha}$ which is a vector with elements α_{ik} equal to $+1$ or -1; one or more of these vectors $\boldsymbol{\alpha}$ will be used to decide between the simple hypotheses:

(i) signal is absent;

(ii) signal is present.

3. The likelihood ratio

The likelihood ratio for each realisation $\boldsymbol{\alpha}$ is

$$LR(\boldsymbol{\alpha}) = \frac{L_1(\boldsymbol{\alpha})}{L_0(\boldsymbol{\alpha})} = \frac{P[\mathbf{A} = \boldsymbol{\alpha} \mid \text{signal is present}]}{P[\mathbf{A} = \boldsymbol{\alpha} \mid \text{signal is absent}]}. \tag{2}$$

The Neyman-Pearson lemma requires that the realisations $\boldsymbol{\alpha}$ be allotted to one of two disjoint classes according to the value of $LR(\boldsymbol{\alpha})$. A constant K is chosen based on an acceptable probability of false alarm and if $LR(\boldsymbol{\alpha}) \geqslant K$ we decide that signal is present, otherwise we decide signal is absent.

Consider first the case where each of the nm elements of $\boldsymbol{\alpha}$ is $+1$. Then the denominator of $LR(\boldsymbol{\alpha})$, the probability that all the I_{ik} are positive, is

$$L_0(\boldsymbol{\alpha}) = \left(\frac{1}{2\pi}\right)^{mn/2} |R_0|^{-1/2} \int_0^\infty \cdots \int_0^\infty \exp\{-\tfrac{1}{2}(\mathbf{x}^T R_0^{-1} \mathbf{x})\}\, d\mathbf{x}, \tag{3}$$

where R_0 is the correlation matrix with entries given by equation (1) when signal is absent; that is, the (u, v)th element of R_0 is

$$\rho^I_{ik,jl} = \rho^N_{ij}(l-k),$$

where $u = (i-1)m + k$ and $v = (j-1)m + l$. Similarly the numerator of $LR(\boldsymbol{\alpha})$ is

$$L_1(\boldsymbol{\alpha}) = \left(\frac{1}{2\pi}\right)^{mn/2} |R_1|^{-1/2} \int_0^\infty \cdots \int_0^\infty \exp\{-\tfrac{1}{2}(\mathbf{x}^T R_1^{-1} \mathbf{x})\}\, d\mathbf{x},$$

where the (u, v)th element of R_1 is given by equation (1) when signal is present.

In general $\boldsymbol{\alpha}$ has entries $+1$ and -1, corresponding to some of the I_{ik} being positive and some negative. The orthant probabilities L_1 and L_0 are then, after some re-arrangement, of the form

$$L = \left(\frac{1}{2\pi}\right)^{mn/2} |R|^{-1/2} \int_0^\infty \cdots \int_0^\infty \cdots \int_{-\infty}^0 \cdots \int_{-\infty}^0 \exp\left\{-\tfrac{1}{2}\left(\mathbf{y}^T R^{-1} \mathbf{y}\right)\right\} d\mathbf{y}$$

$$= \left(\frac{1}{2\pi}\right)^{mn/2} |R^*|^{-1/2} \int_0^\infty \cdots \int_0^\infty \exp\left\{-\tfrac{1}{2}\left(\mathbf{x}^T (R^*)^{-1} \mathbf{x}\right)\right\} d\mathbf{x},$$

where R is either R_0 or R_1 and the elements of R^* are

$$r_{uv} = \begin{cases} \rho^I_{ik,jl}, & \text{if } \alpha_{ik} \text{ and } \alpha_{jl} \text{ have the same sign,} \\ -\rho^I_{ik,jl}, & \text{otherwise.} \end{cases}$$

The product $\alpha_{ik}\alpha_{jl}$ will perform the required change of sign so, from equation (1),

$$r_{uv} = \alpha_{ik}\alpha_{jl}\left[a_i a_j \rho^S_{ij}(l-k) + \left(1 - a_i^2\right)^{1/2}\left(1 - a_j^2\right)^{1/2} \rho^N_{ij}(l-k)\right]. \tag{4}$$

There is a large literature on orthant probabilities which is discussed in Johnson and Kotz [7, pages 45–58]. Approximate methods for evaluating such probabilities exist and exact calculations are possible in a number of special cases.

Since the expression for $LR(\boldsymbol{\alpha})$ will generally be intractable, we need to use some approximation method. In most cases of interest the a_i are small, so we expand $L_1(\boldsymbol{\alpha}) = \Phi(\mathbf{a})$ as a Taylor series in $\mathbf{a} = (a_1, \ldots, a_n)$ about $\mathbf{a} = \mathbf{0}$ to give

$$L_1(\boldsymbol{\alpha}) = \Phi(\mathbf{a}) = \Phi(\mathbf{0}) + \sum_{p=1}^n a_p \left[\frac{\partial \Phi}{\partial a_p}\right]_{\mathbf{a}=\mathbf{0}} + \frac{1}{2} \sum_{p=1}^n a_p^2 \left[\frac{\partial^2 \Phi}{\partial a_p^2}\right]_{\mathbf{a}=\mathbf{0}}$$

