Can. J. Math., Vol. XXIII, No. 3, 1971, pp. 495–502

HOMEOMORPHIC SETS OF REMOTE POINTS

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Let X be a completely regular Hausdorff space, and let βX denote the Stone-Čech compactification of X. A point $p \in \beta X$ is called a *remote point* of βX if p does not belong to the βX-closure of any discrete subspace of X. Remote points were first defined and studied by Fine and Gillman, who proved that if the continuum hypothesis is assumed then the set of remote points of $\beta\mathbf{R}(\beta\mathbf{Q})$ is dense in $\beta\mathbf{R} - \mathbf{R}\,(\beta\mathbf{Q} - \mathbf{Q})$ ($\mathbf{R}$ denotes the space of reals, $\mathbf{Q}$ the space of rationals). Assuming the continuum hypothesis, Plank has proved that if X is a locally compact, non-compact, separable metric space without isolated points, then βX has a set of remote points that is dense in $\beta X - X$. Robinson has extended this result by dropping the assumption that X is separable. Let δX denote the smallest cardinal m with the property that X has a dense subset of cardinality m. In this note it is proved that if X and Y are locally compact, non-compact metric spaces without isolated points, and if $\delta X = \delta Y$, then the set of remote points of βX is homeomorphic to the set of remote points of βY.

1. Preliminaries. Throughout this paper we shall use the notation and terminology of Gillman and Jerison [**4**]. In particular, the cardinality of a set S will be denoted by $|S|$, and the set of positive integers will be denoted by N. In this section we record some known results that we shall need later.

1.1. THEOREM. *Let X be a locally compact, non-compact metric space. Then either:*

(i) $\delta X = \aleph_0$ *and X is σ-compact, or:*

(ii) $\delta X > \aleph_0$ *and X is the free union of precisely δX locally compact, σ-compact, non-compact metric spaces.*

Proof. A. H. Stone has proved that every metric space is paracompact (see, for example, [**1**, Theorem 9.5.3]. It is well-known (see, for example, [**1**, Theorem 11.7.3] that every locally compact paracompact space is the free union of a collection of locally compact σ-compact spaces. Suppose that there are m spaces in this collection. If $m \leqq \aleph_0$, then X is σ-compact, and the fact that a compact metric space is separable implies that $\delta X = \aleph_0$. Suppose that $m > \aleph_0$. Then $m \leqq \delta X$ as any dense subset of X must include at least one point from each member of the collection. Conversely, as each locally compact σ-compact metric space is separable, X contains a dense set of cardinality $m \cdot \aleph_0 = m$. Thus $\delta X \leqq m$ and so $\delta X = m$.

Received September 29, 1970 and in revised form February 11, 1971.

https://doi.org/10.4153/CJM-1971-052-1 Published online by Cambridge University Press

Recall that a closed subset of a space X is called *regular closed* if it is the closure of some open subset of X.

1.2. THEOREM [**12**, § 20*C*]. *The family* $\mathrm{R}(X)$ *of all regular closed subsets of* X *is a complete Boolean algebra under the following operations:*

(i) $A \leqq B$ if and only if $A \subseteq B$

(ii) $\vee_\alpha A_\alpha = \mathrm{cl}_X[\bigcup_\alpha A_\alpha]$

(iii) $\wedge_\alpha A_\alpha = \mathrm{cl}_X[\mathrm{int}_X \bigcap_\alpha A_\alpha]$

(iv) $A' = \mathrm{cl}_X(X - A)$ (A' denotes the complement of A).

The following result is a well-known theorem of Marshall Stone (see, for example, [**12**, 8.2].

1.3. THEOREM. *Let* U *be a Boolean algebra, and let* $\mathrm{S}(U)$ *be the set of all ultrafilters on* U. *For each* $x \in U$ *put* $\lambda(x) = \{\alpha \in \mathrm{S}(U): x \in \alpha\}$. *If a topology* τ *is assigned to* $\mathrm{S}(U)$ *by letting* $\{\lambda(x): x \in U\}$ *be an open base for* τ, *then* $(\mathrm{S}(U), \tau)$ *is a compact Hausdorff totally disconnected space and the map* $x \rightarrow \lambda(x)$ *is a Boolean algebra isomorphism from* U *onto the Boolean algebra of open-and-closed subsets of* $\mathrm{S}(U)$.

The space $\mathrm{S}(U)$ is called the *Stone space* of U.

