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Composition operators on μ-Bloch spaces

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Abstract. Given a positive continuous function μ on the interval $0 < t \leq 1$, we consider the space of so-called μ-Bloch functions on the unit ball. If $\mu(t) = t$, these are the classical Bloch functions. For μ, we define a metric $F_z^\mu(u)$ in terms of which we give a characterization of μ-Bloch functions. Then, necessary and sufficient conditions are obtained in order that a composition operator be a bounded or compact operator between these generalized Bloch spaces. Our results extend those of Zhang and Xiao.

1 Introduction

Let D denote the unit disk in the complex plane $\mathbb{C}$, and $H(D)$ the class of all holomorphic functions on D. A function $f \in H(D)$ is called a *Bloch function* if

$$\|f\| = \sup\{(1 - |z|^2)|f'(z)| : z \in D\} < \infty.$$

The Bloch functions, with the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + \|f\|, \tag{1.1}$$

form a Banach space, which is called the *Bloch space* and denoted by $\mathcal{B}$. The Bloch space of the unit disk has been investigated extensively, see [1].

The notion of Bloch function has been generalized to Riemann surfaces and domains in complex spaces of higher dimension. Let

$$B^n = \{z = (z_1, \dots, z_n) : |z_1|^2 + \cdots + |z_n|^2 < 1\}$$

denote the unit ball in the complex space $\mathbb{C}^n$, and $H(B^n)$ the class of all holomorphic functions on B^n. For $f \in H(B^n)$, as in [8, 9], we define

$$Q_f(z) = \sup\Big\{ \frac{|\nabla f(z)u|}{H_z(u,u)^{1/2}} : 0 \neq u \in \mathbb{C}^n \Big\},$$

where $\nabla f(z) = (\partial f/\partial z_1, \dots, \partial f/\partial z_n)$ denotes the complex gradient of f, $\nabla f(z)u$ denotes the inner product $\langle \nabla f(z), \overline{u}\rangle$ of $\nabla f(z)$ and $\overline{u}$ and $H_z(u,u)$ is the Bergman metric on B^n which is defined by

$$H_z(u,u) = \frac{n+1}{2} \frac{(1-|z|^2)|u|^2 + |\langle u, z\rangle|^2}{(1-|z|^2)^2}.$$

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We remark that $Q_f^\mu(z)$ is the norm of $u \to \nabla f(z)u$ as a linear functional on the tangent space at z ($u \in \mathbb{C}^n$ regarded as a tangent vector to the unit ball at z, taking the norm of u to be the norm on tangent vectors associated with the Bergman metric). A function $f \in H(B^n)$ is called a Bloch function on B^n if

$$\|f\| = \sup\{Q_f(z) : z \in B^n\} < \infty, \tag{1.2}$$

and the Bloch space of B^n consists of all Bloch functions on B^n with the same norm (1.1) and is also denoted by $\mathcal{B}$.

Let ϕ be a holomorphic mapping of D into itself. The composition operator C_ϕ on $H(D)$, induced by ϕ, is defined by $C_\phi(f) = f \circ \phi$ for $f \in H(D)$. Since the classical Schwarz–Pick lemma [2] asserts that

$$\frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2} \leq 1 \quad \text{for } z \in D,$$

C_ϕ is always a bounded operator on $\mathcal{B}$. In 1995, K. Madigan and A. Matheson [4] proved that a composition operator C_ϕ is compact if and only if

$$\frac{(1-|z|^2)|\phi'(z)|}{1-|\phi(z)|^2} \to 0 \quad \text{as } \phi(z) \to \partial D.$$

We recall that a linear operator is compact if the image of a bounded sequence contains a convergent subsequence.

In the case of higher dimension, for a holomorphic mapping ϕ of B^n into itself the composition operator C_ϕ induced by ϕ is defined in the same way. It is also a bounded operator on $\mathcal{B}$, because by the Schwarz–Pick lemma for the unit ball B^n,

$$\frac{H_{\phi(z)}(\phi'(z)u, \phi'(z)u)}{H_z(u,u)} \leq 1 \tag{1.3}$$

holds for $z \in B^n$ and $0 \neq u \in \mathbb{C}^n$. Similarly to the case of one dimension, the necessary and sufficient condition for C_ϕ to be compact on $\mathcal{B}$ should be

$$\frac{H_{\phi(z)}(\phi'(z)u, \phi'(z)u)}{H_z(u,u)} \to 0 \quad \text{as } \phi(z) \to \partial B^n.$$

This has been proved by J. Shi and L. Luo [7]. Instead of the unit ball, Z. Zhou and J. Shi [13] consider the composition operators of the Bloch space on the polydisc.

The so-called α-Bloch spaces have been introduced and studied by a number of authors (for the general theory of α-Bloch functions see [14]). For $\alpha > 0$, a holomorphic function f on the unit disk D is called an α-Bloch function, if

$$\sup\Big\{ (1-|z|^2)^\alpha |f(z)| : z \in D\Big\} < \infty.$$

The α-Bloch space $\mathcal{B}^\alpha$ is defined in the same way. S. Ohno, K. Stroethoff and R. Zhao [6] studied the boundedness and compactness of a composition operator C_ϕ between

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α-Bloch spaces, and proved that C_ϕ is a bounded operator of $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$ if and only if

$$\sup\Big\{\frac{(1-|z|^2)^\beta|\phi'(z)|}{(1-|\phi(z)|^2)^\alpha} : z \in D\Big\} < \infty,$$

and that a bounded composition operator C_ϕ of $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$ is compact if and only if

$$\frac{(1-|z|^2)^\beta|\phi'(z)|}{(1-|\phi(z)|^2)^\alpha} \to 0 \quad \text{as } \phi(z) \to \partial D.$$

Let $\alpha > 0$. We may call an $f \in H(B^n)$ an α-Bloch function on B^n, if

$$\|f\|_{\alpha,1} = \sup\big\{ (1-|z|^2)^\alpha |\nabla f(z)| : z \in B^n \big\} < \infty.$$

Meanwhile, we define

$$\|f\|_{\alpha,2} = \sup\big\{ (1-|z|^2)^\alpha |\mathcal{R} f(z)| : z \in B^n \big\} < \infty,$$

where $\mathcal{R}f(z) = \nabla f(z) z = \langle \nabla f(z), \overline{z}\rangle$ is the radial derivative of f. The equivalence of these two norms is proved by W. Yang and C. Ouyang [11]. For $\alpha = 1$, they are equivalent to the norm (1.2), see [8, 9]. Now, the question is how to define the third equivalent norm, like (1.2), for an arbitrary α. For $\alpha > 1/2$, the answer can be found in [15]. In this paper, we solve this problem in a more general situation.

Let $\mathcal{M}$ be the class of all positive and non-decreasing continuous functions $\mu(t)$, $0 < t \le 1$, such that $\mu(t) \to 0$ as $t \to 0$. In addition, we assume that every function in μ possesses the property

(†) there exists a $\delta > 0$ such that $\mu(t)/t^\delta$ is decreasing for small t.

As a consequence of property (†), we have

$$(\dagger\dagger) \qquad \mu(\sigma t) \ge \frac{\mu(t)}{C_{\mu,\sigma}} \quad \text{for } 0 < \sigma < 1,\ 0 < t \le 1.$$

For $\mu \in \mathcal{M}$, a function $f \in H(B^n)$ is called a μ-Bloch function if

$$\|f\|_{\mu,1} = \sup\big\{ \mu(1-|z|^2) |\nabla f(z)| : z \in B^n \big\} < \infty.$$

As in the case of α-Bloch functions, for $f \in H(B^n)$ and $\mu \in \mathcal{M}$, we define

$$\|f\|_{\mu,2} = \sup\big\{ \mu(1-|z|^2) |\mathcal{R} f(z)| : z \in B^n \big\} < \infty.$$

μ-Bloch functions were recently studied by Z. Hu [3] for the polydisc, and by X. Zhang and J. Xiao for the unit ball [12]. Since μ-Bloch functions are not invariant under Möbius mappings of B^n, it is more difficult to treat these function spaces. Zhang and Xiao gave another definition of μ-Bloch function and set necessary and sufficient conditions for the boundedness and compactness of C_ϕ, as a composition

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operator between μ-Bloch spaces, under an appropriate assumption on μ such that the equivalence of their definition and the above is guaranteed.

In Section 2 of this paper, for $\mu \in \mathcal{M}$, we give an estimate of the tangential derivative of a function $f \in H(B^n)$ in terms of the norm $\|f\|_{\mu,2}$. In Section 3, we define a metric $F_z^\mu(u)$, by which the third equivalent norm $\|f\|_{\mu,3}$ is defined. The equivalence of these norms is proved in Section 4. In Section 5, interesting examples of μ-Bloch functions are constructed by gap series for an arbitrary $\mu \in \mathcal{M}$. They will be used in the proof of the necessity of the conditions for boundedness and compactness in Sections 6 and 7. One of them will show that our estimate for the tangential derivative in Section 2 is precise. Sections 6 and 7 are devoted to the discussion of boundedness and compactness. Necessary and sufficient conditions for the boundedness and compactness of C_ϕ as a composition operator between μ-Bloch spaces are obtained. Under an appropriate assumption on μ, our results become those of Zhang and Xiao [12].

2 The Radial Derivative and Tangential Derivative

In the following theorem and throughout this paper, C_μ denotes a positive number depending on μ only, which may assume different values when appearing at different places.