$$+ \sum_{p<q} a_p a_q \left[\frac{\partial^2 \Phi}{\partial a_p \partial a_q}\right]_{\mathbf{a}=\mathbf{0}} + \text{higher order terms},$$

where

$$\Phi(\mathbf{0}) = L_0(\boldsymbol{\alpha}).$$

Also

$$\frac{\partial \Phi}{\partial a_p} = \sum_{u<v} \left(\frac{\partial \Phi}{\partial r_{uv}}\right)\left(\frac{\partial r_{uv}}{\partial a_p}\right)^{\dagger}$$

and

$$\frac{\partial^2 \Phi}{\partial a_p \partial a_q} = \sum_{u<v} \left[\left(\frac{\partial r_{uv}}{\partial a_p}\right) \frac{\partial}{\partial a_q}\left(\frac{\partial \Phi}{\partial r_{uv}}\right) + \left(\frac{\partial \Phi}{\partial r_{uv}}\right)\left(\frac{\partial^2 r_{uv}}{\partial a_p \partial a_q}\right)\right].$$

†In this paper we follow Plackett [12] and use the symmetry of R_0, R_1 and R^* so that there are only $\frac{1}{2}n(n-1)$ distinct off-diagonal elements r_{uv} to consider.

From equation (4), $[\partial r_{uv}/\partial a_p]_{\mathbf{a}=\mathbf{0}} = 0$, so that, omitting higher order terms

$$L_1(\boldsymbol{\alpha}) = L_0(\boldsymbol{\alpha}) + \frac{1}{2}\sum_{p=1}^{n} a_p^2\left[\sum_{u<v}\left(\frac{\partial\Phi}{\partial r_{uv}}\right)\left(\frac{\partial^2 r_{uv}}{\partial a_p^2}\right)\right]_{\mathbf{a}=\mathbf{0}} + \sum_{p<q} a_p a_q\left[\sum_{u<v}\left(\frac{\partial\Phi}{\partial r_{uv}}\right)\left(\frac{\partial^2 r_{uv}}{\partial a_p \partial a_q}\right)\right]_{\mathbf{a}=\mathbf{0}}. \tag{5}$$

The expression involving the derivatives of Φ can be further simplified using a result of Plackett [12],

$$\frac{\partial\Phi}{\partial r_{uv}} = \left(\frac{1}{2\pi}\right)^{mn/2}|R^*|^{-1/2}\int_0^\infty\cdots\int_0^\infty \frac{\partial}{\partial x_u}\frac{\partial}{\partial x_v}\exp\left\{-\tfrac{1}{2}\left(\mathbf{x}^T(R^*)^{-1}\mathbf{x}\right)\right\}d\mathbf{x}. \tag{6}$$

In the sequel the higher order terms in the approximation for $L_1(\boldsymbol{\alpha})$ have been omitted, but we will not change the notation. Thus the statistic hereafter called $LR(\cdot)$ is not strictly the likelihood ratio.

4. An example

The following example is of theoretical interest and also may be of practical importance; accordingly it is discussed in some detail.

We assume that the noise processes are uncorrelated so that

$$\rho_{ij}^N(l-k) = \begin{cases} 1, & \text{if } i = j \text{ and } l-k = 0,\\ 0, & \text{otherwise,}\end{cases}$$

and the signals at each input are identical, that is,

$$\rho_{ij}^S(l-k) = \rho^S(\tau) \quad \text{for all } i \text{ and } j,$$

where $\tau = |l-k|$ takes the values $0, 1, 2, \ldots, m-1$, and $\rho^S(0) = 1$.

In this case if signal is absent, then by equation (4) R_0^* is the identity matrix so that $L_0(\boldsymbol{\alpha}) = 2^{-mn}$.

When signal is present we have, from (4),

$$\frac{\partial^2 r_{uv}}{\partial a_i^2} = \begin{cases} 2\alpha_{ik}\alpha_{il}\rho^S(\tau), & \text{if } i = j \text{ and } k \neq l,\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\frac{\partial^2 r_{uv}}{\partial a_i \partial a_j} = \alpha_{ik}\alpha_{jl}\rho^S(\tau), \qquad i \neq j.$$

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Also from (4), at $\mathbf{a} = \mathbf{0}$, R_1^* is the identity matrix so (6) yields:

$$\left[\frac{\partial \Phi}{\partial r_{uv}}\right]_{\mathbf{a}=\mathbf{0}} = \left(\frac{1}{2\pi}\right)^{mn/2} \int_0^\infty \cdots \int_0^\infty \frac{\partial}{\partial x_u} \frac{\partial}{\partial x_v} \exp\{-\tfrac{1}{2}(\mathbf{x}^T\mathbf{x})\}\, d\mathbf{x} = (2/\pi)2^{-mn}.$$

Hence (5) becomes

$$L_1(\boldsymbol{\alpha}) = 2^{-mn} + (1/\pi)2^{-mn} \sum_{i=1}^{n} a_i^2 \sum_{k<l} 2\alpha_{ik}\alpha_{il}\rho^S(l-k)$$
$$+ (2/\pi)2^{-mn} \sum_{i<j} a_i a_j \sum_{k=1}^{m} \sum_{l=1}^{m} \alpha_{ik}\alpha_{jl}\rho^S(l-k).$$