Recall that a continuous map f from a space X onto a space Y is said to be *irreducible* if the image under f of each proper closed subset of X is a proper closed subset of Y. The following result is Theorem 2.18 of [**13**].

1.4. THEOREM. *Let* X *be a compact Hausdorff space and let* $\mathscr{U}$ *be a subalgebra of* $\mathrm{R}(X)$ *that is also a basis for the closed subsets of* X. *Then the map* $f:\mathrm{S}(\mathscr{U}) \rightarrow X$ *given by*

$$f(\alpha) = \bigcap \{A \in \mathscr{U}: \alpha \in \lambda(A)\} \quad (\alpha \in \mathrm{S}(\mathscr{U}))$$

is a well-defined irreducible continuous map from $\mathrm{S}(\mathscr{U})$ *onto* X (λ *is as defined in* 1.3).

The proof of 1.4 is essentially the same as the proof of Theorem 3.2 of [**5**].

As stated above, a point $p \in \beta X$ is a remote point of βX if p is not in the βX-closure of any discrete subspace of X. In [**3**] Fine and Gillman, assuming the continuum hypothesis, demonstrated the existence of a set of remote points of $\beta\mathbf{R}$ that is dense in $\beta\mathbf{R} - \mathbf{R}$ ($\mathbf{R}$ denotes the real line). Let $\mathrm{T}(\beta X)$ denote the set of remote points of βX. The following result comprises a portion of Theorems 5.3 and 5.4 of [**9**].

1.5. THEOREM. *Let* X *be a metric space without isolated points. Then*

$$\mathrm{T}(\beta X) = \bigcap \{(\beta X - X) - \mathrm{cl}_{\beta X} A: A \text{ is closed and nowhere dense in } X\}.$$

If in addition X *is locally compact,* σ*-compact, and non-compact, and if the continuum hypothesis is assumed (i.e.* $\aleph_1 = 2^{\aleph_0}$*), then* $\mathrm{T}(\beta X)$ *has cardinality* $2^{\aleph_1}$ *and is dense in* $\beta X - X$.

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Robinson [**10**] extended Plank's results to show that if the continuum hypothesis is assumed, and if X is a locally compact non-compact metric space without isolated points, then $T(\beta X)$ is dense in $\beta X - X$. We shall not use this result, but it does give us the assurance that $T(\beta X)$ is non-empty when $\delta X > \aleph_0$.

1.6. LEMMA. *Let f be an irreducible mapping from Y onto X. If S is dense in X, then $f^{\leftarrow}[S]$ is dense in Y.*

Proof. If $f^{\leftarrow}[S]$ were not dense in Y, then $\text{cl}_Y f^{\leftarrow}[S]$ would be a proper closed subset of Y. As f is irreducible, $f[\text{cl}_Y f^{\leftarrow}[S]]$ would be a proper closed subset of X containing the dense set S, which is impossible.

2. The Main Results. In this section we prove the theorem quoted in the last sentence of the first paragraph of this paper. We proceed as follows: if X is a locally compact, non-compact metric space without isolated points, we let Y be the free union of δX copies of the Cantor set and construct an irreducible mapping f from Y onto X. The Stone extension of f, namely, f^β, takes βY onto βX and we show that f^β maps the remote points of βY homeomorphically onto the remote points of βX.

2.1. LEMMA. *Let K be a compact metric space without isolated points. Then there exists an irreducible map f from the Cantor set C onto K with the following property: If D is a discrete subspace of C, then there exists a discrete subspace F of K such that $f[D] \subseteq \text{cl}_K F$.*

Proof. As K is a compact metric space it has a countable basis $\mathscr{D}$ of closed subsets. As K is a regular Hausdorff space, the family $\{\text{cl}_K(\text{int}_K B): B \in \mathscr{D}\} = \mathscr{D}^*$ is also a countable basis for the closed subsets of K. Let $\mathscr{A}$ be the subalgebra of $R(K)$ generated by $\mathscr{D}^*$ (see [**12**, 1.3 and § 4]. Then $|\mathscr{A}| = \aleph_0$ since $|\mathscr{D}| = \aleph_0$. Hence $S(\mathscr{A})$ is a compact Hausdorff space with a countable basis, so $S(\mathscr{A})$ is a compact totally disconnected metric space. Since K has no isolated points, $\mathscr{A}$ has no atoms (see [**12**, § 9] and so $S(\mathscr{A})$ has no isolated points. But any compact totally disconnected metric space without isolated points is homeomorphic to the Cantor set C (see [**6**, 2.97]); hence $S(\mathscr{A})$ and C are homeomorphic. Hence the irreducible map f defined in 1.4 takes C onto K.