Theorem 2.1 *Let $\mu \in \mathcal{M}$ and $f \in H(B^n)$. Then, for any $z \in B^n$ and $\zeta \in \partial B^n$ with $\zeta \perp z$, we have*

$$|\nabla f(z)\zeta| \le C_\mu \|f\|_{\mu,2}\Big(1 + \int_{1-|z|^2}^1 \frac{dt}{t^{1/2}\mu(t)}\Big). \tag{2.1}$$

If

$$I_\mu = \int_0^1 \frac{dt}{t^{1/2}\mu(t)} < \infty, \tag{2.2}$$

then (2.1) *becomes*

$$|\nabla f(z)\zeta| \le C_\mu \|f\|_{\mu,2}. \tag{2.3}$$

Proof To prove (2.1) and (2.3) we may, by a unitary change of coordinates, assume that $z = (r_0, 0, \dots, 0)$ with $0 \le r_0 < 1$ and $\zeta = (0, 1, 0, \dots, 0)$. Then

$$\nabla f(z)\zeta = \frac{\partial f}{\partial z_2}(r_0, 0, \dots, 0). \tag{2.4}$$

Let $f(z) = \sum_\lambda a_\lambda z^\lambda$, where $\lambda = (\lambda_1, \dots, \lambda_n)$ with integers $\lambda_k \ge 0$ and $z^\lambda = z_1^{\lambda_1} \cdots z_n^{\lambda_n}$. Then,

$$\frac{\partial f(z)}{\partial z_2} = \sum_{\lambda_2 \ne 0} a_\lambda \lambda_2 z^\lambda / z_2, \quad \mathcal{R} f(z) = \sum_\lambda a_\lambda |\lambda| z^\lambda,$$

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where $|\lambda| = \lambda_1 + \cdots + \lambda_n$, and

$$\frac{\partial f}{\partial z_2}(z_1, 0, \ldots, 0) = \sum_{\lambda_1=0}^{\infty} a_{(\lambda_1,1,0,\ldots,0)} z_1^{\lambda_1},$$
$$\frac{\partial \mathcal{R} f}{\partial z_2}(z_1, 0, \ldots, 0) = \sum_{\lambda_1=0}^{\infty} (\lambda_1 + 1) a_{(\lambda_1,1,0,\ldots,0)} z_1^{\lambda_1}.$$

Thus,

$$r_0 \cdot \frac{\partial f}{\partial z_2}(r_0, 0, \ldots, 0) = \int_0^{r_0} \frac{\partial \mathcal{R} f}{\partial z_2}(r, 0, \ldots, 0) dr. \tag{2.5}$$

For a fixed $r \geq 0$, the function $g(z_2) = \mathcal{R} f(r, z_2, 0, \ldots, 0)$ is estimated by

$$|g(z_2)| \leq \frac{\|f\|_{\mu,2}}{\mu(3(1 - r^2)/4)} \leq \frac{C_\mu \|f\|_{\mu,2}}{\mu(1 - r^2)} \quad \text{for } |z_2| < \frac{1}{2}(1 - r^2)^{1/2}.$$

Here property †† is used. Using Cauchy's inequality, we have

$$\Big| \frac{\partial \mathcal{R} f}{\partial z_2}(r, 0, \ldots, 0) \Big| = |g'(0)| \leq \frac{C_\mu \|f\|_{\mu,2}}{(1 - r^2)^{1/2} \mu(1 - r^2)}, \tag{2.6}$$

and, by (2.4) – (2.6),

$$|\nabla f(z)\zeta| \leq \frac{C_\mu \|f\|_{\mu,2}}{|z|} \int_0^{|z|} \frac{dr}{(1 - r^2)^{1/2} \mu(1 - r^2)}. \tag{2.7}$$

Since

$$\frac{1}{|z|} \int_0^{|z|} \frac{dr}{(1 - r^2)^{1/2} \mu(1 - r^2)} \leq C_\mu + 2 \int_{1/2}^{|z|^2} \frac{dr}{(1 - r)^{1/2} \mu(1 - r)}$$
$$= C_\mu + 2 \int_{1-|z|^2}^{1} \frac{dt}{t^{1/2} \mu(t)} \quad \text{for } |z| \geq 1/2,$$

and

$$\frac{1}{|z|} \int_0^{|z|} \frac{dr}{(1 - r^2)^{1/2} \mu(1 - r^2)} \leq C_\mu \quad \text{for } 0 \neq |z| \leq 1/2,$$

(2.1) follows from (2.7) if $z \neq 0$. By continuity, (2.1) also holds for $z = 0$. (2.3) follows from (2.1) under the assumption (2.2). The theorem is proved. ■

The estimate (2.1) for $\mu(t) = t^\alpha$ with $0 < \alpha < 1/2$ or $1/2 < \alpha < 1$ can be found in Rudin's book [5]. In Section 5, we will give an example to show that the estimate (2.1) is sharp.

Lemma 2.2 *Let $\mu \in \mathcal{M}$. Then, we have*

$$1+\int_t^1 \frac{d\tau}{\tau^{1/2}\mu(\tau)} \geq \frac{1}{C_\mu}\cdot\frac{t^{1/2}}{\mu(t)} \quad \text{for } 0<t\leq 1. \tag{2.8}$$

Proof According to the property (†), there exists a $\delta > 0$ such that $\mu(t)/t^\delta$ is decreasing for $0 < t \leq t_0 < 1$. Then, $t^{1/2+\delta}/\mu(t)$ is increasing for $0 < t \leq t_0$ and, consequently,

$$\int_t^1 \frac{d\tau}{\tau^{1/2}\mu(\tau)} > \int_t^{t_0} \frac{\tau^{1/2+\delta}d\tau}{\tau^{1+\delta}\mu(\tau)} > \frac{1}{\delta}\frac{t^{1/2+\delta}}{\mu(t)}\Big(\frac{1}{t^\delta}-\frac{1}{t_0^\delta}\Big).$$

Thus, there exists a positive $t' < t_0$ such that

$$\int_t^1 \frac{dt}{t^{1/2}\mu(t)} > \frac{1}{2\delta}\frac{t^{1/2}}{\mu(t)} \quad \text{for } 0<t<t'.$$

This shows that (2.8) holds for $0 < t < t'$. (2.8) is obviously true for $t' \leq t \leq 1$. The lemma is proved. ∎

Lemma 2.3 *Let $\mu \in \mathcal{M}$. If there exists $\delta > 0$ such that $\mu(t)/t^{1/2+\delta}$ is increasing for sufficiently small t, or $1/M \leq \mu(t)/t^{1/2+\delta} \leq M$ for $0 < t \leq 1$, then $I_\mu = \infty$ and*

$$1+\int_t^1 \frac{d\tau}{\tau^{1/2}\mu(\tau)} \leq C_\mu\cdot\frac{t^{1/2}}{\mu(t)} \quad \text{for } 0<t\leq 1. \tag{2.9}$$

Proof Let $\mu(t)/t^{1/2+\delta}$ be increasing for $0 < t \leq t_0 < 1$. Then,

$$I_\mu > \int_0^{t_0} \frac{d\tau}{\tau^{1/2}\mu(t)} \geq \frac{t_0^{1/2+\delta}}{\mu(t_0)}\int_0^{t_0}\frac{d\tau}{\tau^{1+\delta}} = \infty.$$

As in the proof of the preceding lemma, for $0 < t < t_0$, we have

$$\int_t^{t_0} \frac{d\tau}{\tau^{1/2}\mu(\tau)} = \int_t^{t_0} \frac{\tau^{1/2+\delta}d\tau}{\tau^{1+\delta}\mu(\tau)} < \frac{t^{1/2+\delta}}{\mu(t)}\int_t^{t_0}\frac{d\tau}{\tau^{1+\delta}} < \frac{1}{\delta}\frac{t^{1/2}}{\mu(t)}.$$

Thus, there exists a positive $t' < t_0$ such that

$$1+\int_t^1 \frac{d\tau}{\tau^{1/2}\mu(\tau)} < \frac{2}{\delta}\frac{t^{1/2}}{\mu(t)} \quad \text{for } 0<t<t',$$

since $t^{1/2}/\mu(t) \to \infty$ as $t \to 0$ by the assumption that $\mu(t)/t^{1/2+\delta}$ is increasing for small t. This shows that (2.9) holds for $0 < t < t'$. (2.9) is obviously true for $t' \leq t \leq 1$.

Now, assume that $1/M \leq \mu(t)/t^{1/2+\delta} \leq M$ for $0 < t \leq 1$. Then,

$$I_\mu = \int_0^1 \frac{\tau^{1/2+\delta}d\tau}{\tau^{1+\delta}\mu(\tau)} \geq \frac{1}{M}\int_0^1 \frac{d\tau}{\tau^{1+\delta}} = \infty,$$

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and there exist a $t' < 1$ such that

$$1 + \int_t^1 \frac{d\tau}{\tau^{1/2}\mu(\tau)} \leq 1 + M \int_t^1 \frac{d\tau}{\tau^{1+\delta}}$$
$$= 1 + \frac{M}{\delta}\Big(\frac{1}{t^\delta} - 1\Big) \leq \frac{2M}{\delta t^\delta} \leq \frac{2M}{\delta}\frac{t^{1/2}}{\mu(t)}, \quad \text{for } 0 < t < t'.$$

This shows that (2.9) holds for $0 < t < t'$. (2.9) is obviously true for $t' \leq t \leq 1$. The lemma is proved. ∎

The above lemmas show that if $\mu \in \mathcal{M}$ satisfies the condition formulated in Lemma 2.3, then

$$\text{(2.10)} \qquad \frac{1}{C_\mu}\frac{t^{1/2}}{\mu(t)} \leq 1 + \int_t^1 \frac{d\tau}{\tau^{1/2}\mu(\tau)} \leq C_\mu \cdot \frac{t^{1/2}}{\mu(t)}, \quad \text{for } 0 < t \leq 1,$$

and (2.1) can be replaced by

$$|\nabla f(z)\zeta| \leq \frac{C_\mu(1-|z|^2)^{1/2}}{\mu(1-|z|^2)} \cdot \|f\|_{\mu,2}.$$

3 μ-Metrics

Let $\mu \in \mathcal{M}$. If the integral I_μ defined in Theorem 2.1 is divergent, we denote

$$\nu(t) = \nu_\mu(t) = \Big(\frac{1}{\mu(1)} + \int_t^1 \frac{dt}{t^{1/2}\mu(t)}\Big)^{-1};$$

otherwise, let $\nu_\mu(t) \equiv \mu(1)$. The metric $F_z^\mu(u)$ corresponding to μ is defined by

$$F_z^\mu(u) = \sqrt{\frac{n+1}{2}}\frac{1}{\mu(1-|z|^2)}\Big\{\frac{\mu(1-|z|^2)^2}{\nu(1-|z|^2)^2}|u|^2 + \Big(1 - \frac{\mu(1-|z|^2)^2}{\nu(1-|z|^2)^2}\Big)\frac{|\langle u, z\rangle|^2}{|z|^2}\Big\}^{1/2}$$

for $0 \neq z \in B^n$ and $u \in \mathbb{C}^n$. For $z = 0$, we put $F_0^\mu(u) = \sqrt{(n+1)/2}|u|/\mu(1)$.

It is easy to verify that for $z \in B^n$, we have

$$\text{(3.1)} \qquad \frac{\sqrt{n+1}|u|}{\sqrt{2}\max\{\mu(1-|z|^2), \nu(1-|z|^2)\}} \leq F_z^\mu(u) \leq \frac{\sqrt{n+1}|u|}{\sqrt{2}\min\{\mu(1-|z|^2), \nu(1-|z|^2)\}}.$$

Indeed, if $z \neq 0$, we may write $u = u_1 z/|z| + u_2\zeta$, where $z \perp \zeta$ and $|\zeta| = 1$. Thus, $|u_1|^2 = |\langle u, z\rangle|^2/|z|^2$, $|u_2|^2 = |u|^2 - |u_1|^2$ and

$$F_z^\mu(u) = \sqrt{\frac{n+1}{2}}\Big(\frac{|u_1|^2}{\mu(1-|z|^2)^2} + \frac{|u_2|^2}{\nu(1-|z|^2)^2}\Big)^{1/2},$$

from which (3.1) follows. Note that

$$\frac{1}{\nu(t)} \leq \frac{1}{\mu(1)} + \frac{1}{\mu(t)} \int_0^1 \frac{d\tau}{\tau^{1/2}} = \frac{1}{\mu(1)} + \frac{2}{\mu(t)} \leq \frac{3}{\mu(t)}.$$

Thus, (3.1) becomes

$$\frac{\sqrt{n+1}|u|}{3\sqrt{2}\nu(1-|z|^2)} \leq F_z^\mu(u) \leq \frac{3\sqrt{n+1}|u|}{\sqrt{2}\mu(1-|z|^2)} \quad \text{for } z \in B^n. \tag{3.2}$$

It follows from (3.2) that

$$F_z^\mu(u) \geq \frac{\sqrt{n+1}|u|}{\sqrt{2}\mu(1)} \quad \text{for } z \in B^n, \tag{3.3}$$

and since μ is non-decreasing,

$$F_z^\mu(u) \leq \frac{3\sqrt{n+1}|u|}{\sqrt{2}\mu(1-r^2)} \quad \text{for } |z| \leq r < 1.$$

Lemma 3.1 *If μ satisfies the condition in Lemma 2.3, then $F_z^\mu(u)$ is equivalent to*

$$((1-|z|^2)/\mu(1-|z|^2))H_z(u,u)^{1/2},$$

where $H_z(u,u)$ is the Bergman metric of B^n formulated in the Introduction.