Note that $u < v$ requires $k < l$ when $i = j$, but when $i \neq j$, since $0 \leqslant |k - l| \leqslant m - 1$, $u < v$ requires $i < j$ only; also the a_i are non-zero since signal is present for L_1. Therefore

$$LR(\boldsymbol{\alpha}) = 1 + \frac{2}{\pi} \sum_{i=1}^{n} a_i^2 \sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{k=1}^{m-\tau} \alpha_{ik}\alpha_{i,k+\tau}$$
$$+ \frac{2}{\pi} \sum_{i<j} a_i a_j \left[\sum_{k=1}^{m} \rho^S(0)\alpha_{ik}\alpha_{jk} + \sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{k=1}^{m-\tau} (\alpha_{ik}\alpha_{j,k+\tau} + \alpha_{i,k+\tau}\alpha_{jk}) \right].$$

To simplify the right hand side write $q_k^* = \sum_{i=1}^{n} a_i \alpha_{ik}$ and note $\rho^S(0) = 1$ so that

$$LR(\boldsymbol{\alpha}) = 1 + \frac{2}{\pi} \sum_{k=1}^{m} \frac{1}{2}\left(q_k^{*2} - \sum_{i=1}^{n} a_i^2 \right) + \frac{2}{\pi} \sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{k=1}^{m-\tau} q_k^* q_{k+\tau}^*$$
$$= 1 + \frac{1}{\pi}\left(\sum_{k=1}^{m} q_k^{*2} - m \sum_{i=1}^{n} a_i^2 \right) + \frac{2}{\pi} \sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{k=1}^{m-\tau} q_k^* q_{k+\tau}^*. \tag{7}$$

Hence $LR(\boldsymbol{\alpha}) \geqslant K$ if

$$\frac{1}{2} \sum_{k=1}^{m} q_k^{*2} + \sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{k=1}^{m-\tau} q_k^* q_{k+\tau}^* \geqslant K_1, \tag{8}$$

where $2K_1 = \pi(K-1) + m\sum_{i=1}^{n} a_i^2$.

The terms on the left hand side of (8) may be written as $\mathbf{v}^T\mathbf{u}$ where the elements of $\mathbf{v}$ depend only on the correlation function of the signal, and

$$\mathbf{v}^T = \{\tfrac{1}{2}\rho^S(0), \rho^S(1), \rho^S(2) \cdots \rho^S(m-1)\}.$$

The elements of $\mathbf{u}$ are closely related to the estimate of the correlation function of the sequence $\{q_k^*\}$, the terms of which are the sums of weighted clipped inputs at the times $k = 1, \ldots, m$.

To simplify the problem further we take $\sigma_i^S = \sigma^S$ and $\sigma_i^N = \sigma^N$ for all i, so

$$a_i = a \text{ for all } i$$

$$= \begin{cases} \dfrac{\sigma^S}{(\sigma^{2^S} + \sigma^{2^N})^{1/2}}, & \text{signal present,} \\ 0, & \text{signal absent.} \end{cases}$$

If we write $q_k = \sum_{i=1}^n \alpha_{ik}$, equation (7) then becomes

$$LR(\boldsymbol{\alpha}) = 1 + \frac{a^2}{\pi}\left[\sum_{k=1}^m q_k^2 - mn + 2\sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{k=1}^{m-\tau} q_k q_{k+\tau}\right], \tag{9}$$

(where a is non-zero, since it arose in L_1). Thus if the standard deviations and correlation functions of the signal and noise are known and satisfy the assumptions made in deriving (7), then $LR(\boldsymbol{\alpha})$ may be calculated for each observed value $\boldsymbol{\alpha}$ of $\mathbf{A}$.

$LR(\boldsymbol{\alpha})$ is an observed value of the statistic $LR(\mathbf{A})$ which is defined as follows: if

$$Q_k = \sum_{i=1}^n A_{ik}$$

and

$$T = \sum_{k=1}^m Q_k^2 + 2\sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{k=1}^{m-\tau} Q_k Q_{k+\tau}$$

then

$$LR(\mathbf{A}) = 1 + (c^2/\pi)(T - mn) \tag{10}$$

where the constant c is the value of a when signal is present.

