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ON A PROBLEM IN PARTIAL DIFFERENCE EQUATIONS(1)

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The purpose of this paper is not to solve a problem but to pose one that may be of some interest, depth, and consequence.

Given that the positive integer n has the canonical representation $n=\prod_{i=1}^{h} p_i^{\alpha_i}$, the problem of finding the number $F(n)=f(\alpha_1, \alpha_2, \ldots, \alpha_h)$ of ordered factorizations of n into positive nontrivial integral factors is equivalent to that of finding the number of ordered partitions of the vector $(\alpha_1, \alpha_2, \ldots, \alpha_h)$ into nonzero vectors with nonnegative integral components. This problem was solved as early as 1893 by P. A. MacMahon [3], who proved that

$$
\begin{aligned}
F(n) &= f(\alpha_1, \alpha_2, \ldots, \alpha_h) \\
&= \sum_{j=1}^{q} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} \prod_{k=1}^{h} \binom{\alpha_h+j-i-1}{\alpha_k}
\end{aligned} \tag{1}
$$

where $q=\sum_{i=1}^{h} \alpha_i$. While this formula gives $f(\alpha_1, \alpha_2, \ldots, \alpha_h)$ in closed form, it clearly is not particularly useful for calculation. A much more useful result which allows for the recursive calculation of the $F(n)$ was given by Long [2] and by Carlitz and Moser [1], who proved that, for $n > 1$,

$$
\tfrac{1}{2} \sum_{d \mid n} F(d) = F(n) = 2 \sum_{d \mid n} \mu(d) F(n/d) - \mu(n). \tag{2}
$$

In terms of the function f, (2) becomes a partial difference equation in $\alpha_1, \alpha_2, \ldots \alpha_h$ For example, for $h=1$, we obtain

$$
f(\alpha_1)-2f(\alpha_1-1) = 0, \qquad f(0) = 1, \tag{3}
$$

which has the solution $f(\alpha_1)=2^{\alpha_1-1}$. For $h=2$, we obtain

$$
f(\alpha_1, \alpha_2)-2f(\alpha_1-1, \alpha_2)-2f(\alpha_1, \alpha_2-1)+2f(\alpha_1-1, \alpha_2-1) = 0 \tag{4}
$$

with $f(0, 0)=1$, $f(\alpha_1, 0)=2^{\alpha_1-1}$ for $\alpha_1 \geq 1$, and $f(0, \alpha_2)=2^{\alpha_2-1}$ for $\alpha_2 \geq 1$, and it is not difficult to show directly that the solution is given by

$$
f(\alpha_1, \alpha_2) = 2^{\alpha_1+\alpha_2-1} \sum_{i \geq 0} 2^{-i} \binom{\alpha_1}{i} \binom{\alpha_2}{i}. \tag{5}
$$

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For $h=3$, we obtain

$$\begin{aligned} &f(\alpha_1, \alpha_2, \alpha_3)-2f(\alpha_1-1, \alpha_2, \alpha_3)-2f(\alpha_1, \alpha_2-1, \alpha_3)-2f(\alpha_1, \alpha_2, \alpha_3-1)\\ &\qquad +2f(\alpha_1-1, \alpha_2-1, \alpha_3)+2f(\alpha_1-1, \alpha_2, \alpha_3-1)+2f(\alpha_1, \alpha_2-1, \alpha_3-1)\\ &\qquad -2f(\alpha_1-1, \alpha_2-1, \alpha_3-1) = 0 \end{aligned} \tag{6}$$

with $f(0, 0, 0)=1$, $f(\alpha_1, 0, 0)=2^{\alpha_1-1}$ for $\alpha_1 \geq 1$, $f(0, \alpha_2, 0)=2^{\alpha_2-1}$ for $\alpha_2 \geq 1$, and $f(0, 0, \alpha_3)=2^{\alpha_3-1}$ for $\alpha_3 \geq 1$ and the general pattern is now clear. I now assert that the solution to (6) can be obtained in the following intriguing way: Fully expand the polynomial

$$\begin{aligned} &2^{\alpha_1-1}(2x_1+1)^{\alpha_2}(2x_1x_2+x_1+x_2+1)^{\alpha_3}\\ &\qquad = 2^{\alpha_1-1}\{x_1+(x_1+1)\}^{\alpha_2}\{x_1x_2+(x_1+1)(x_2+1)\}^{\alpha_3} \end{aligned} \tag{7}$$

and then replace x_1^k by

$$\binom{\alpha_1}{\alpha_2+\alpha_3-k} \quad \text{for } 0 \leq k \leq \alpha_2+\alpha_3$$

and replace x_2^k by

$$\binom{\alpha_2}{\alpha_3-k} \quad \text{for } 0 \leq k \leq \alpha_3.$$

The resulting function of $\alpha_1, \alpha_2, \alpha_3$ is the desired solution to (6).

In general, in the n variable case, one fully expands the polynomial

$$2^{\alpha_1-1}\prod_{i=2}^{n}\left\{\prod_{j=1}^{i-1}x_j+\prod_{j=1}^{i-1}(x_j+1)\right\}^{\alpha_i} \tag{8}$$

and then replaces x_i^k by the binomial coefficient

$$\binom{\alpha_i}{\sum_{j=i+1}^{n}\alpha_j-k} \quad \text{for } 0 \leq k \leq \sum_{j=i+1}^{n}\alpha_j$$

to obtain the desired function $f(\alpha_1, \alpha_2, \ldots, \alpha_n)$. Thus, for example, in the two variable case, we expand

$$\begin{aligned} 2^{\alpha_1-1}(2x_1+1)^{\alpha_2} &= 2^{\alpha_1-1}\sum_{i=0}^{\alpha_2}\binom{\alpha_2}{i}(2x_1)^{\alpha_2-i}\\ &= 2^{\alpha_1+\alpha_2-1}\sum_{i=0}^{\alpha_2}\binom{\alpha_2}{i}x_1{}^{\alpha_2-i}\,2^{-i} \end{aligned}$$

and replace $x_1^{\alpha_2-i}$ by

$$\binom{\alpha_1}{\alpha_2-(\alpha_2-i)} = \binom{\alpha_1}{i}$$

to obtain the solution (5) noted above.

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Of course, the difficulty is that I can prove my claim only in the cases $n=1, 2, 3$ and have checked it in particular cases for $n=4, 5, 6$. But then, in a very real sense, the solution is quite beside the point; MacMahon has already provided that. What may be of considerable importance is that the conjectured method of solution suggests the existence of a transform method of solution which may be applicable to a reasonably large class of finite partial difference equations. Hopefully, some reader may be able to decide the issue.

References

1. L. Carlitz and L. Moser, *On some special factorizations of* $(1 - x^n)/(1 - x)$, Canad. Math. Bull. **9** (1966), 421–426.

2. C. T. Long, *Addition theorems for sets of Integers*, Pac. J. Math. **23** (1967), 107–112.

3. P. A. MacMahon, *The theory of perfect partitions and the compositions of multipartite numbers*, Philos. Trans. Roy. Soc. London (A), **184** (1893), 835–901.

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