Proof Assume that μ satisfies the condition in Lemma 2.3. Then, by (2.10),

$$\frac{1}{C_\mu}\frac{t^{1/2}}{\mu(t)} \leq \frac{1}{\nu(t)} \leq C_\mu \cdot \frac{t^{1/2}}{\mu(t)}, \quad \text{for } 0 < t \leq 1,$$

and

$$\begin{aligned}
F_z^\mu(u)^2 &= \frac{n+1}{2}\frac{1}{\mu(1-|z|^2)^2}\Big\{\frac{\mu(1-|z|^2)^2}{\nu(1-|z|^2)^2}\Big(|u|^2 - \frac{|\langle u,z\rangle|^2}{|z|^2}\Big) + \frac{|\langle u,z\rangle|^2}{|z|^2}\Big\} \\
&\leq \frac{n+1}{2}\frac{C_\mu}{\mu(1-|z|^2)^2}\Big\{(1-|z|^2)\Big(|u|^2 - \frac{|\langle u,z\rangle|^2}{|z|^2}\Big) + \frac{|\langle u,z\rangle|^2}{|z|^2}\Big\} \\
&= \frac{n+1}{2}\frac{C_\mu}{\mu(1-|z|^2)^2}\Big\{(1-|z|^2)|u|^2 + |\langle u,z\rangle|^2\Big\} \\
&= C_\mu\Big(\frac{1-|z|^2}{\mu(1-|z|^2)}\Big)^2 H_z(u,u).
\end{aligned}$$

For the same reason

$$F_z^\mu(u)^2 \geq \frac{1}{C_\mu}\Big(\frac{1-|z|^2}{\mu(1-|z|^2)}\Big)^2 H_z(u,u).$$

This proves the lemma. ■

Note that in terms of the function ν, (2.1) in Theorem 2.1 can be written in

$$|\nabla f(z)\zeta| \leq \frac{C_\mu \|f\|_{\mu,2}}{\nu(1-|z|^2)}. \tag{3.4}$$

4 Equivalent Norms of μ-Bloch Functions

For $\mu \in \mathcal{M}$ and $f \in H(B^n)$, we define

$$Q_f^\mu(z) = \sup\left\{ \frac{|\nabla f(z)u|}{F_z^\mu(u)} : 0 \neq u \in \mathbb{C}^n \right\}, \quad \text{for } z \in B^n,$$

and

$$\|f\|_{\mu,3} = \sup\left\{ Q_f^\mu(z) : z \in B^n \right\}.$$

If μ satisfies the condition in Lemma 2.3, by Lemma 3.1 $F_z^\mu(u)$ is equivalent to

$$((1-|z|^2)/\mu(1-|z|^2))H_z(u,u)^{1/2},$$

and $\|f\|_{\mu,3}$ is equivalent to

$$\sup\left\{ \frac{\mu(1-|z|^2)|\nabla f(z)u|}{(1-|z|^2)H_z(u,u)} : 0 \neq u \in \mathbb{C}^n \right\},$$

It is the norm that was defined by Zhang and Xiao in [12].

Theorem 4.1 *For $\mu \in \mathcal{M}$, the norms $\|f\|_{\mu,1}$, $\|f\|_{\mu,2}$ and $\|f\|_{\mu,3}$ are equivalent.*

Proof Assume that $f \in B^n$ and $\mu \in \mathcal{M}$. It is obvious that $\|f\|_{\mu,2} \leq \|f\|_{\mu,1}$. Let $z \in B^n$. If $\nabla f(z) \neq 0$, letting $u = \nabla f(z)/|\nabla f(z)|$, we have

$$\begin{aligned}\mu(1-|z|^2)|\nabla f(z)| &= \mu(1-|z|^2)|\nabla f(z)\overline{u}| \\ &\leq \mu(1-|z|^2)Q_f^\mu(z)F_z^\mu(\overline{u},\overline{u})^{1/2} \leq 3\sqrt{\frac{n+1}{2}}Q_f^\mu(z),\end{aligned}$$

where (3.2) is used. This shows that

$$\|f\|_{\mu,1} \leq 3\sqrt{(n+1)/2}\|f\|_{\mu,3}. \tag{4.1}$$

Now, let $1/2 \leq |z| < 1$ and $0 \neq u \in \mathbb{C}^n$. There exists a ζ such that $|\zeta| = 1$, $\langle \zeta, z\rangle = 0$ and $u = u_1 z/|z| + u_2\zeta$. Then, $|u|^2 = |u_1|^2 + |u_2|^2$ and $u_1 = \langle u, z\rangle/|z|$. By (3.4), we have

$$\begin{aligned}|\nabla f(z)u|^2 &= |u_1\nabla f(z)(z/|z|) + u_2\nabla f(z)\zeta|^2 \leq 8(|u_1|^2|\nabla f(z)z|^2 + |u_2|^2|\nabla f(z)\zeta|^2) \\ &\leq \frac{8C_\mu^2\|f\|_{\mu,2}^2}{\mu(1-|z|^2)^2}\left(|u_1|^2 + |u_2|^2\frac{\mu(1-|z|^2)^2}{\nu(1-|z|^2)^2}\right) \\ &= \frac{8C_\mu^2\|f\|_{\mu,2}^2}{\mu(1-|z|^2)2}\left(\frac{\mu(1-|z|^2)^2}{\nu(1-|z|^2)^2}|u|^2 + \left(1 - \frac{\mu(1-|z|^2)^2}{\nu(1-|z|^2)^2}\right)|u_1|^2\right) \\ &= \frac{8C_\mu^2\|f\|_{\mu,2}^2}{\mu(1-|z|^2)^2}\left(\frac{\mu(1-|z|^2)^2}{\nu(1-|z|^2)^2}|u|^2 + \left(1 - \frac{\mu(1-|z|^2)^2}{\nu(1-|z|^2)^2}\right)\frac{|\langle u,z\rangle|^2}{|z|^2}\right) \\ &= \frac{16C_\mu^2\|f\|_{\mu,2}^2}{n+1}F_z^\mu(u)^2.\end{aligned}$$

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It is proved that

$$\frac{|\nabla f(z)u|}{F_z^\mu(u)} \le \frac{C_\mu}{\sqrt{n+1}}\|f\|_{\mu,2} \tag{4.2}$$

holds for $1/2 \le |z| < 1$ and $0 \ne u \in \mathbb{C}^n$. Combining (4.2) with (3.2) gives

$$|\nabla f(z)u| \le C_\mu \|f\|_{\mu,2}|u| \tag{4.3}$$

for $|z| = 1/2$ and $0 \ne u \in \mathbb{C}^n$. Since $|\nabla f(z)u|$ is subharmonic for a fixed u, (4.3) holds for $|z| \le 1/2$. It follows from (4.3) and (3.3) that (4.2) holds for $|z| \le 1/2$ and $0 \ne u \in \mathbb{C}^n$ also. This shows that

$$\|f\|_{\mu,3} \le \frac{C_\mu}{\sqrt{n+1}} \cdot \|f\|_{\mu,2}. \tag{4.4}$$

The theorem is proved. ■

The equivalence of the norms for $\mu(t) = t^\alpha$ with $\alpha > 1/2$ was indicated in [14].

5 Examples of μ-Bloch functions

The following lemma is due to Z. Hu [3]. For the convenience of our readers, we include the proof.

Lemma 5.1 *Let $\gamma(\rho)$, $0 \le \rho < 1$, be an non-decreasing and positive continuous function with the property that $\gamma(\rho) \to \infty$ as $\rho \to 1$ and there exist positive numbers δ and ρ_0, $\rho_0 < 1$, such that $\gamma(\rho)(1-\rho)^\delta$ is decreasing for $\rho_0 \le \rho < 1$. Then, there exists a function $\Gamma(\omega)$, holomorphic in the unit disk D and represented by a gap series with positive coefficients, such that $\gamma(\rho)/M \le \Gamma(\rho) \le M\gamma(\rho)$ with $M > 0$ for $0 \le \rho < 1$.*

Proof Let ρ_k be the smallest ρ such that

$$\frac{\gamma(\rho_{k+1})}{\gamma(\rho_k)} = 8^\delta \quad \text{for } k = 0, 1, 2, \dots. \tag{$*$}$$

Let $n_k = [A/\log(1/\rho_k)]$ for $k = 0, 1, 2, \dots$, where $A = \log(4 \cdot 8^\delta)$. Then there exists a positive integer K such that for $k \ge K$, we have

$$\frac{1-\rho_k}{1-\rho_{k+1}} \ge \left(\frac{\gamma(\rho_{k+1})}{\gamma(\rho_k)}\right)^{1/\delta} = 8,$$

since $\gamma(\rho)(1-\rho)^\delta$ is decreasing for $\rho_0 \le \rho < 1$, and

$$e^{-A} = \rho_k^{A/\log(1/\rho_k)} \le \rho_k^{n_k} < \rho_k^{A/\log(1/\rho_k)-1} < 2e^{-A} = \frac{8^{-\delta}}{2}, \tag{$**$}$$

$$(***)\qquad \frac{n_{k+1}}{n_k} \geq \frac{A/\log(1/\rho_{k+1})-1}{A/\log(1/\rho_k)} > \frac{A/(2(1-\rho_{k+1}))-1}{A/(1-\rho_k)}$$
$$= \frac{(1/2-(1-\rho_{k+1})/A)(1-\rho_k)}{1-\rho_{k+1}}$$
$$\geq 8(1/2-(1-\rho_{k+1})/A) > 2.$$