4.1 The expectation and variance of $LR(\mathbf{A})$.

From equation (10),

$$Q_k^2 = \left(\sum_{i=1}^n A_{ik}\right)^2 = n + \sum_{i \neq j} A_{ik} A_{jk}$$

and

$$Q_k Q_{k+\tau} = \sum_{i=1}^n \sum_{j=1}^n A_{ik} A_{j,k+\tau}.$$

For distinct pairs (i, k), (j, l), we have from (1), using a well known result (which can be readily obtained from [12]),

$$E(A_{ik} A_{jl}) = (2/\pi)\arcsin \rho_{ik,jl}^I = (2/\pi)\arcsin\{a^2 \rho^S(\tau)\},$$

since $a_i = a$ and $\rho^N_{ik}(k-l) = 0$ when $(i,k) \neq (j,l)$; so that

$$E(Q_k^2) = n + (2/\pi)n(n-1)\arcsin a^2,$$

and

$$E(Q_k Q_{k+\tau}) = (2n^2/\pi)\arcsin\{a^2\rho^S(\tau)\}.$$

Using Taylor expansions for the arcsin terms and omitting terms which are $o(a^4)$, we obtain

$$E(T) = mn + (2a^2n/\pi)\left\{m(n-1) + 2n\sum_{\tau=1}^{m-1}(m-\tau)\{\rho^S(\tau)\}^2\right\}.$$

Therefore

$$E\{LR(\mathbf{A})\} = 1 + (2a^2c^2n/\pi^2)\left\{m(n-1) + 2n\sum_{\tau=1}^{m-1}(m-\tau)\{\rho^S(\tau)\}^2\right\} \quad (11)$$

and when signal is absent $E\{LR(\mathbf{A})\} = 1$ since $a = 0$. The variance of $LR(\mathbf{A})$ is

$$\sigma^2_{LR(\mathbf{A})} = (c^4/\pi^2)\sigma_T^2 = (c^4/\pi^2)\{E\{T^2\} - E\{T\}^2\}.$$

The derivation of $E\{T^2\}$ is outlined in the Appendix; a full account will appear in [6].

The result, omitting terms in a of degree 4 and higher, is

$$\begin{aligned}
E\{T^2\} &= mn\{mn + 2n - 2\} + 4n^2\sum_{\tau=1}^{m-1}(m-\tau)\rho^S(\tau)^2 \\
&+ \frac{4a^2}{\pi}\Bigg\{mn(n-1)(mn+2n-4) + \{2mn^3 + 12n^2(n-1)\}\sum_{\tau=1}^{m-1}(m-\tau)\rho^S(\tau)^2 \\
&\quad + 4n^3\sum_{\tau=1}^{[(m-1)/2]}(m-2\tau)\rho^S(2\tau)\rho^S(\tau)^2 \\
&\quad + 8n^3\Bigg\{\sum_{\tau=2}^{m-1}\sum_{\lambda=1}^{\tau-1}(m-\tau)\rho^S(\tau-\lambda)\rho^S(\tau)\rho^S(\lambda) \\
&\qquad + \sum_{\tau=2}^{[m/2]}\sum_{\lambda=1}^{\tau-1}(\mathrm{m}-\tau-\lambda)\rho^S(\tau+\lambda)\rho^S(\tau)\rho^S(\lambda) \\
&\qquad + \sum_{\tau=[m/2]+1}^{m-2}\sum_{\lambda=1}^{m-\tau-1}(m-\tau-\lambda)\rho^S(\tau+\lambda)\rho^S(\tau)\rho^S(\lambda)\Bigg\}\Bigg\},
\end{aligned} \quad (12)$$

where [·] means "integer part of" in the summation limits. Squaring the result for $E\{T\}$, again omitting terms in a of degree 4 and higher, gives

$$E^2\{T\} = m^2n^2 + (4a^2/\pi)mn^2\left\{m(n-1) + 2n\sum_{\tau=1}^{m-1}(m-\tau)\rho^S(\tau)^2\right\}.$$

These two results give

$$\begin{aligned}\sigma^2_{LR(A)} = (2c^4/\pi^2)\Bigg\{&mn(n-1) + 2n^2\sum_{\tau=1}^{m-1}(m-\tau)\rho^S(\tau)^2 \\ &+ (4a^2/\pi)\Bigg\{mn(n-1)(n-2) + 6n^2(n-1)\sum_{\tau=1}^{m-1}(m-\tau)\rho^S(\tau)^2 \\ &+2n^3\sum_{\tau=1}^{[(m-1)/2]}(m-2\tau)\rho^S(2\tau)\rho^S(\tau)^2 \\ &+4n^3\sum_{\tau=2}^{m-1}\sum_{\lambda=1}^{\tau-1}(m-\tau)\rho^S(\tau-\lambda)\rho^S(\tau)\rho^S(\lambda) \\ &+4n^3\sum_{\tau=2}^{[m/2]}\sum_{\lambda=1}^{\tau-1}(m-\tau-\lambda)\rho^S(\tau+\lambda)\rho^S(\tau)\rho^S(\lambda) \\ &+4n^3\sum_{\tau=[m/2]+1}^{m-2}\sum_{\lambda=1}^{m-\tau-1}(m-\tau-\lambda)\rho^S(\tau+\lambda)\rho^S(\tau)\rho^S(\lambda)\Bigg\}\Bigg\}.\end{aligned} \tag{13}$$

In the absence of signal $a = 0$ so the expression (13) becomes

$$\sigma^2_{LR(A)} = (2c^4/\pi^2)\left\{mn(n-1) + 2n^2\sum_{\tau=1}^{m-1}(m-\tau)\rho^S(\tau)^2\right\}.$$

