Glasgow Math. J. **56** (2014) 57–63. © Glasgow Mathematical Journal Trust 2013.
doi:10.1017/S0017089513000074.

ON THE SPECIAL VALUES OF L-FUNCTIONS OF CM-BASE CHANGE FOR HILBERT MODULAR FORMS

CRISTIAN VIRDOL

Department of Mathematics, Yonsei University, Seodaemun-gu, Seoul, South Korea
e-mail: virdol@yonsei.ac.kr

(Received 15 April 2012; revised 2 November 2012; accepted 20 December 2012;
first published online 13 August 2013)

Abstract. In this paper we generalize some results, obtained by Shimura, on the special values of L-functions of l-adic representations attached to quadratic CM-base change of Hilbert modular forms twisted by finite order characters. The generalization is to the case of the special values of L-functions of arbitrary base change to CM-number fields of l-adic representations attached to Hilbert modular forms twisted by some finite-dimensional representations.

2010 *Mathematics subject classification*. 11F41, 11F80, 11R42, 11R80.

1. Introduction. For F, a totally real number field, let J_F be the set of infinite places of F, and let $\Gamma_F := \mathrm{Gal}(\bar{\mathbb{Q}}/F)$. Let f be a normalized Hecke eigenform of $\mathrm{GL}(2)/F$ of weight $k = (k(\tau))_{\tau\in J_F}$, where all $k(\tau)$ have the same parity and $k(\tau) \geq 2$. We denote by Π the cuspidal automorphic representation of $\mathrm{GL}(2)/F$ generated by f. In this paper we assume that Π is non-CM. We denote by ρ_Π the l-adic representation attached to Π, for some prime number l (by fixing an isomorphism $\iota : \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ one can regard ρ_Π as a complex valued representation). Define $k_0 = \max\{k(\tau)|\tau \in J_F\}$ and $k^0 = \min\{k(\tau)|\tau \in J_F\}$. In this paper we write $a \sim b$ for $a, b \in \mathbb{C}$ if $b \neq 0$ and $a/b \in \bar{\mathbb{Q}}$. By a CM-field we mean a quadratic totally imaginary extension of a totally real number field.

In this paper we prove the following result.

THEOREM 1.1. *Assume $k(\tau) \geq 3$ for all $\tau \in J_F$, and $k(\tau)$ mod 2 is independent of τ. Let M be a CM-field which contains F, and let ψ be a finite-dimensional complex-valued continuous representation of $\Gamma_M := Gal(\bar{\mathbb{Q}}/M)$ such that $K := \bar{\mathbb{Q}}^{ker\psi}$ is an abelian extension of a CM number field. Then*

$$L(m, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi) \sim \pi^{(m+1-k_0)[M:\mathbb{Q}]\dim\psi} \langle f, f\rangle^{\frac{[M:F]}{2}\dim\psi},$$

for any integer m satisfying

$$(k_0 + 1)/2 \leq m < (k_0 + k^0)/2.$$

Theorem 1.1 is a generalization of Theorem 5.7 of [**7**] (i.e. Proposition 2.1; and the inner product $\langle f, f\rangle$ is normalized as in Section 2). In the proof of Theorem 1.1 we use some results on the behaviour (see [**10**, **11**]) of the inner product $\langle f, f\rangle$ under base change of f to some large ([**1**]) totally real extensions of F (see formula (3.1)).

We remark that Deligne's [**4**] conjecture for motives predicts that

$$L(n, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi) \sim_{\mathbb{Q}(\Pi_{/M},\psi)} c^+(\mathrm{Res}_{M/\mathbb{Q}}(M(f)_{/M} \otimes M(\psi)(n))),$$

for any integer n satisfying $(k_0 - k^0)/2 < n < (k_0 + k^0)/2$, where $M(f)$ is the motive conjecturally associated to f and $M(\psi)$ is the motive associated to ψ, $\mathbb{Q}(\Pi_{/M}, \psi)$ is the field of rationality of $M(f)_{/M} \otimes M(\psi)$, '$\sim_{\mathbb{Q}(\Pi_{/M},\psi)}$' means up to multiplication by an element in the number field $\mathbb{Q}(\Pi_{/M}, \psi)$, and $c^+(\mathrm{Res}_{M/\mathbb{Q}}(M(f)_{/M} \otimes M(\psi)(n)))$ is Deligne's period associated to the n-Tate twist of $\mathrm{Res}_{M/\mathbb{Q}}(M(f)_{/M} \otimes M(\psi))$. In this paper we cannot say anything about Deligne's conjecture, as we do not know how to relate $c^+(\mathrm{Res}_{M/\mathbb{Q}}(M(f)_{/M} \otimes M(\psi)(n)))$ to $\langle f, f\rangle^{\frac{[M:F]}{2}\dim\psi}$ (i.e. we do not know how to obtain even an equality up to an algebraic number times a power of π between these two periods; not even when $F = \mathbb{Q}$, ψ is a character and M is an imaginary quadratic number field).