We define

$$\Gamma(\omega) = \sum_{k=K}^{\infty} \gamma(\rho_k)\omega^{n_k}.$$

Let $\rho_K \leq \rho_{m-1} \leq \rho < \rho_m$. By $(*)$, $(**)$, and $(***)$,

$$\Gamma(\rho) < \Gamma(\rho_m) = \sum_{k=K}^{\infty} \gamma(\rho_k)\rho_m^{n_k} = \sum_{k=K}^{m-1} \gamma(\rho_k)\rho_m^{n_k} + \sum_{k=m}^{\infty} \gamma(\rho_k)\rho_m^{n_k}$$
$$< \sum_{k=K}^{m-1} \gamma(\rho_k) + \sum_{k=m}^{\infty} \gamma(\rho_k)(\rho_m^{n_m})^{n_k/n_m}$$
$$< \gamma(\rho_m)\sum_{k=K}^{m-1} 8^{-(m-k)\delta} + \gamma(\rho_m)\sum_{k=m}^{\infty} 8^{(k-m)\delta}\Big(\frac{8^{-\delta}}{2}\Big)^{2^{k-m}}$$
$$< \gamma(\rho_m)\sum_{k=K}^{m-1} 8^{-(m-k)\delta} + \gamma(\rho_m)\sum_{k=m}^{\infty} 8^{(k-m)\delta}\Big(\frac{8^{-\delta}}{2}\Big)^{k-m+1}$$
$$< \gamma(\rho_m)\Big(\frac{8^{-\delta}}{1-8^{-\delta}} + 8^{-\delta}\Big) < \frac{2\cdot 8^{-\delta}}{1-8^{-\delta}}\cdot\gamma(\rho_m).$$

On the other hand, by $(**)$,

$$\Gamma(\rho) \geq \Gamma(\rho_{m-1}) > \gamma(\rho_{m-1})\rho_{m-1}^{n_{m-1}} \geq e^{-A}\gamma(\rho_{m-1}) = \frac{8^{-\delta}}{4}\cdot\gamma(\rho_{m-1}).$$

Thus, since γ is non-decreasing, we have

$$\frac{8^{-2\delta}}{4} = \frac{8^{-\delta}}{4}\cdot\frac{\gamma(\rho_{m-1})}{\gamma(\rho_m)} \leq \frac{\Gamma(\rho)}{\gamma(\rho)} \leq \frac{2\cdot 8^{-\delta}}{1-8^{-\delta}}\cdot\frac{\gamma(\rho_m)}{\gamma(\rho_{m-1})} = \frac{2}{1-8^{-\delta}}.$$

The above estimate has been proved for $\rho \geq \rho_K$. For $0 \leq \rho \leq \rho_K$, the ratio $\Gamma(\rho)/\gamma(\rho)$ is bounded above and has a positive lower bound, since both $\Gamma(\rho)$ and $\gamma(\rho)$ are positive and continuous. This shows that $\Gamma(\omega)$ is the function required and the lemma is proved. ■

By using the above lemma, we may construct useful examples of μ-Bloch functions.

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Example 1 For $\mu \in \mathcal{M}$, let $\Gamma_\mu(\omega)$ be the function constructed for $\gamma(\rho) = 1/\mu(1-\rho)$ in the above lemma. Let

$$G_\mu(\omega) = \int_0^\omega \Gamma_\mu(w)dw \quad \text{for } \omega \in D.$$

For $z_0 \in \partial B^n$, define $g(z) = g_{\mu,z_0}(z) = G_\mu(\langle z, z_0\rangle)$ for $z \in B^n$. Then, for $z \in B^n$,

$$\nabla g(z) = \Gamma_\mu(\langle z, z_0\rangle)\bar{z}_0 \tag{5.1}$$

and

$$\mu(1-|z|^2)|\nabla g(z)| = \mu(1-|z|^2)|\Gamma_\mu(\langle z, z_0\rangle)| \le \mu(1-|z|^2)\Gamma_\mu(|z|) \le \frac{C_\mu \mu(1-|z|^2)}{\mu(1-|z|)}.$$

It follows from (††) that

$$\frac{\mu(1-r^2)}{\mu(1-r)} \le \frac{\mu(1-r^2)}{\mu\big((1-r^2)/2\big)} \le C_\mu \quad \text{for } 0 \le r < 1. \tag{5.2}$$

Thus,

$$\|g\|_{\mu,1} = \sup_{z\in B^n} \mu(1-|z|^2)|\nabla g(z)| \le C_\mu. \tag{5.3}$$

This means that $g \in \mathcal{B}^\mu$.

On the other hand, taking $z = rz_0$ with $0 \le r < 1$, we have $\nabla g(z)\zeta = 0$ and

$$\begin{aligned}\mu(1-|z|^2)|\nabla g(z)| &= \mu(1-|z|^2)|\nabla g(z) z_0| \\ &= \mu(1-r^2)\Gamma_\mu(r) \ge \frac{1}{C_\mu} \cdot \frac{\mu(1-r^2)}{\mu(1-r)} \ge \frac{1}{C_\mu}.\end{aligned}$$

This shows that on the line $z = rz_0$ with $0 \le r < 1$, all tangential derivatives of g are equal to 0, and the radial derivative attains $1/\mu(1-|z|^2)$ up to a constant factor depending on μ only.

Example 2 For $\mu \in \mathcal{M}$, let $\Gamma_\mu(\omega)$ be the function formulated in Example 1,

$$\Lambda_\mu(\omega) = \frac{\Gamma(\omega)}{(1-\omega)^{1/2}}$$

and

$$L_\mu(\omega) = 1 + \int_0^\omega \Lambda_\mu(z)dz \quad \text{for } \omega \in D.$$

Then, for $0 \le r < 1$, since $1/(C_\mu \mu(1-\rho)) \le \Gamma(\rho) \le C_\mu/\mu(1-\rho)$ by Lemma 5.1, we have

$$L_\mu(r) \le 1 + C_\mu \int_0^r \frac{d\rho}{(1-\rho)^{1/2}\mu(1-\rho)} \tag{5.4}$$

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and

$$L_\mu(r) \geq \frac{1}{C_\mu}\Big(1 + \int_0^r \frac{d\rho}{(1-\rho)^{1/2}\mu(1-\rho)}\Big) = \frac{1}{C_\mu}\Big(1 + \int_{1-r}^1 \frac{dt}{t^{1/2}\mu(t)}\Big). \tag{5.5}$$

For $z_0, \zeta \in \partial B^n$ with $\zeta \perp z_0$, define $l(z) = l_{\mu,z_0,\zeta} = \langle z,\zeta\rangle L_\mu(\langle z,z_0\rangle)$ for $z \in B^n$. Then, for $z \in B^n$,

$$\nabla l(z) = L_\mu(\langle z,z_0\rangle)\overline{\zeta} + \langle z,\zeta\rangle \Lambda_\mu(\langle z,z_0\rangle)\overline{z}_0 \tag{5.6}$$

and

$$\mu(1-|z|^2)|\nabla l(z)| \leq \mu(1-|z|^2)L_\mu(|z|) + \mu(1-|z|^2)|\langle z,\zeta\rangle|\Lambda_\mu(|\langle z,z_0\rangle|). \tag{5.7}$$

Since $\Lambda_\mu(\rho) \leq C_\mu/((1-\rho)^{1/2}\mu(1-\rho))$, by (5.2), we have

$$\begin{aligned}\mu(1-|z|^2)|\langle z,\zeta\rangle|\Lambda_\mu(|\langle z,z_0\rangle|) &\leq \frac{C_\mu|\langle z,\zeta\rangle|}{(1-|\langle z,z_0\rangle|)^{1/2}}\frac{\mu(1-|z|^2)}{\mu(1-|\langle z,z_0\rangle|)}\\ &\leq \frac{C_\mu(1-|\langle z,z_0\rangle|^2)^{1/2}}{(1-|\langle z,z_0\rangle|)^{1/2}}\frac{\mu(1-|z|^2)}{\mu(1-|z|)} \leq \frac{C_\mu\sqrt{2}\mu(1-|z|^2)}{\mu(1-|z|)} \leq C'_\mu,\end{aligned} \tag{5.8}$$

where the inequality $|\langle z,\zeta\rangle|^2 + |\langle z,z_0\rangle|^2 \leq |z|^2 < 1$ is used, and by (5.4) and (5.2),

$$\begin{aligned}\mu(1-|z|^2)L_\mu(|z|) &\leq \mu(1) + C_\mu\mu(1-|z|^2)\int_0^{|z|}\frac{dr}{(1-r)^{1/2}\mu(1-r)}\\ &\leq \mu(1) + \frac{C_\mu\mu(1-|z|^2)}{\mu(1-|z|)}\int_0^1\frac{dr}{(1-r)^{1/2}} \leq C'_\mu.\end{aligned} \tag{5.9}$$

It follows from (5.2), (5.7), (5.8), and (5.9) that

$$\|l\|_{\mu,1} = \sup_{z\in B^n}\mu(1-|z|^2)|\nabla l(z)| \leq C_\mu \tag{5.10}$$

and $l \in \mathcal{B}^\mu$.

On the other hand, taking $z = rz_0$ with $r \geq 0$, we have $\nabla l(z)z_0 = 0$ and by (5.5),

$$\nabla l(z)\zeta = L_\mu(r) > L_\mu(r^2) \geq \frac{1}{C_\mu}\Big(1 + \int_{1-r^2}^1\frac{dt}{t^{1/2}\mu(t)}\Big).$$

This shows that on the line $z = rz_0$ with $r \geq 0$, the radial derivative of l is equal to 0 and the tangential derivative along ζ attains the upper bound (2.1) in Theorem 2.1 up to a constant factor depending only on μ. So (2.1) is sharp.

6 Bounded Composition Operators Between μ-Bloch Spaces

Theorem 6.1 *Let $\mu_1, \mu_2 \in \mathcal{M}$, and let ϕ be a holomorphic mapping of B^n into itself. Then the following conditions are equivalent:*

(i) $C_\phi\colon \mathcal{B}^{\mu_1} \longrightarrow \mathcal{B}^{\mu_2}$ *is bounded;*
(ii) $\sup\{\mu_2(1-|z|^2)F^{\mu_1}_{\phi(z)}(\phi'(z)z) : z \in B^n\} = M_1 < \infty$;
(iii)
$$\sup\Big\{\frac{F^{\mu_1}_{\phi(z)}(\phi'(z)u)}{F^{\mu_2}_z(u)} : z \in B^n,\ 0 \neq u \in \mathbb{C}^n\Big\} = M_2 < \infty.$$

Proof It is immediate that (iii) implies (ii). In fact, for $0 \neq z \in B^n$, we have $F^{\mu_2}_z(z) = |z|/\mu_2(1-|z|^2)$ and, by (iii),

$$M_2 \geq \frac{F^{\mu_1}_{\phi(z)}(\phi'(z)z)}{F^{\mu_2}_z(z)} > \mu_2(1-|z|^2)F^{\mu_1}_{\phi(z)}(\phi'(z)z).$$

Now assume that (ii) holds. Let $f \in \mathcal{B}^{\mu_1}$ and $z \in B^n$. If $\phi'(z)z = 0$,

$$\mu_2(1-|z|^2)|\nabla(f\circ\phi)(z)z| = \mu_2(1-|z|^2)|\nabla f(\phi(z))\phi'(z)z| = 0.$$