4.2 Computer simulation results.

Simulation of the signal and noise processes was carried out using the parameter values $n = 6$, $m = 10$, $\sigma^N = 1.0$, $\sigma^S = 0.1$ and $\rho^S(\tau) = (0.9)^\tau$ so that $c = 0.1/(1.01)^{1/2}$. Realisations of the noise process were obtained using the polar method of Marsaglia and Bray [11] to transform uniformly distributed random numbers to independent normally distributed random numbers. An extension of this method was used to produce realisations of the signal process. First a sequence $\{\xi_j\}$, being a realisation of a process which was uncorrelated and $N\{0, \{1-(\rho^S(1))^2\}\sigma^{2^S}\}$, was generated. The sequence of signal values $\{S_n\}$ was then obtained using the recurrence relation (first order autoregression):

$$S_1 = \xi_1; \quad S_n = \rho^S(1)S_{n-1} + \xi_n.$$

Providing the early terms in $\{S_n\}$ are discarded this procedure generates a sequence having the required signal characteristics.

The simulated noise and signal at each receiver were then added and clipped to produce the realisation $\boldsymbol{\alpha}$ and hence $LR(\boldsymbol{\alpha})$ from equation (9). Table 1 shows the empirical means and standard deviations of $LR(\mathrm{A})$ obtained in 12 trials each of 1000 observations of $LR(\mathrm{A})$.

TABLE 1
Empirical mean and standard deviation of $LR(\mathrm{A})$

	Signal	absent	Signal	present
Trial	Mean	S.D.	Mean	S.D.
1	1.005	.198	1.045	.242
2	.996	.194	1.035	.237
3	.986	.171	1.037	.230
4	1.002	.196	1.045	.261
5	.986	.175	1.024	.231
6	.989	.185	1.028	.231
7	.997	.193	1.034	.244
8	.994	.196	1.027	.238
9	.994	.185	1.031	.232
10	1.006	.205	1.044	.258
11	1.000	.187	1.041	.244
12	.993	.201	1.025	.229

These results compare well with the corresponding theoretical moments calculated using equations (11) and (13), namely,

$$E\{LR(\mathrm{A})\} = \begin{cases} 1, & \text{signal absent,} \\ 1.039, & \text{signal present,} \end{cases}$$

and

$$\sigma_{LR(A)} = \begin{cases} 0.197, & \text{signal absent,} \\ 0.242, & \text{signal present.} \end{cases}$$

However the variances of $LR(A)$ are too large relative to the difference between the means to produce a powerful test for distinguishing between the hypotheses signal present and signal absent.*

If sufficiently many independent values of $LR(A)$ are averaged to give

$$\overline{LR} = \frac{1}{M} \sum_{i=1}^{M} LR(A_i)$$

then, by the Central Limit Theorem, $\overline{LR}$ will be approximately normal with variance proportional to M^{-1}. Therefore the appropriate sample size M can be calculated for a test with probability of false alarm, α, and probability of detection $1 - \beta$. For the case where $M = 150$, Table 2 provides a comparison between the theoretical mean and standard deviation of $\overline{LR}$ and empirical values calculated from 1000 observations of $\overline{LR}$.

TABLE 2

Theoretical and empirical means and standard deviations of $\overline{LR}$

	Signal absent	Signal present
Empirical mean	0.999	1.038
$E\{\overline{LR}\}$	1.00000	1.03874
Empirical S. D.	0.016	0.020
Theoretical S. D.	0.01607	0.01976

Figure 1 shows a histogram for $\overline{LR}$ with $M = 150$ in the cases signal absent and signal present. If a critical region is $\overline{LR} > 1.02$ as indicated on the figure then the corresponding empirical values $\hat{\alpha} = 0.097$ and $\hat{\beta} = 0.172$ agree very closely with the theoretical values of $\alpha = 0.106$ and $\beta = 0.171$ based on the assumption that $\overline{LR}$ is distributed normally.

*These results may be compared with those for the same problem using unclipped data, which gave a difference between the mean (signal present) and the mean (signal absent) of about 0.2 standard deviations.