2. Known results. Consider F a totally real number field and let J_F be the set of infinite places of F. If Π is a cuspidal automorphic representation (discrete series at infinity) of weight $k = (k(\tau))_{\tau\in J_F}$ of $\mathrm{GL}(2)/F$, where all $k(\tau)$ have the same parity and all $k(\tau) \geq 2$, let $k_0 = \max\{k(\tau)|\tau \in J_F\}$ and $k^0 = \min\{k(\tau)|\tau \in J_F\}$. Then there exists ([**8**]) a λ-adic representation

$$\rho_\Pi := \rho_{\Pi,\lambda} : \Gamma_F \to \mathrm{GL}_2(O_\lambda) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_l),$$

which satisfies $L(s, \iota\rho_{\Pi,\lambda}) = L(s - \frac{(k_0-1)}{2}, \Pi) = L(s - \frac{(k_0-1)}{2}, f)$, where $\iota : \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ is a specific isomorphism, and the above equality of L-functions is up to finitely many Euler factors; also, because the line of convergence of $L(s, \Pi)$ is Re(s)=1, we get that the line of convergence of $L(s, \rho_{\Pi,\lambda})$ is $\mathrm{Re}(s) = (k_0 + 1)/2$); the representation ρ_Π is unramified outside the primes dividing $\mathbf{n}l$. Here f is the normalized Hecke eigenform of $\mathrm{GL}(2)/F$ of weight k corresponding to Π, O is the coefficients ring of Π (i.e. O is the ring of integers of the field generated over $\mathbb{Q}$ by the eigenvalues $a_\wp$ defined by $T_\wp f = a_\wp f$, where $T_\wp$ is the Hecke operator at $\wp$, and $\wp$ runs over the prime ideals of F (see [**8**] for details)), λ is a prime ideal of O above some prime number l and $\mathbf{n}$ is the level of Π. We define

$$\langle f, f\rangle = \pi^{\sum_{\tau\in J_F} k(\tau)} \int_{Z_{\infty+}\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)} f(x)\overline{f(x)}dx$$

where $Z_{\infty+} \simeq \mathbb{R}_+^\times$ is the connected component of the center of $\mathrm{GL}_2(\mathbb{R})$, and the measure is normalized such that $\mathrm{vol}(Z_{\infty+}\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)) = 1$.

Proposition 2.1 follows from Proposition 5.2 and Theorem 5.7 of [**7**]. We actually use the fact that $L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi) = L(s, \iota\rho_\Pi \otimes \mathrm{Ind}_{\Gamma_M}^{\Gamma_F}\psi))$ in order to reduce Proposition 2.1 to a particular case of Theorem 5.7 of [**7**] where a convolution of two cuspidal automorphic representations (one non-CM, and the other CM) of $\mathrm{GL}(2)/F$ was considered. We remark that $\mathrm{Ind}_{\Gamma_M}^{\Gamma_F}\psi$ corresponds to a CM cuspidal automorphic representation of $\mathrm{GL}(2)/F$ of weight 1.

PROPOSITION 2.1. *Assume $k(\tau) \geq 2$ for all $\tau \in J_F$ and $k(\tau)$ mod 2 is independent of τ. Let M be a quadratic CM-extension of F, and let ψ be a continuous one-dimensional* *representation of* Γ_M*. Then*

$$L(m, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi) \sim \pi^{(m+1-k_0)[M:\mathbb{Q}]}\langle f, f\rangle$$

for any integer m satisfying

$$(k_0+1)/2 \leq m < (k_0+k^0)/2.$$

3. The proof of Theorem 1.1 for ψ a character. We fix a non-CM cuspidal automorphic representation Π of GL(2)/F as in Theorem 1.1, and let M/F be a CM-finite extension. In this section we assume that ψ is an arbitrary one-dimensional continuous representation of Γ_M and prove Theorem 1.1 in this case.

We know the following result (Theorem 1.1 of [**12**] or Theorem 2.1 of [**13**] or Theorem A of [**1**]).