If $\phi'(z)z \neq 0$, then

$$\begin{aligned}\mu_2(1-|z|^2)&|\nabla(f\circ\phi)(z)z| \\ &= \mu_2(1-|z|^2)F^{\mu_1}_{\phi(z)}(\phi'(z)z)\cdot\frac{|\nabla f(\phi(z))\phi'(z)z|}{F^{\mu_1}_{\phi(z)}(\phi'(z)z)} \leq M_1\|f\|_{\mu_1,3}.\end{aligned}$$

It is proved that $\|C_\phi(f)\|_{\mu_2,2} \leq M_1\|f\|_{\mu_1,3}$. Consequently, by (4.1) and (4.4),

$$\|C_\phi(f)\|_{\mu_2,1} \leq \frac{C_{\mu_1}C_{\mu_2}M_1}{\sqrt{n+1}}\cdot\|f\|_{\mu_1,1} \leq \frac{C_{\mu_1}C_{\mu_2}M_1}{\sqrt{n+1}}\cdot\|f\|_{\mathcal{B}^{\mu_1}}.$$

On the other hand,

$$\begin{aligned}|f(\phi(0))| &\leq |f(0)| + \int_0^{\phi(0)}|\nabla f(\zeta)||d\zeta| \\ &\leq |f(0)| + \|f\|_{\mu_1,1}\int_0^{|\phi(0)|}\frac{dr}{\mu_1(1-r^2)} = C_{\mu_1,\phi}\|f\|_{\mathcal{B}^{\mu_1}}.\end{aligned}$$

Thus,

$$\|C_\phi(f)\|_{\mathcal{B}^{\mu_2}} = |f(\phi(0))| + \frac{C_{\mu_1}C_{\mu_2}M_1}{\sqrt{n+1}}\cdot\|f\|_{\mu_1,1} \leq C_{\mu_2}C_{\mu_1,\phi}(1+M_1)\|f\|_{\mathcal{B}^{\mu_1}}.$$

This shows that $C_\phi : \mathcal{B}^{\mu_1} \longrightarrow \mathcal{B}^{\mu_2}$ is bounded. It is proved that (ii) implies (i).

Finally, assume that $C_\phi\colon \mathcal{B}^{\mu_1} \longrightarrow \mathcal{B}^{\mu_2}$ is bounded. For $z' \in B^n$ and $0 \neq u \in \mathbb{C}^n$ with $\phi(z') \neq 0$ and $\phi'(z')u \neq 0$, let $w' = \phi(z')$, $z_0 = w'/|w'|$, $v' = \phi'(z')u = v_1 z_0 + v_2\zeta = e^{i\theta_1}|v_1|z_0 + e^{i\theta_2}|v_2|\zeta$ with $\zeta \perp w'$ and $|\zeta| = 1$. Define

$$f(z) = f_{z',u}(z) = e^{-i\theta_1} g_{\mu_1,z_0}(z) + e^{-i\theta_2} l_{\mu_1,z_0,\zeta}(z) \quad \text{for } z \in B^n,$$

where g_{μ_1,z_0} and $l_{\mu_1,z_0,\zeta}(z)$ are the functions defined in Examples 1 and 2. Then,

$$f(0) = 0 \quad \text{and } \|f\|_{\mu_1,1} \leq C_{\mu_1} \tag{6.1}$$

by (5.3) and (5.10). On the other hand, it follows from (5.1) and (5.6) that

$$\nabla f(w') = e^{-i\theta_1}\Gamma_{\mu_1}(|w'|)\overline{z}_0 + e^{-i\theta_2} L_{\mu_1}(|w'|)\overline{\zeta}$$

and

$$\nabla f(w')v' = |v_1|\Gamma_{\mu_1}(|w'|) + |v_2| L_{\mu_1}(|w'|).$$

We have

$$\Gamma(|w'|) \geq \frac{1}{C_{\mu_1}\mu_1(1-|w'|)}, \quad L_{\mu_1}(|w'|) \geq \frac{1}{C_{\mu_1}\nu_{\mu_1}(1-|w'|^2)}.$$

The last inequality follows from (5.2). Thus,

$$\begin{aligned} |\nabla f(w')v'| &\geq \frac{1}{C_{\mu_1}}\Big(\frac{|v_1|}{\mu_1(1-|w'|)} + \frac{|v_2|}{\nu_{\mu_1}(1-|w'|^2)}\Big) \\ &\geq \frac{1}{C_{\mu_1}}\Big(\frac{|v_1|^2}{\mu_1(1-|w'|^2)^2} + \frac{|v_2|^2}{\nu_{\mu_1}(1-|w'|^2)^2}\Big)^{1/2} = \frac{\sqrt{2}}{C_{\mu_1}\sqrt{n+1}} F^{\mu_1}_{w'}(v'). \end{aligned}$$

This shows that

$$\frac{|\nabla f(w')v'|}{F^{\mu_1}_{w'}(v')} \geq \frac{1}{C_{\mu_1}\sqrt{n+1}}. \tag{6.2}$$

Since C_ϕ is bounded, by (6.1) and (6.2), we have

$$\begin{aligned} C_{\mu_1}\|C_\phi\| &\geq \|C_\phi\|\cdot\|f\|_{\mu_1,1} = \|C_\phi\|\cdot\|f\|_{\mathcal{B}^{\mu_1}} \geq \|C_\phi(f)\|_{\mathcal{B}^{\mu_2}} \\ &\geq \|C_\phi(f)\|_{\mu_2,1} \geq \frac{\sqrt{n+1}}{C_{\mu_2}}\|C_\phi(f)\|_{\mu_2,3} \geq \frac{\sqrt{n+1}}{C_{\mu_2}}\frac{|\nabla f(\phi(z'))\phi'(z')u|}{F^{\mu_2}_{z'}(u)} \\ &= \frac{\sqrt{n+1}}{C_{\mu_2}}\frac{|\nabla f(\phi(z'))\phi'(z')u|}{F^{\mu_1}_{\phi(z')}(\phi'(z')u)}\frac{F^{\mu_1}_{\phi(z')}(\phi'(z')u)}{F^{\mu_2}_{z'}(u)} \\ &= \frac{\sqrt{n+1}}{C_{\mu_2}}\frac{|\nabla f(w')v'|}{F^{\mu_1}_{w'}(v')}\frac{F^{\mu_1}_{\phi(z')}(\phi'(z')u)}{F^{\mu_2}_{z'}(u)} \geq \frac{1}{C_{\mu_2}C_{\mu_1}}\frac{F^{\mu_1}_{\phi(z')}(\phi'(z')u)}{F^{\mu_2}_{z'}(u)}. \end{aligned}$$

Thus,

$$\frac{F^{\mu_1}_{\phi(z')}(\phi'(z')u)}{F^{\mu_2}_{z'}(u)} \leq C_{\mu_2}C_{\mu_1}\|C_\phi\|,$$

when $\phi(z') \neq 0$ and $\phi'(z')u \neq 0$. The same inequality also holds if $\phi(z') = 0$ and $\phi'(z')u = 0$ by continuity. This shows that (i) implies (iii). The theorem is proved. ∎

Lemma 6.2 *$C_\phi \colon \mathcal{B}^\mu \longrightarrow \mathcal{B}^\mu$ is bounded for any $\phi \in Aut(B^n)$ and $\mu \in \mathcal{M}$.*

Proof Let $\phi \in \mathrm{Aut}(B^n)$ and $\mu \in \mathcal{M}$. Assume that $\phi = \psi \circ \phi_a$, where ψ is a mapping defined by a unitary matrix and ϕ_a is a mapping in $\mathrm{Aut}(B^n)$ which exchanges a with the origin. A well-known identity asserts that

$$1 - |\phi(z)|^2 = 1 - |\phi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a\rangle|^2}.$$

Thus,

$$\frac{1-|\phi(z)|^2}{1-|z|^2} \geq \frac{1-|a|^2}{2} \quad \text{for } z \in B^n. \tag{6.3}$$

Let $z \in B^n$. If $|\phi(z)| \leq |z|$, by (3.2), we have

$$\mu(1-|z|^2)F^\mu_{\phi(z)}(\phi'(z)z) \leq \frac{3\sqrt{n+1}\mu(1-|z|^2)|\phi'(z)z|}{\sqrt{2}\mu(1-|\phi(z)|^2)} \leq C_n|\phi'(z)|,$$

where $|\phi'(z)|$ is the operator norm of $\phi'(z)$, which is defined by

$$|\phi'(z)| = \sup\{|\phi'(z)u| : u \in \partial B^n\}.$$

In the case $|\phi(z)| \geq |z|$, because of (6.3) and (††),

$$\begin{aligned}\mu(1-|z|^2)F^\mu_{\phi(z)}(\phi'(z)z) &\leq \frac{C_n\mu(1-|z|^2)|\phi'(z)|}{\mu(1-|\phi(z)|^2)} \\ &\leq \frac{C_n\mu(1-|z|^2)|\phi'(z)|}{\mu((1-|a|^2)(1-|z|^2)/2)} \leq C_nC_{a,\mu}|\phi'(z)|.\end{aligned}$$

Now ϕ is holomorphic on the closed ball $\overline{B}^n$ and so $|\phi'(z)|$ is bounded on B^n. This shows that the condition (ii) in Theorem 6.1 is satisfied. By Theorem 6.1, $C_\phi \colon \mathcal{B}^\mu \longrightarrow \mathcal{B}^\mu$ is bounded and the lemma is proved. ∎

Lemma 6.3 *Let $\mu \in \mathcal{M}$ with the property that $\mu(t)/t$ is increasing for small t or there is a $\delta \geq 0$ such that $mt^{1+\delta} \leq \mu(t) \leq Mt^{1+\delta}$ for $0 < t \leq 1$, and let ϕ be a holomorphic mapping of B^n into itself such that $\phi(0) = 0$. Then $C_\phi \colon \mathcal{B}^\mu \longrightarrow \mathcal{B}^\mu$ is bounded.*

Proof Assume that $\mu(t)/t$ is increasing for $0 < t \leq t_0 < 1$. Then μ satisfies the assumption in Lemma 2.3. By the Schwarz–Pick lemma, $|\phi(z)| \leq |z|$ and $1-|z|^2 \leq 1-|\phi(z)|^2$ since $\phi(0) = 0$. For $z \in B^n$ and $0 \neq u \in \mathbb{C}^n$, applying Lemma 3.1 and (1.3), we have

$$\frac{F^\mu_{\phi(z)}(\phi'(z)u)}{F^\mu_z(u)} \leq C_\mu \cdot \frac{\mu(1-|z|^2)}{(1-|z|^2)} \frac{1-|\phi(z)|^2}{\mu(1-|\phi(z)|^2)}.$$

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If $1-|\phi(z)|^2 \le t_0$, since $\mu(t)/t$ is increasing for $0 < t \le t_0$, we have

$$\frac{\mu(1-|z|^2)}{1-|z|^2} \le \frac{\mu(1-|\phi(z)|^2)}{1-|\phi(z)|^2}.$$

If $1-|\phi(z)|^2 \ge t_0$, then

$$\frac{1-|\phi(z)|^2}{\mu(1-|\phi(z)|^2)} \le \max\{t/\mu(t) : t_0 \le t \le 1\}.$$

If $1-|z|^2 \le t_0$, since $\mu(t)/t$ is increasing for $0 < t \le t_0$, we have

$$\frac{\mu(1-|z|^2)}{(1-|z|^2)} \le \frac{\mu(t_0)}{t_0}.$$

If $1-|z|^2 \ge t_0$, then

$$\frac{\mu(1-|z|^2)}{(1-|z|^2)} \le \max\{\mu(t)/t : t_0 \le t \le 1\}.$$

Combining the above estimates we conclude that the condition (iii) in Theorem 6.1 is satisfied and $C_\phi \colon \mathcal{B}^\mu \longrightarrow \mathcal{B}^\mu$ is bounded.