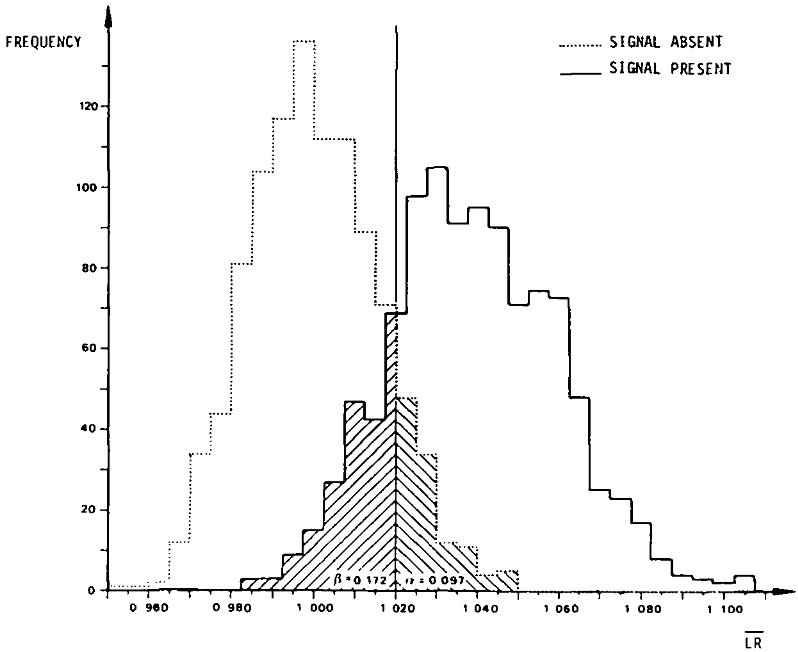

Figure 1. The histogram of $\overline{LR}$

The above approach involving the averaging of several independent values of $LR(\mathbf{A})$ has been adopted here as one method of comparing the performance of this processor with theory. Another approach which would probably be more effective in a practical situation would involve the techniques of sequential analysis [16].

5. Conclusion

Some progress has been made in the development of a statistic, based on a likelihood ratio, for the detection of signals in additive noise when the input data have been clipped. The main restriction imposed, to simplify the problem of validating the theory by simulation, is an assumption that the noise processes are uncorrelated. Under such restrictions the results of a simulation trial are in close agreement with theoretical approximations to the expectation and variance obtained in Section 4.1.

In many practical cases, however, the assumption that the noise processes are uncorrelated is not realistic and it will therefore be desirable to extend the present work to cover such cases. Additional work is also required to determine the effect of other assumptions and approximations made in the work presented in this paper.

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Appendix: the derivation of $E\{T^2\}$

Immediately before equation (10) a random variable T used in the expression for $LR(\mathbf{A})$ was defined as

$$T = \sum_{k=1}^{m} Q_k^2 + 2\sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{k=1}^{m-\tau} Q_k Q_{k+\tau},$$

where

$$Q_k = \sum_{\iota=1}^{n} A_{\iota k} \quad \text{and} \quad A_{\iota k} = \pm 1.$$

In order to determine the variance of $LR(\mathbf{A})$ we require $E[T^2]$.

Now

$$\begin{aligned} T^2 &= \left\{\sum_{k=1}^{m} Q_k^2\right\}^2 + 4\sum_{k=1}^{m} Q_k^2 \sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{l=1}^{m-\tau} Q_l Q_{l+\tau} + 4\left\{\sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{k=1}^{m-\tau} Q_k Q_{k+\tau}\right\}^2 \\ &= \sum_{k=1}^{m} Q_k^4 + \sum_{\substack{k,l=1 \\ (k\neq l)}}^{m} Q_k^2 Q_l^2 + 4\sum_{\tau=1}^{m-1} \rho^S(\tau) \sum_{k=1}^{m} \sum_{l=1}^{m-\tau} Q_k^2 Q_l Q_{l+\tau} \\ &\quad + 4\sum_{\tau=1}^{m-1} \rho^S(\tau)^2 \left\{\sum_{k=1}^{m-\tau} Q_k^2 Q_{k+\tau}^2 + \sum_{\substack{k,l=1 \\ (k\neq l)}}^{m-\tau} Q_k Q_{k+\tau} Q_l Q_{l+\tau}\right\} \\ &\quad + 8\sum_{\substack{\tau,\lambda=1 \\ (\tau>\lambda)}}^{m-1} \rho^S(\tau)\rho^S(\lambda) \sum_{k=1}^{m-\tau} \sum_{l=1}^{m-\lambda} Q_k Q_{k+\tau} Q_l Q_{l+\lambda}. \end{aligned} \tag{14}$$

So we require the expectation of

(i) Q_k^4, $(1 \leqslant k \leqslant m)$;

(ii) $Q_k^2 Q_l^2$, $(k \neq l,\ 1 \leqslant k \leqslant m \text{ and } 1 \leqslant l \leqslant m)$;

(iii) $Q_k^2 Q_l Q_{l+\tau}$, $(1 \leqslant k \leqslant m \text{ and } 1 \leqslant l \leqslant m-\tau)$;

(iv) $Q_k^2 Q_{k+\tau}^2$, $(1 \leqslant k \leqslant m-\tau)$;

(v) $Q_k Q_{k+\tau} Q_l Q_{l+\tau}$, $(k \neq l,\ 1 \leqslant k \leqslant m-\tau,\ 1 \leqslant l \leqslant m-\tau)$;

(vi) $Q_k Q_{k+\tau} Q_l Q_{l+\lambda}$, $(\tau > \lambda,\ 1 \leqslant k \leqslant m-\tau,\ 1 \leqslant l \leqslant m-\lambda)$.