THEOREM 3.1. *Let* Π *be a cuspidal automorphic representation of weight* $k = (k(\tau))_{\tau\in J_F}$ *of GL(2)/F, where all* $k(\tau)$ *have the same parity and all* $k(\tau) \geq 2$*. Let* F' *be a totally real extension of F. Then there exists a totally real Galois extension* F'' *of* F' *such that* $\rho_\Pi|_{\Gamma_{F''}}$ *is cuspidal automorphic, i.e. there exists a cuspidal automorphic representation* Π'' *of weight* k'' *of GL(2)/*F'' *such that* $\rho_\Pi|_{\Gamma_{F''}} \cong \rho_{\Pi''}$.

We denote by F' the maximal totally real subfield of M; hence M is a quadratic CM-extension of F'. Then from Theorem 3.1 we know that we can find a totally real Galois extension F'' of F', and a cuspidal automorphic representation Π'' of GL(2)/F'' such that $\rho_\Pi|_{\Gamma_{F''}} \cong \rho_{\Pi''}$. Because Π is non-CM, we get that Π'' is non-CM.

From Theorem 15.10 of [**3**] we know that there exist some subfields $M_i \subseteq MF''$ such that $M \subseteq M_i$ and Gal(MF''/M_i) are solvable, and some integers n_i, such that the trivial representation

$$1_M : \mathrm{Gal}(MF''/M) \to \mathbb{C}^\times$$

can be written as

$$1_M = \sum_{i=1}^{u} n_i \mathrm{Ind}_{\mathrm{Gal}(MF''/M_i)}^{\mathrm{Gal}(MF''/M)} 1_{M_i}$$

(an equality in the character ring of Gal(MF''/M)), where

$$1_{M_i} : \mathrm{Gal}(MF''/M_i) \to \mathbb{C}^\times$$

is the trivial representation. In particular, we have $1 = \sum_{i=1}^{u} n_i[M_i : M]$. Then (for the equality between the second and the third terms below, we use Corollary 10.20 of [**3**], which says that if G is a finite group and H a subgroup, and if ρ and ϕ are k-linear

https://doi.org/10.1017/S0017089513000074 Published online by Cambridge University Press

representations of G and H, where k is a field, then $\rho \otimes \mathrm{Ind}_H^G \phi \simeq \mathrm{Ind}_H^G(\rho|_H \otimes \phi)$),

$$\begin{aligned} L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi) &= \prod_{i=1}^{u} L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \mathrm{Ind}_{\Gamma_{M_i}}^{\Gamma_M} 1_{M_i} \otimes \psi)^{n_i} \\ &= \prod_{i=1}^{u} L(s, \mathrm{Ind}_{\Gamma_{M_i}}^{\Gamma_M}(\iota\rho_\Pi|_{\Gamma_{M_i}} \otimes 1_{M_i} \otimes \psi|_{\Gamma_{M_i}}))^{n_i} \\ &= \prod_{i=1}^{u} L(s, \iota\rho_\Pi|_{\Gamma_{M_i}} \otimes \psi|_{\Gamma_{M_i}})^{n_i}. \end{aligned}$$

Since $\rho_\Pi|_{\Gamma_{F''}}$ is cuspidal automorphic and MF'' is a quadratic extension of F'', we get ([**5**]) that $\rho_\Pi|_{\Gamma_{MF''}}$ is cuspidal automorphic, and because the group $\mathrm{Gal}(MF''/M_i)$ is solvable, one easily gets (see Section 4 of [**9**]) that $\rho_\Pi|_{\Gamma_{M_i}}$ is cuspidal automorphic.

Hence, the function $L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi)$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation because each function $L(s, \iota\rho_\Pi|_{\Gamma_{M_i}} \otimes \psi|_{\Gamma_{M_i}})$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation. Moreover, since each function $L(s, \iota\rho_\Pi|_{\Gamma_{M_i}} \otimes \psi|_{\Gamma_{M_i}})$ has no poles or zeros for $\mathrm{Re}(s) \geq (k_0+1)/2$ (see Proposition 5.2 of [**7**] and Proposition 4.16 of [**6**]), we get that the function $L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi)$ has no poles or zeros for $\mathrm{Re}(s) \geq (k_0+1)/2$. Thus, for any integer m satisfying

$$(k_0+1)/2 \leq m,$$

we get the identity

$$L(m, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi) = \prod_{i=1}^{u} L(m, \iota\rho_\Pi|_{\Gamma_{M_i}} \otimes \psi|_{\Gamma_{M_i}})^{n_i}.$$