If there is a $\delta \ge 0$ such that $mt^{1+\delta} \le \mu(t) \le Mt^{1+\delta})$ for $0 < t \le 1$, then μ satisfies the assumption in Lemma 2.3 also and, for $z \in B^n$ and $0 \ne u \in \mathbb{C}^n$,

$$\frac{F^\mu_{\phi(z)}(\phi'(z)u)}{F^\mu_z(u)} \le \frac{C_\mu M}{m} \cdot \frac{(1-|z|^2)^\delta}{(1-|\phi(z)|^2)^\delta} \le \frac{C_\mu M}{m}.$$

The condition (iii) is satisfied and C_ϕ is bounded. The lemma is proved. ■

As a consequence of the above two lemmas, we have the following theorem.

Theorem 6.4 *Let $\mu \in \mathcal{M}$ with the property that $\mu(t)/t$ is increasing for small t or there is a $\delta \ge 0$ such that $mt^{1+\delta} \le \mu(t) \le Mt^{1+\delta}$ for $0 < t \le 1$, and let ϕ be a holomorphic mapping of B^n into itself. Then C_ϕ is a bounded operator of $\mathcal{B}^\mu$ into itself. Further, if $\mu_1 \in \mathcal{M}$ and $\mu_1(t) \ge m\mu(t)$ for small t with $m > 0$, then $C_\phi : \mathcal{B}^{\mu_1} \longrightarrow \mathcal{B}^\mu$ is bounded.*

Proof Let $\phi = \psi \circ \sigma$, where $\psi \in \operatorname{Aut}(B^n)$ and $\sigma(0) = 0$. Then $C_\phi = C_\sigma \circ C_\psi$. By the above lemmas, C_σ and C_ψ are both bounded operators of $\mathcal{B}^\mu$ into itself and, consequently, C_ϕ is.

If $\mu_1(t) \ge m\mu(t)$ for $0 < t \le t_0 = 1 - r_0^2$, then, for $f \in H(B^n)$, we have

$$\sup_{|z|\ge r_0} \mu(1-|z|^2)|\nabla f(z)| \le \frac{1}{m} \sup_{|z|\ge r_0} \mu_1(1-|z|^2)|\nabla f(z)| \le \frac{1}{m}\|f\|_{\mu_1,1}.$$

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On the other hand,

$$\sup_{|z|\le r_0} \mu(1-|z|^2)|\nabla f(z)| \le \mu(1)\max_{|z|=r_0}|\nabla f(z)|$$
$$\le \frac{\mu(1)}{\mu_1(t_0)}\max_{|z|=r_0}\mu_1(1-|z|^2)|\nabla f(z)| \le \frac{\mu(1)}{\mu_1(t_0)}\|f\|_{\mu_1,1}.$$

It is proved that $\|f\|_{\mu,1} \le \max\{1/m,\ \mu(1)/\mu_1(t_0)\}\|f\|_{\mu_1,1}$. So, if we let i be the identity mapping of B^n, then C_i is a bounded operator $\mathcal{B}^{\mu_1}$ into $\mathcal{B}^{\mu}$. It follows that $C_\phi = C_\phi \circ C_i$ is a bounded operator of $\mathcal{B}^{\mu_1}$ into $\mathcal{B}^{\mu}$, since we have proved that C_ϕ is a bounded operator of $\mathcal{B}^{\mu}$ into itself. The theorem is proved. ∎

7 Compact Composition Operators Between μ-Bloch Spaces

Lemma 7.1 *For $\mu \in \mathcal{M}$ with $I_\mu = \infty$, $0 \ne w \in B^n$ and $0 \ne v \in \mathbb{C}^n$, there exists a function $f_{\mu,w,v}$ such that*

(i) $f_{\mu,w,v}(0) = 0$ *and* $\|f_{\mu,w,v}\|_{\mu,1} \le C_\mu$;
(ii) $|\nabla f_{\mu,w,v}(w)v|/F_w^\mu(v) \ge 1/C_{\mu,n}$.

Further, for a fixed μ, $f_{\mu,w,v}(z) \to 0$ as $w \to \partial B^n$ locally uniformly in B^n. Precisely speaking, for $\epsilon > 0$, $0 < r < 1$, there exists an $r'_{\mu,\epsilon,r}$ such that $|f_{\mu,w,v}(z)| < \epsilon$ for $|w| > r'$, $|z| \le r$ and $0 \ne v \in \mathbb{C}^n$.

Proof Let $\mu \in \mathcal{M}$, $0 \ne w \in B^n$ and $0 \ne v \in \mathbb{C}^n$ be fixed, let $v = v_1 w/|w| + v_2\zeta$ with $\zeta \perp w$ and $|\zeta| = 1$, and let $v_1 = |v_1|e^{i\theta_1}$ and $v_2 = |v_2|e^{i\theta_2}$. We define

$$f(z) = f_{\mu,w,v}(z) = e^{-i\theta_1}(1-|w|^2)^{1/2}L_\mu(\langle z,w\rangle)/|w|$$
$$+\frac{e^{-i\theta_2}\langle z,\zeta\rangle L_\mu(\langle z,w\rangle)^2}{L_\mu(|w|^2)} - \frac{e^{-i\theta_1}(1-|w|^2)^{1/2}}{|w|},$$

where $L(\omega) = L_\mu(\omega)$ is the function defined in Example 2. Then, $f(0) = 0$ and

$$\nabla f(z) = e^{-i\theta_1}(1-|w|^2)^{1/2}\Lambda(\langle z,w\rangle)\overline{w}/|w|$$
$$+\frac{e^{-i\theta_2}L(\langle z,w\rangle)^2\overline{\zeta}}{L(|w|^2)} + \frac{2e^{-i\theta_2}\langle z,\zeta\rangle L(\langle z,w\rangle)\Lambda(\langle z,w\rangle)\overline{w}}{L(|w|^2)}.$$

It is obvious that

$$|\Lambda(\langle z,w\rangle)| \le \Lambda(|\langle z,w\rangle|) \le \Lambda(|z||w|) \le \Lambda(|w|),$$
$$|L(\langle z,w\rangle)| \le L(|w|), \quad |L(\langle z,w\rangle)| \le L(|z|).$$

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Thus, since $\Lambda(\rho) \le C_\mu/((1-\rho)^{1/2}\mu(1-\rho))$ for $0 \le \rho < 1$, we have

$$\begin{aligned}|\nabla f(z)| &\le (1-|w|^2)^{1/2}\Lambda(|z||w|) + \frac{L(|z|)L(|w|)}{L(|w|^2)} + \frac{2|\langle z,\zeta\rangle| L(|w|)\Lambda(|\langle z,w\rangle|)}{L(|w|^2)} \\ &\le \frac{C_\mu(1-|w|^2)^{1/2}}{(1-|z||w|)^{1/2}\mu(1-|z||w|)} + \frac{L(|z|)L(|w|)}{L(|w|^2)} \\ &\quad + \frac{2C_\mu|\langle z,\zeta\rangle|L(|w|)}{(1-|\langle z,w\rangle|)^{1/2}\mu(1-|\langle z,w\rangle|)L(|w|^2)}\end{aligned}$$

and

$$\begin{aligned}(7.1)\quad \mu(1-|z|^2)|\nabla f_{\mu,w,v}(z)| &\le \frac{C_\mu\mu(1-|z|^2)}{\mu(1-|z|)}\cdot\frac{(1-|w|^2)^{1/2}}{(1-|w|)^{1/2}} \\ &+ \mu(1-|z|^2)L(|z|)\cdot\frac{L(|w|)}{L(|w|^2)} + \frac{2C_\mu\mu(1-|z|^2)}{\mu(1-|z|)}\cdot\frac{|\langle z,\zeta\rangle|}{(1-|\langle z,w\rangle|)^{1/2}}\cdot\frac{L(|w|)}{L(|w|^2)}.\end{aligned}$$

If $|w| \ge 1/2$, since

$$\begin{aligned}\int_{1/2}^{|w|}\frac{d\rho}{(1-\rho)^{1/2}\mu(1-\rho)} &\le \int_{1/4}^{|w|^2}\frac{d\rho}{(1-\sqrt{\rho})^{1/2}\mu(1-\sqrt{\rho})} \\ &\le \sqrt{2}\int_{1/4}^{|w|^2}\frac{d\rho}{(1-\rho)^{1/2}\mu\Big((1-\rho)/2\Big)} \\ &\le \sqrt{2}C_\mu\int_{1/4}^{|w|^2}\frac{d\rho}{(1-\rho)^{1/2}\mu((1-\rho)},\end{aligned}$$

where the property (††) is used, we have, by (5.4) and (5.5),

$$(7.2)\qquad L(|w|) \le C'_\mu\Big(1+\int_0^{|w|^2}\frac{d\rho}{(1-\rho)^{1/2}\mu((1-\rho)}\Big) \le C'_\mu L(|w|^2).$$

The above estimate is evidently true for $|w| \le 1/2$.

It is obvious that

$$(7.3)\qquad \frac{(1-|w|^2)^{1/2}}{(1-|w|)^{1/2}} \le \sqrt{2}$$

and, by (5.2),

$$(7.4)\qquad \frac{\mu(1-|z|^2)}{\mu(1-|z|)} \le C_\mu \quad \text{for } z \in B^n.$$

For $z \in B^n$, let $u = \langle z, w/|w|\rangle w/|w| + \langle z,\zeta\rangle\zeta$. Then, $(z-u) \perp u$ and

$$1 > |z|^2 \ge |u|^2 = |\langle z, w/|w|\rangle|^2 + |\langle z,\zeta\rangle|^2 > |\langle z,w\rangle|^2 + |\langle z,\zeta\rangle|^2.$$

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Consequently,

$$\frac{|\langle z,\zeta\rangle|}{(1-|\langle z,w\rangle|)^{1/2}} < \frac{\sqrt{2}|\langle z,\zeta\rangle|}{(1-|\langle z,w\rangle|^2)^{1/2}} < \sqrt{2}. \tag{7.5}$$

Now, replacing (5.9), (7.2)–(7.5), in (7.1), we obtain

$$\mu(1-|z|^2)|\nabla f(z)| \le C_\mu \quad \text{for } z \in B^n.$$

This shows that $\|f\|_{\mu,1} \le C_\mu$, and (i) is proved.