In the expansion of each of these products terms of the form $A_{ik}A_{jl}A_{pq}A_{rw}$ occur in which all the subscript pairs are distinct. It may be proved that in this case the expectation $E\{A_{ik}A_{jl}A_{pq}A_{rw}\}$ consists of terms in a whose lowest degree is 4. Consistent with previous practice for LR itself these terms are neglected. If any two of the subscript pairs are equal, say $(i, k) = (p, q)$, terms of the form

$$A_{ik}^2 A_{jl} A_{rw} = A_{jl} A_{rw} \text{ occur;}$$

or if $(i, k) = (p, q)$ and $(j, l) = (r, w)$ we get

$$A_{ik}^2 A_{jl}^2 = 1;$$

if any three of the subscript pairs are equal, say $(i, k) = (p, q) = (r, w)$, terms of the form

$$A_{ik}^3 A_{jl} = A_{ik} A_{jl} \text{ occur;}$$

and if all four subscript pairs are equal, we get

$$A_{ik}^4 = 1.$$

We examine case (vi) only as it is the most complicated. (The other cases may be done similarly, but care must be taken with the conditions for each.)

$$Q_k Q_{k+\tau} Q_l Q_{l+\lambda}|_{(\tau>\lambda)} = \sum_{i,j,p,r=1}^{n} A_{ik} A_{j,k+\tau} A_{pl} A_{r,l+\lambda};$$

the product $A_{ik}A_{j,k+\tau}A_{pl}A_{r,l+\lambda}$ has distinct subscript pairs except in the following mutually exclusive cases:

(a) $k = l$ and $i = p$,
(b) $k = l + \lambda$ and $i = r$,
(c) $k + \tau = l$ and $j = p$,
(d) $k + \tau = l + \lambda$ and $j = r$.

In case (a), for each set of pairs (k, l) and (τ, λ) there are n^3 quadruples (i, j, i, r) for which

$$\begin{aligned} E\{A_{ik}A_{j,k+\tau}A_{pl}A_{r,l+\lambda}\} &= E\{A_{j,k+\tau}A_{r,l+\lambda}\} \\ &= (2/\pi)\arcsin \rho^{I}_{jq,rw} \quad (\text{where } q = k+\tau \text{ and } w = l+\lambda) \\ &= (2/\pi)\arcsin\{a^2\rho^S\,|\,q - w\,|\} \quad (\text{by assumptions in the text}) \\ &= (2/\pi)a^2\rho^S(\tau - \lambda). \end{aligned}$$

The same result is obtained for (d).

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In case (b), for each set of pairs (k, l) and (τ, λ) there are n^3 quadruples (i, j, p, i) for which

$$\begin{aligned} E\{A_{ik}A_{j,k+\tau}A_{pl}A_{r,l+\lambda}\} &= E\{A_{j,k+\tau}A_{pl}\} \\ &= (2/\pi)\arcsin\{a^2\rho^S|k+\tau-l|\} \\ &= (2/\pi)\arcsin\{a^2\rho^S(\tau+\lambda)\} \\ &= (2/\pi)a^2\rho^S(\tau+\lambda), \text{ ignoring higher order terms.} \end{aligned}$$

The same result is obtained for case (c).

Also since $1 \leqslant k \leqslant m-\tau$ and $1 \leqslant l \leqslant m-\lambda$,

in (a) $k = l$ for $1 \leqslant k \leqslant \min\{m-\tau, m-\lambda\} = m-\tau$;
so there are $m-\tau$ pairs (k, l) where $k = l$ for each (τ, λ);

in (b) $k = l+\lambda$ for $1+\lambda \leqslant k \leqslant m-\tau$,
so there are $m-\tau-\lambda$ pairs (k, l) where $k = l+\lambda$ for each (τ, λ);

in (c) $k+\tau = l$ for $1+\tau \leqslant k \leqslant m-\lambda$,
so there are $m-\tau-\lambda$ pairs (k, l) where $k+\tau = l$ for each (τ, λ); and

in (d) $k+\tau = l+\lambda$ for $\max\{1+\tau, 1+\lambda\} \leqslant k \leqslant m$, i.e., $1+\tau \leqslant k \leqslant m$,
so there are $m-\tau$ pairs (k, l) where $k+\tau = l+\lambda$ for each (τ, λ).

In all four cases, since $\tau > \lambda$, then $\tau \geqslant \lambda+1 \geqslant 2$. In cases (b) and (c), $m-\tau-\lambda$ must be $\geqslant 1$, so we have

$$\tau \leqslant m-\lambda-1 \leqslant m-2;$$

$$\text{and also } \left.\begin{aligned} \lambda &\leqslant \tau-1 \\ \text{and } \lambda &\leqslant m-\tau-1 \end{aligned}\right\};$$

$$\text{i.e., } \lambda \leqslant \min\{\tau-1, m-\tau-1\}.$$

But $\tau-1 \leqslant m-\tau-1$ for $\tau \leqslant [m/2]$, and $\tau-1 > m-\tau-1$ for $\tau > [m/2]$. So the limits for τ, λ in cases (b) and (c) are:

$$1 \leqslant \lambda \leqslant \tau-1 \quad \text{when } 2 \leqslant \tau \leqslant [m/2]$$

and

$$1 \leqslant \lambda \leqslant m-\tau-1 \quad \text{when } [m/2]+1 \leqslant \tau \leqslant m-2.$$