Let F_i be the maximal totally real subfield of M_i. Since $\rho_\Pi|_{\Gamma_{M_i}}$ is cuspidal automorphic and M_i/F_i is quadratic, one can easily prove that $\rho_\Pi|_{\Gamma_{F_i}}$ is cuspidal automorphic (see Lemma 1.3 of [**2**]), so $\rho_\Pi|_{\Gamma_{F_i}} \cong \rho_{\Pi_i}$ for some cuspidal automorphic representation Π_i of $\mathrm{GL}(2)/F_i$. We denote by f_i the normalized Hecke eigenform of $\mathrm{GL}(2)/F_i$ associated to Π_i. Then f_i has weight $k_i = (k_i(\tau))_{\tau \in J_{F_i}}$, where J_{F_i} is the set of infinite places of F_i, and $k_i(\tau) = k(\tau|F)$ for any $\tau \in J_{F_i}$.

Now from Proposition 2.1 we get

$$L(m, \iota\rho_\Pi|_{\Gamma_{M_i}} \otimes \psi|_{\Gamma_{M_i}}) \sim \pi^{(m+1-k_0)[M_i:\mathbb{Q}]}\langle f_i, f_i\rangle,$$

for any integer m satisfying

$$(k_0+1)/2 \leq m < (k_0+k^0)/2.$$

But we know that (see the paragraph just before Remark 5.1 of [**10**])

$$\langle f_i, f_i\rangle \sim \langle f, f\rangle^{[F_i:F]}, \tag{3.1}$$

and using the fact that $1 = \sum_{i=1}^{u} n_i[M_i : M]$, we obtain

$$L(m, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi) \sim \pi^{\sum_{i=1}^{u}(m+1-k_0)[M_i:\mathbb{Q}]n_i} \prod_{i=1}^{u} \langle f_i, f_i\rangle^{n_i}$$

$$\sim \pi^{\sum_{i=1}^{u}(m+1-k_0)[M_i:\mathbb{Q}]n_i}\langle f,f\rangle^{\sum_{i=1}^{u}[F_i:F]n_i} \sim \pi^{(m+1-k_0)[M:\mathbb{Q}]}\langle f,f\rangle^{\frac{[M:F]}{2}}$$

for any integer m satisfying

$$(k_0+1)/2 \leq m < (k_0+k^0)/2,$$

which proves Theorem 1.1 for ψ one-dimensional representation. ■

4. The proof of Theorem 1.1 for general ψ. Let ψ be a finite-dimensional representation of Γ_M as in Theorem 1.1. We denote by M' the maximal CM-subfield of $K := \bar{\mathbb{Q}}^{\ker\psi}$. Obviously, M'/M is Galois and K is an abelian extension of M'.

From the beginning of Section 15 in [**3**] we know that there exist some subfields $E_i \subseteq M'$ such that $M \subseteq E_i$ and $\mathrm{Gal}(M'/E_i)$ are cyclic, and some integers n_i such that the trivial representation

$$1_M : \mathrm{Gal}(M'/M) \to \mathbb{C}^\times$$

can be written as

$$[M':M]1_M = \sum_{i=1}^{u} n_i \mathrm{Ind}_{\mathrm{Gal}(M'/E_i)}^{\mathrm{Gal}(M'/M)} 1_{E_i},$$

where $1_{E_i} : \mathrm{Gal}(M'/E_i) \to \mathbb{C}^\times$ is the trivial representation. In particular, we have $[M':M] = \sum_{i=1}^{u} n_i[E_i:M]$. Then

$$\begin{aligned} L(s,\iota\rho_\Pi|_{\Gamma_M}\otimes\psi)^{[M':M]} &= \prod_{i=1}^{u} L(s,\iota\rho_\Pi|_{\Gamma_M}\otimes\psi\otimes\mathrm{Ind}_{\Gamma_{E_i}}^{\Gamma_M}1_{E_i})^{n_i} \\ &= \prod_{i=1}^{u} L(s,\mathrm{Ind}_{\Gamma_{E_i}}^{\Gamma_M}(\iota\rho_\Pi|_{\Gamma_{E_i}}\otimes\psi|_{\Gamma_{E_i}}\otimes 1_{E_i}))^{n_i} \\ &= \prod_{i=1}^{u} L(s,\iota\rho_\Pi|_{\Gamma_{E_i}}\otimes\psi|_{\Gamma_{E_i}})^{n_i}. \end{aligned}$$