On the other hand, since $\Lambda(\rho) \ge 1/\Big(C_\mu(1-\rho)^{1/2}\mu(1-\rho)\Big)$ for $0 \le \rho < 1$ and $L(r) \ge 1/(C_\mu \nu(1-r))$ by (5.5), we have

$$\begin{aligned} |\nabla f(w)v| &= |v_1|(1-|w|^2)^{1/2}\Lambda(|w|^2) + |v_2|L(|w|^2) \\ &\ge \frac{1}{C_\mu}\Big(\frac{|v_1|}{\mu(1-|w|^2)} + \frac{|v_2|}{\nu_\mu(1-|w|^2)}\Big) \\ &\ge \frac{1}{C_\mu}\Big(\frac{|v_1|^2}{\mu(1-|w|^2)^2} + \frac{|v_2|^2}{\nu_\mu(1-|w|^2)^2}\Big)^{1/2} \\ &= \frac{1}{C_{\mu,n}} \cdot F_w^\mu(v). \end{aligned}$$

This shows (ii).

Let $0 < r < 1$ be given. For $|z| \le r$, we have

$$|f_{\mu,w,v}(z)| \le \frac{(1-|w|^2)^{1/2}L_\mu(r)}{|w|} + \frac{L_\mu(r)^2}{L_\mu(|w|^2)} + \frac{(1-|w|^2)^{1/2}}{\mu(1)|w|}.$$

The right side of the above tends to 0 as $|w| \to 1$ since $L_\mu(|w|) \to \infty$ as $|w| \to 1$ for $I_\mu = \infty$. The second part of the lemma is proved. ■

Lemma 7.2 *For $\mu \in \mathcal{M}$ with $I_\mu < \infty$ and $0 \ne w \in B^n$, there exists a function $f_{\mu,w}$ such that*

(i) $f_{\mu,w}(0) = 0$ *and* $\|f_{\mu,w,v}\|_\mu \le C_\mu$;
(ii) $\mu(1-|w|^2)|\nabla f_{\mu,w}(w)v|/|\langle v,w\rangle| \ge 1/C_\mu$.

Further, for a fixed μ, $f_{\mu,w}(z) \to 0$ as $w \to \partial B^n$ locally uniformly in B^n.

Proof For $\mu \in \mathcal{M}$ with $I_\mu < \infty$ and $0 \ne w \in B^n$, let

$$f(z) = f_{\mu,w}(z) = (1-|w|^2)^{1/2}L_\mu(\langle z,w\rangle)/|w| - \frac{(1-|w|^2)^{1/2}}{|w|}.$$

Then, as in the proof of Lemma 7.1, we have $f_{\mu,w}(0) = 0$, $\|f_{\mu,w}\|_{\mu,1} \le C_\mu$ and, for $0 \ne v = v_1 w/|w| + v_2\zeta$ with $\zeta \perp w$ and $|\zeta| = 1$,

$$|\nabla f(w)v| = |v_1|(1-|w|^2)^{1/2}\Lambda(|w|^2) \ge \frac{1}{C_\mu}\frac{|v_1|}{\mu(1-|w|^2)} = \frac{1}{C_\mu}\frac{|\langle v,w\rangle|}{\mu(1-|w|^2)}.$$

The second part of the lemma is obvious. ■

Lemma 7.3 *Let $f \in H(B^n)$ and $\mu \in \mathcal{M}$ with $I_\mu < \infty$. If $|\nabla f(z)| \le m$ for $|z| \le r_0$, $1/2 \le r_0 < 1$, then for $r_0 \le |z| < 1$ and $\zeta \perp z$ with $|\zeta| = 1$, we have*

$$|\nabla f(z)\zeta| \le m + C_{\mu,r_0}\|f\|_{\mu,1},$$

where $C_{\mu,r_0} \to 0$ as $r_0 \to 1$.

Proof It is sufficient to prove the lemma for $z = (\rho, 0, \dots, 0)$ with $\rho \ge r_0$ and $\zeta = (0, 1, 0, \dots, 0)$. As in the proof of Theorem 2.1,

$$\rho\frac{\partial f}{\partial z_2}(\rho, 0, \dots, 0) - r_0\frac{\partial f}{\partial z_2}(r_0, 0, \dots, 0) = \int_{r_0}^{\rho} \frac{\partial \Re f}{\partial z_2}(z_1, 0, \dots, 0)dz_1,$$

$$\begin{aligned}|\nabla f(z)\zeta| &= \Big|\frac{\partial f}{\partial z_2}(\rho, 0, \dots, 0)\Big| \\ &\le \Big|\frac{\partial f}{\partial z_2}(r_0, 0, \dots, 0)\Big| + C_\mu\|f\|_{\mu,2}\int_{r_0}^{\rho}\frac{dr}{(1-r^2)^{1/2}\mu(1-r^2)} \\ &\le m + C_\mu\|f\|_{\mu,1}\int_0^{1-r_0^2}\frac{dt}{t^{1/2}\mu(t)}. \quad \blacksquare\end{aligned}$$

Theorem 7.4 *Let $\mu_1, \mu_2 \in \mathcal{M}$, and let ϕ be a holomorphic mapping of B^n into itself and $C_\phi : \mathcal{B}^{\mu_1} \longrightarrow \mathcal{B}^{\mu_2}$ be bounded. If $I_{\mu_1} = \infty$, then the following conditions are equivalent:*

(i) $C_\phi : \mathcal{B}^{\mu_1} \longrightarrow \mathcal{B}^{\mu_2}$ *is compact;*
(ii) $\mu_2(1-|z|^2)F^{\mu_1}_{\phi(z)}(\phi'(z)z) \longrightarrow 0$ *as* $\phi(z) \to \partial B^n$*;*
(iii) $\frac{F^{\mu_1}_{\phi(z)}(\phi'(z)u)}{F^{\mu_2}_z(u)} \longrightarrow 0$ *as* $\phi(z) \to \partial B^n$.

If $I_{\mu_1} < \infty$, then the following conditions and (i) are equivalent:

(ii′)
$$\frac{\mu_2(1-|z|^2)|\langle\phi'(z)z, \phi(z)\rangle|}{\mu_1(1-|\phi(z)|^2)} \longrightarrow 0 \quad \text{as } \phi(z) \to \partial B^n;$$

(iii′)
$$\frac{|\langle\phi'(z)u, \phi(z)\rangle|}{F^{\mu_2}_z(u)\mu_1(1-|\phi(z)|^2)} \longrightarrow 0 \quad \text{as } \phi(z) \to \partial B^n.$$

Proof As in the proof of Theorem 6.1, it is obvious that (iii) implies (ii) and (iii′) implies (ii′). Since C_ϕ is bounded, by Theorem 6.1,

$$\sup\{\mu_2(1-|z|^2)F^{\mu_1}_{\phi(z)}(\phi'(z)z) : z \in B^n\} = M < \infty. \tag{7.6}$$

First assume that $I_\mu = \infty$. Let (ii) hold. Let $f_k \in B^{\mu_1}$ and $\|f_k\|_{B^{\mu_1}} = 1$, for $k = 1, 2, \dots$. Applying Montel's theorem, by choosing a subsequence, we may assume that f_k converges to a function f locally uniformly in B^n. It is easy to see that $\|f\|_{B^{\mu_1}} \le 1$. Let $g_k = f_k - f$. Then, $g_k \to 0$ locally uniformly in B^n and

$$\|g_k\|_{\mathcal{B}^{\mu_1}} \le 2 \quad \text{for } k = 1, 2, \dots. \tag{7.7}$$

Let $\epsilon > 0$ be given. By the assumption (ii), there exists an $r_0 < 1$ such that

$$\mu_2(1-|z|^2)F^{\mu_1}_{\phi(z)}(\phi'(z)z) < \epsilon \quad \text{if } |\phi(z)| > r_0. \tag{7.8}$$

Since $g_k(w) \to 0$ uniformly for $|w| \le r_0$, by (3.3), there exists a K such that

$$\frac{|\nabla g_k(w)v|}{F^{\mu_1}_w(v,v)^{1/2}} \le \frac{3\sqrt{2}\mu(1)}{\sqrt{n+1}}|\nabla g_k(w)| < \epsilon \tag{7.9}$$

for $k > K$, $|w| \le r_0$ and $0 \ne v \in \mathbb{C}^n$.

Let $k > K$ and $z \in B^n$. To estimate $\mu_2(1-|z|^2)|\mathcal{R}C_\phi(g_k)(z)|$, we distinguish three cases.

(a) If $\phi'(z)z = 0$, $\mu_2(1-|z|^2)|\mathcal{R}C_\phi(g_k)(z)| = \mu_2(1-|z|^2)|\nabla g_k(\phi(z))\phi'(z)z| = 0$.
(b) If $\phi'(z)z \ne 0$ and $|\phi(z)| \le r_0$, then, by (7.6) and (7.9),

$$\mu_2(1-|z|^2)|\mathcal{R}C_\phi(g_k)(z)| = \mu_2(1-|z|^2)F^{\mu_1}_{\phi(z)}(\phi'(z)z)\frac{|\nabla g_k(\phi(z))\phi'(z)z|}{F^{\mu_1}_{\phi(z)}(\phi'(z)z)} < M\epsilon.$$

(c) If $\phi'(z)z \ne 0$ and $|\phi(z)| > r_0$, it follows from (7.7) and (7.8) that

$$\mu_2(1-|z|^2)|\mathcal{R}C_\phi(g_k)(z)| \le \epsilon\|g_k\|_{\mu_1,3} < C_{\mu_1}\epsilon.$$

We conclude that $\|C_\phi(g_k)\|_{\mu_2,2} < \epsilon\max\{M, C_{\mu_1}\}$ for $k > K$. This shows that

$$\|C_\phi(g_k)\|_{\mu_2,2} \to 0$$

and, consequently, $\|C_\phi(g_k)\|_{\mathcal{B}^{\mu_2}} \to 0$ as $k \to \infty$, since

$$\|C_\phi(g_k)\|_{\mu_2,1} \le C_{\mu_2}\|C_\phi(g_k)\|_{\mu_2,2}$$

and $g_k(\phi(0)) \to 0$ as $k \to \infty$. Thus, $f_k \circ \phi \to f \circ \phi$ according to the $\mathcal{B}^{\mu_2}$ norm. The compactness of C_ϕ is proved. This shows that (ii) implies (i).