The complete result for case (vi) is: for each (τ, λ) where $2 \leqslant \tau \leqslant m-1$ and $1 \leqslant \lambda \leqslant \tau-1$ there are $2(m-\tau)$ pairs (k, l) satisfying the given conditions for which $E\{Q_kQ_{k+\tau}Q_lQ_{l+\lambda}\} = 2n^3a^2\rho^S(\tau-\lambda)/\pi$; and for each (τ, λ) where either

$$2 \leqslant \tau \leqslant [m/2] \quad \text{and} \quad 1 \leqslant \lambda \leqslant \tau-1$$

or

$$[m/2]+1 \leqslant \tau \leqslant m-2 \quad \text{and} \quad 1 \leqslant \lambda \leqslant m-\tau-1$$

there are $2(m-\tau-\lambda)$ pairs (k, l) satisfying the given conditions for which

$$E\{Q_k Q_{k+\tau} Q_l Q_{l+\lambda}\} = 2n^3 a^2 \rho^S(\tau+\lambda)/\pi.$$

The results for cases (i)...(v) are:

(i) There are m values of k for which

$$E\{Q_k^4\} = n(3n-2) + 4a^2\{n(n-1)(3n-4)\}/\pi.$$

(ii) There are $m(m-1)$ pairs (k, l) for which

$$E\{Q_k^2 Q_l^2\} = n^2 + 4a^2 n^2 (n-1)/\pi.$$

(iii) For each τ there are $4(m-\tau)$ pairs (k, l) for which

$$E\{Q_k^2 Q_l Q_{l+\tau}\} = 2n^3 a^2 \rho^S(\tau)/\pi + 2n^2(n-1)a^2\rho^S(\tau)/\pi.$$

The other $(m-4)(m-\tau)$ pairs (k, l) give

$$E\{Q_k^2 Q_l Q_{l+\tau}\} = 2n^3 a^2 \rho^S(\tau)/\pi.$$

(iv) For each τ there are $(m-\tau)$ values of k for which

$$E\{Q_k^2 Q_{k+\tau}^2\} = n^2 + 4a^2 n^2(n-1)/\pi.$$

(v) For each τ where $1 \leqslant \tau \leqslant [(m-1)/2]$, there are $2(m-2\tau)$ pairs (k, l) for which

$$E\{Q_k Q_{k+\tau} Q_l Q_{l+\tau}\} = 2n^3 a^2 \rho^S(2\tau)/\pi.$$

Taking expectations in equation (14) and using these results, gives the value of $E\{T^2\}$ appearing as equation (12) in the main text.

References

[1] N. M. Blachman, "The signal × signal, noise × noise, and signal × noise output of a nonlinearity", *IEEE Trans. Inform. Theory* IT-14 (1968), 21–27.

[2] D. J. Edelblute, J. M. Fisk and G. L. Kinnison, "Criteria for optimum signal detection for arrays", *J. Acoust. Soc. Amer.* **41** (1967), 199–205.

[3] H. Ekre, "Polarity coincidence correlation detection of a weak noise source", *IEEE Trans. Inform. Theory* IT-9 (1963), 18–23.

[4] J. J. Faran and R. Hills, Jr., "The application of correlation techniques to acoustic receiving systems", *Technical Memo No.* 28, *Acoust. Res. Lab., Harvard*, (1952).

[5] L. E. Franks, *Signal theory* (Prentice-Hall, Englewood Cliffs, N. J , 1969).

[6] W. Frost, "A likelihood ratio statistic for the detection of signals using clipped data", M. Math. Thesis, University of Newcastle (in preparation).

[7] N L. Johnson and S. Kotz, *Distributions in statistics: continuous multivariate distributions* (John Wiley, New York, 1972).

[8] M. Kanefsky, "Detection of weak signals with polarity coincidence arrays", *IEEE Trans. Inform. Theory* IT-12 (1966), 260–268.

[9] R. G. Keats and J. Cooper, "The optimum processing of clipped signals; an approach based on minimum signal distortion", *J. Austral. Math. Soc. Ser. B* 22 (1980), 175–184.

[10] R. G. Keats and V. K.-K. Yu, "A generalisation of the study of sum and square law signal processors with multiple clipped inputs", *J. Austral. Math. Soc. Ser. B* 19 (1976), 294–315.
[11] G. Marsaglia and T. A. Bray, "A convenient method for generating normal variables", *SIAM Rev.* 6 (1964), 260–264.
[12] R. L. Plackett, "A reduction formula for normal multivariate integrals", *Biometrika* 41 (1954), 351–360.
[13] P. M. Schultheiss and F. B. Tuteur, "Optimum and sub-optimum detection of Gaussian signals in an isotropic Gaussian noise field, Part II: degradation of detectability due to clipping", *IEEE Trans. on Military Electronics* MIL-9 (1965), 208–211.
[14] J. B. Thomas and T. R. Williams, "On the detection of signals in nonstationary noise by product arrays", *J. Acoust. Soc. Amer.* 31 (1959), 453–462.
[15] T. Usher, Jr., "Signal detection by arrays in noise fields with local variations", *J. Acoust. Soc. Amer.* 36 (1964), 1444–1449.
[16] A. Wald, *Sequential analysis* (John Wiley, New York, 1947).