We write

$$\psi|_{\Gamma_{E_i}} = \bigoplus_{j=1}^{u_i}\psi_{ij},$$

where ψ_{ij} are irreducible representations of Γ_{E_i}. Since $\mathrm{Gal}(M'/E_i)$ is cyclic, $\psi_{ij}|_{\Gamma_{M'}}$ is abelian and ψ_{ij} is irreducible, we get that the following:

LEMMA 4.1. *We have*

$$\psi_{ij} \simeq Ind_{\Gamma_{E_{ij}}}^{\Gamma_{E_i}}\phi_{ij}$$

for some continuous character

$$\phi_{ij} : \Gamma_{E_{ij}} \to \mathbb{C}^\times,$$

where E_{ij} is a subfield of M' which contains E_i.

Proof: Let σ be a generator of $\mathrm{Gal}(M'/E_i)$. Then, since M'/E_i is Galois, σ permutes the irreducible components of $\psi_{ij}|_{\Gamma_{M'}}$. The representation $\psi_{ij}|_{\Gamma_{M'}}$ is abelian, and thus a direct sum of characters. Let ϕ be one of these characters. We denote by E_{ij} the subfield of M' which contains E_i having the property that $\mathrm{Gal}(M'/E_{ij})$ is the stabiliser of ϕ under the action of $\mathrm{Gal}(M'/E_i) = \langle\sigma\rangle$. The character ϕ extends to a character ϕ_{ij} of $\Gamma_{E_{ij}}$. Then, because ψ_{ij} is irreducible, $\sigma \in \mathrm{Gal}(E_{ij}/E_i)$ permutes simply transitively all the components of the abelian representation $\psi_{ij}|_{\Gamma_{E_{ij}}}$ and we have $[E_{ij} : E_i] = \dim \psi_{ij}$. Let $V_{\psi_{ij}}$ be the space corresponding to ψ_{ij}, and $V_{\phi_{ij}}$ be the space corresponding to ϕ_{ij}. Since $\mathrm{Hom}_{\Gamma_{E_{ij}}}(V_{\psi_{ij}}, V_{\phi_{ij}})$ is non-trivial, by Frobenius reciprocity we get that $\mathrm{Hom}_{\Gamma_{E_i}}(V_{\psi_{ij}}, \mathrm{Ind}_{\Gamma_{E_{ij}}}^{\Gamma_{E_i}} V_{\phi_{ij}})$ is also non-trivial. But $\dim \mathrm{Ind}_{\Gamma_{E_{ij}}}^{\Gamma_{E_i}} \phi_{ij} = \dim \psi_{ij}$, and thus we obtain $\psi_{ij} \simeq \mathrm{Ind}_{\Gamma_{E_{ij}}}^{\Gamma_{E_i}} \phi_{ij}$. ■

Therefore, we obtain

$$\begin{aligned}
L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi)^{[M':M]} &= \prod_{i=1}^{u} L(s, \iota\rho_\Pi|_{\Gamma_{E_i}} \otimes \psi|_{\Gamma_{E_i}})^{n_i} \\
&= \prod_{i=1}^{u}\prod_{j=1}^{u_i} L(s, \iota\rho_\Pi|_{\Gamma_{E_i}} \otimes \mathrm{Ind}_{\Gamma_{E_{ij}}}^{\Gamma_{E_i}} \phi_{ij})^{n_i} \\
&= \prod_{i=1}^{u}\prod_{j=1}^{u_i} L(s, \mathrm{Ind}_{\Gamma_{E_{ij}}}^{\Gamma_{E_i}} (\iota\rho_\Pi|_{\Gamma_{E_{ij}}} \otimes \phi_{ij}))^{n_i} \\
&= \prod_{i=1}^{u}\prod_{j=1}^{u_i} L(s, \iota\rho_\Pi|_{\Gamma_{E_{ij}}} \otimes \phi_{ij})^{n_i}.
\end{aligned}$$

Hence, the function $L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi)^{[M':M]}$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation because from Section 3 we know that each function $L(s, \iota\rho_\Pi|_{\Gamma_{E_{ij}}} \otimes \phi_{ij})$ has a meromorphic continuation to the entire complex plane and satisfies a functional equation. Also, since each function $L(s, \iota\rho_\Pi|_{\Gamma_{E_{ij}}} \otimes \phi_{ij})$ has no poles or zeros for $\mathrm{Re}(s) \geq (k_0 + 1)/2$, we get that the function $L(s, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi)^{[M':M]}$ has no poles or zeros for $\mathrm{Re}(s) \geq (k_0 + 1)/2$. Thus, for any integer m satisfying

$$(k_0 + 1)/2 \leq m,$$

we get the identity

$$L(m, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi)^{[M':M]} = \prod_{i=1}^{u}\prod_{j=1}^{u_i} L(m, \iota\rho_\Pi|_{\Gamma_{E_{ij}}} \otimes \phi_{ij})^{n_i}.$$