Now, assume that (i) holds. Suppose on the contrary that (iii) doesn't hold. Then, there exist $\delta > 0$, sequences z_k and $u_k \ne 0$, such that

$$\frac{F^{\mu_1}_{\phi(z_k)}(\phi'(z_k)u_k)}{F^{\mu_2}_{z_k}(u_k)} > \delta, \quad \text{for } k = 1, 2, \dots, \tag{7.10}$$

where $w_k = \phi(z_k) \to \partial B^n$ as $k \to \infty$. For $k = 1, 2, \dots$, let $v_k = \phi'(z_k)u_k$ and $f_k = f_{\mu_1, w_k, v_k}$ be functions defined in Lemma 7.1. Then, f_k and, consequently, $C_\phi(f_k)$ converge to 0 locally uniformly in B^n. Since C_ϕ is compact and f_k is a bounded sequence in $\mathcal{B}^{\mu_1}$ by (i) in Lemma 7.1, by choosing a subsequence, we may assume that there is a function $g \in \mathcal{B}^{\mu_2}$ such that $\|C_\phi(f_k) - g\|_{\mathcal{B}^{\mu_2}} \to 0$. g must be equal to 0 identically for $C_\phi(f_k)$ converges to 0 locally uniformly in B^n. Thus, $\|C_\phi(f_k)\|_{\mathcal{B}^{\mu_2}} \to 0$. In particular,

$$\frac{|\nabla C_\phi(f_k)(z_k)u_k|}{F^{\mu_2}_{z_k}(u_k)} = \frac{|\nabla f_k(\phi(z_k))\phi'(z_k)u_k|}{F^{\mu_2}_{z_k}(u_k)} \to 0. \tag{7.11}$$

However, by (ii) in Lemma 7.1,

$$\frac{|\nabla f_k(\phi(z_k))\phi'(z_k)u_k|}{F^{\mu_1}_{\phi(z_k)}(\phi'(z_k)u_k)} \geq \frac{1}{C_{n,\mu_1}}, \quad \text{for } k = 1, 2, \dots. \tag{7.12}$$

Combining (7.10) and (7.12), we have

$$\frac{|\nabla f_k(\phi(z_k))\phi'(z_k)u_k|}{F^{\mu_2}_{z_k}(u_k)} \geq \frac{\delta}{C_{n,\mu_1}}.$$

This contradicts (7.11). This shows that (i) implies (iii). The theorem is proved for $I_\mu = \infty$.

Now we consider the case that $I_\mu < \infty$. Assume that (ii′) holds. As above, for a bounded sequence in $\mathcal{B}^{\mu_1}$, we have subsequence $f_k \in \mathcal{B}^{\mu_1}$ and an $f \in \mathcal{B}^{\mu_1}$ such that $g_k = f_k - f \to 0$ locally uniformly in the unit disk, $\|f_k\|_{\mathcal{B}^{\mu_1}} \leq 1$ and (7.7) holds. Let $\epsilon > 0$ be given. By Lemma 7.3 and the assumption (ii′), there exists an $r_0 \geq 1/2$ such that $C_{\mu_1,r_0} < \epsilon$, where C_{μ_1,r_0} is the number in Lemma 7.3, and

$$\frac{\mu_2(1-|z|^2)|\langle\phi'(z)z, \phi(z)\rangle|}{\mu_1(1-|\phi(z)|^2)} < \epsilon \quad \text{if } |\phi(z)| > r_0. \tag{7.13}$$

Since $g_k(w) \to 0$ uniformly on $|w| \leq r_0$, by (3.3), there exists a K such that

$$|\nabla g_k(w)| < \epsilon \quad \text{for } k > K,\ |w| \leq r_0, \tag{7.14}$$

and

$$\frac{|\nabla g_k(w)v|}{F^{\mu_1}_w(v)} < \epsilon \quad \text{for } k > K,\ |w| \leq r_0,\ 0 \neq v \in \mathbb{C}^n.$$

Let $k > K$ and $z \in B^n$. By the same reasoning as in the case $I_\mu = \infty$, we have

$$\mu_2(1-|z|^2)|\mathcal{R}C_\phi(g_k)(z)z| < M\epsilon$$

if $\phi'(z)z = 0$ or $\phi'(z)z \neq 0$ and $|\phi(z)| \leq r_0$. In the case $\phi'(z)z \neq 0$ and $|\phi(z)| > r_0$, let $\phi'(z)z = u_1\phi(z)/|\phi(z)| + u_2\zeta$ with $\zeta \perp \phi(z)$ and $|\zeta| = 1$. Then $u_1 = \langle\phi'(z)z, \phi(z)/|\phi(z)|\rangle$, $u_2 = \langle\phi'(z)z, \zeta\rangle$, and we have

$$\begin{aligned}|\mathcal{R}C_\phi(g_k)(z)z| &= |\nabla g_k(\phi(z))\phi'(z)z| \\ &= |\nabla g_k(\phi(z))(\langle\phi'(z)z, \phi(z)/|\phi(z)|\rangle\phi(z)/|\phi(z)| + \langle\phi'(z)z, \zeta\rangle\zeta)| \\ &\leq 4|\langle\phi'(z)z, \phi(z)\rangle||\nabla g_k(\phi(z))\phi(z)| + |\langle\phi'(z)z, \zeta\rangle||\nabla g_k(\phi(z))\zeta|\end{aligned}$$

and

$$\begin{aligned}\mu_2(1-|z|^2)|\mathcal{R}C_\phi(g_k)(z)z| \leq{}& \mu_2(1-|z|^2)|\langle\phi'(z)z, \zeta\rangle||\nabla g_k(\phi(z))\zeta| \\ &+ 4\mu_1(1-|\phi(z)|^2)|\nabla g_k(\phi(z))\phi(z)| \cdot \frac{\mu_2(1-|z|^2)|\langle\phi'(z)z, \phi(z)\rangle|}{\mu_1(1-|\phi(z)|^2)}.\end{aligned} \tag{7.15}$$

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Estimating the right side of (7.15), we have, by (3.3) and (7.6),

$$\begin{aligned}(7.16)\quad \mu_2(1-|z|^2)|\langle\phi'(z)z,\zeta\rangle| &\le \mu_2(1-|z|^2)|\phi'(z)z|\\ &\le \mu_1(1)\mu_2(1-|z|^2)F^{\mu_1}_{\phi(z)}(\phi'(z)z) \le \mu_1(1)M,\end{aligned}$$

and, by Lemma 7.3 and (7.14),

$$(7.17)\qquad |\nabla g_k(\phi(z))\zeta| < \epsilon + C_{\mu_1,r_0}\|g_k\|_{\mu_1,1} \le \epsilon + \epsilon\|g_k\|_{\mathcal{B}^{\mu_1}} < 3\epsilon,$$

and, by (7.7) and the definition of F^{μ}_z,

$$\begin{aligned}(7.18)\qquad \mu_1(1-|\phi(z)|^2)|\nabla g_k(\phi(z))\phi(z)| &= \sqrt{\frac{n+1}{2}}\frac{|\phi(z)||\nabla g_k(\phi(z))\phi(z)|}{F^{\mu_1}_{\phi(z)}(\phi(z))}\\ &\le \sqrt{\frac{n+1}{2}}\|g_k\|_{\mu_1,3} \le C_{\mu_1}\|g_k\|_{\mu_1,1}\\ &\le C'_{\mu_1}\|g_k\|_{\mathcal{B}^{\mu_1}} \le 2C'_{\mu_1}.\end{aligned}$$

Thus, substituting in (7.15) by (7.16), (7.17), (7.18) and (7.13), we obtain

$$\mu_2(1-|z|^2)|\mathcal{R}C_\phi(g_k)(z)z| \le (3\mu_1(1)M + 8C'_{\mu_1})\epsilon.$$

Thus, $\|C_\phi(g_k)\|_{\mu_2,2} \to 0$ as $k\to\infty$. As above, this shows that $f_k\circ\phi \to f\circ\phi$ according to the $\mathcal{B}^{\mu_2}$ norm, and $C_\phi\colon \mathcal{B}^{\mu_1} \longrightarrow \mathcal{B}^{\mu_2}$ is compact. We have proved that (ii′) implies (i).

Now, assume that $C_\phi\colon \mathcal{B}^{\mu_1} \longrightarrow \mathcal{B}^{\mu_2}$ is compact. To prove (iii′), suppose on the contrary that there exist $\delta > 0$, sequences z_k and $u_k \neq 0$, such that $\phi(z_k) \to \partial B^n$ and

$$(7.19)\qquad \frac{|\langle\phi'(z_k)u_k,\phi(z_k)\rangle|}{F^{\mu_2}_{z_k}(u_k)\mu_1(1-|\phi(z_k)|^2)} > \delta,\quad \text{for } k=1,2,\dots.$$

For $k = 1,2,\dots$, let $w_k = \phi(z_k)$ and $f_k = f_{\mu_1,w_k}$ be the functions defined in Lemma 7.2. Then, as above, by choosing a subsequence, we may assume that $\|C_\phi(f_k)\|_{\mathcal{B}^{\mu_2}} \to 0$ as $k\to\infty$. In particular,

$$(7.20)\qquad \frac{|\nabla C_\phi(f_k)(z_k)u_k|}{F^{\mu_2}_{z_k}(u_k)} = \frac{|\nabla f_k(w_k)\phi'(z_k)u_k|}{F^{\mu_2}_{z_k}(u_k)} \to 0.$$

However, by (ii) in Lemma 7.2,

$$(7.21)\qquad \mu_1(1-|w_k|^2)\frac{|\nabla f_k(w_k)\phi'(z_k)u_k|}{|\langle\phi'(z_k)u_k,w_k\rangle|} > \frac{1}{C_{\mu_1}}\quad \text{for } k=1,2,\dots.$$

(7.19) and (7.21) contradict (7.20). This shows that (i) implies (iii′). ∎

If $v = v_1 w/|w| + v_2\zeta$ with $\zeta \perp w$ and $|\zeta| = 1$, then

$$F_w^\mu(v) = \sqrt{\frac{n+1}{2}}\Big(\frac{|v_1|^2}{\mu(1-|w|^2)^2} + \frac{|v_2|^2}{\nu_\mu(1-|w|^2)^2}\Big)^{1/2}$$
$$= \sqrt{\frac{n+1}{2}}\Big(\frac{|\langle v, w/|w|\rangle|^2}{\mu(1-|w|^2)^2} + \frac{|\langle v, \zeta\rangle|^2}{\nu_\mu(1-|w|^2)^2}\Big)^{1/2} \geq \sqrt{\frac{n+1}{2}}\frac{|\langle v, w\rangle|}{\mu(1-|w|^2)}.$$

This shows that the conditions (ii′) and (iii′) are weaker than (ii) and (iii) respectively.

If μ_1 and μ_2 satisfy the condition in Lemma 2.3 (then $I_{\mu_1} = I_{\mu_2} = \infty$), then condition (iii) in Theorems 6.1 and condition (iii) in Theorem 7.4 become

$$\sup\Big\{\frac{\mu(1-|z|^2)(1-|\phi(z)|^2)H_{\phi(z)}(\phi'(z)u, \phi'(z)u)}{\mu(1-|\phi(z)|^2)(1-|z|^2)H_z(u,u)} : z \in B^n\ 0 \neq u \in \mathbb{C}^n\Big\} < \infty$$

and

$$\frac{\mu(1-|z|^2)(1-|\phi(z)|^2)H_{\phi(z)}(\phi'(z)u, \phi'(z)u)}{\mu(1-|\phi(z)|^2)(1-|z|^2)H_z(u,u)} \to 0 \quad \text{as } \phi(z) \to \partial B^n,$$

respectively. These are the necessary and sufficient conditions established by Zhang and Xiao in [12].

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