From Section 3 we know that

$$L(m, \iota\rho_\Pi|_{\Gamma_{E_{ij}}} \otimes \phi_{ij}) \sim \pi^{(m+1-k_0)[E_{ij}:\mathbb{Q}]} \langle f, f\rangle^{\frac{[E_{ij}:F]}{2}}$$

for any integer m satisfying

$$(k_0 + 1)/2 \leq m < (k_0 + k^0)/2.$$

Hence, from the fact that $[M' : M] \dim \psi = \sum_{i=1}^{u} \sum_{j=1}^{u_i} n_i [E_{ij} : M]$, we get

$$L(m, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi)^{[M':M]} = \prod_{i=1}^{u} \prod_{j=1}^{u_i} L(m, \iota\rho_\Pi|_{\Gamma_{E_{ij}}} \otimes \phi_{ij})^{n_i}$$

$$\sim \pi^{\sum_{i=1}^{u} \sum_{j=1}^{u_i} (m+1-k_0)[E_{ij}:\mathbb{Q}]n_i} \langle f, f\rangle^{\sum_{i=1}^{u} \sum_{j=1}^{u_i} \frac{[E_{ij}:F]}{2} n_i}$$

$$\sim \pi^{(m+1-k_0)[M':\mathbb{Q}]\dim \psi} \langle f, f\rangle^{\frac{[M':F]}{2} \dim \psi},$$

and thus

$$L(m, \iota\rho_\Pi|_{\Gamma_M} \otimes \psi) \sim \pi^{(m+1-k_0)[M:\mathbb{Q}]\dim \psi} \langle f, f\rangle^{\frac{[M:F]}{2} \dim \psi}$$

for any integer m satisfying

$$(k_0 + 1)/2 \leq m < (k_0 + k^0)/2.$$

This concludes the proof of Theorem 1.1. ■

REFERENCES

1. T. Barnet-Lamb, T. Gee, D. Geraghty and R. Taylor, *Potential automorphy and change of weight* (preprint). arXiv:1010.2561v1 [math.NT].

2. T. Barnet-Lamb, D. Geraghty, M. Harris and R. Taylor, A family of Calabi-Yau varieties and potential automorphy II, *Publ. Res. Inst. Math. Sci.* **47** (2011), 29–98.

3. C. W. Curtis and I. Reiner, *Methods of representation theory*, vol. I (Wiley, New York, NY, 1981).

4. P. Deligne, Valeurs de fonctions L et periodes d'integrales, *Proc. Symp. Pure Math.* **33**(part 2) (1979), 313–346.

5. R. P. Langlands, *Base change for GL(2)*, Ann. of Mathematics Studies, No. 96 (Princeton University Press, Princeton, NJ, 1980).

6. G. Shimura, The special values of the zeta functions associated with Hilbert modular forms, *Duke Math. J.* **45** (1978), 637–679.

7. G. Shimura, Algebraic relations between critical values of zeta functions and inner products, *Amer. J. Math.* **104** (1983), 253–285.

8. R. Taylor, On Galois representations associated to Hilbert modular forms, *Invent. Math.* **98** (1989), 265–280.

9. C. Virdol, Tate classes and poles of L-functions of twisted quaternionic Shimura surfaces, *J. Number Theory* **123**(2) (2007), 315–328.

10. C. Virdol, On the critical values of L-functions of tensor product of base change for Hilbert modular forms, *J. Math. Kyoto Univ.* **49**(2) (2009), 347–357.

11. C. Virdol, On the critical values of L-functions of base change for Hilbert modular forms, *Amer. J. Math.* **132**(4) (2010), 1105–1111.

12. C. Virdol, Non-solvable base change for Hilbert modular forms and zeta functions of twisted quaternionic Shimura varieties, *Annales de la Faculte des Sciences de Toulouse* **19**(3-4) (2010), 831–848.

13. C. Virdol, On the Birch and Swinnerton-Dyer conjecture for abelian varieties attached to Hilbert modular forms, *J. Number Theory* **131**(4) (2011), 681–684.