Let D be a discrete subspace of C. Since C has a countable basis, $|D| \leqq \aleph_0$. Put $D = (d_n)_{n \in N}$. In the notation of 1.3, for each $n \in N$ there exists $A(n) \in \mathscr{A}$ such that $\lambda(A(n)) \cap D = \{d_n\}$. By replacing each $\lambda(A(n))$ by $\lambda(A(n)) - \bigcup_{j<n} \lambda(A(j))$ if necessary, we may assume that $i \neq j$ implies $\lambda(A(i)) \cap \lambda(A(j)) = \emptyset$. Put $H = K - \bigcup_{A \in \mathscr{A}} \text{bd}_K A$ ($\text{bd}_K A$ denotes the topological boundary of A in K). By the Baire category theorem H is dense in K, and so, by 1.6, $f^{\leftarrow}[H]$ is dense in C. It is now easy to see that for each $n \in N$ we can find a subset $E(n)$ of $\lambda(A(n)) \cap f^{\leftarrow}[H]$ such that $E(n)$ is a discrete subspace of C and $d_n \in \text{cl}_C E(n)$. Put $E = \bigcup_{n \in N} E(n)$. Then E is

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a discrete subspace of C and $D \subseteq \text{cl}_C E$. Since E is discrete it is countable; put $E = (x_n)_{n \in N}$. As above, there exists $\{B(n): n \in N\} \subseteq \mathscr{A}$ such that $x_n \in \lambda(B(n))$ and $i \neq j$ implies $\lambda(B(i)) \cap \lambda(B(j)) = \emptyset$, which in turn implies

$$\text{int}_K B(i) \cap \text{int}_K B(j) = \emptyset.$$

It follows from the definition of f that $f(x_n) \in B(n) \cap H \subseteq \text{int}_K B(n)$, so $f[E]$ is a discrete subspace of K. Evidently $f[D] \subseteq f[\text{cl}_C E] = \text{cl}_K f[E]$, so $f[E]$ is the set F whose existence was claimed.

2.2. Lemma. *Let X be a locally compact, σ-compact non-compact Hausdorff space without isolated points. Then there exists a sequence $\{K(n): n \in N\}$ of compact regular closed subsets of X with the following properties:*

(i) $X = \bigcup_{n \in N} K(n)$.

(ii) *Each $K(n)$ has no isolated points.*

(iii) $K(n) \cap K(m) \neq \emptyset$ *implies* $|m - n| \leqq 1$.

(iv) *For each* $n \in N$, $\text{bd}_X K(n) = K(n) \cap [K(n-1) \cup K(n+1)]$.

Proof. It is known (see [**1**, Theorem 11.7.2]) that any locally compact, σ-compact Hausdorff space X can be written in the form $X = \bigcup_{n \in N} V(n)$ where for each $n \in N$, $V(n)$ is open in X, $\text{cl}_X V(n)$ is compact, and

$$\text{cl}_X V(n) \subseteq V(n+1).$$

Since X is non-compact, we may assume this last inclusion to be proper. Without loss of generality we may assume that each $V(n)$ is regular open (i.e. the interior of some closed set), for if we let $U(n) = \text{int}_X \text{cl}_X V(n)$, then the family $\{U(n): n \in N\}$ has the same properties as those listed above for the family $\{V(n): n \in N\}$. Define $V(0)$ to be the empty set, and put $K(n) = \text{cl}_X V(n) - V(n-1)$ for each $n \in N$. Obviously each $K(n)$ is compact, and a straightforward argument shows that $K(n) = \text{cl}_X[V(n) - \text{cl}_X V(n-1)]$. Hence each $K(n)$ is regular closed, and $\text{int}_X K(n) = V(n) - \text{cl}_X V(n-1)$, since this latter set is the intersection of two regular open sets and hence is regular open. Assertion (i) is obviously true. If p were an isolated point of $K(n)$, then there would exist W, open in X, such that $W \cap \text{int}_X K(n) = \{p\}$; this contradicts the assumption that X has no isolated points. Hence (ii) is true. To prove (iii), without loss of generality, suppose that $m \leqq n - 2$. Then $K(m) \subseteq \text{cl}_X V(m) \subseteq V(m+1) \subseteq V(n-1)$, so $K(m) \cap K(n) = \emptyset$. Finally,

$$\begin{aligned} \text{bd}_X K(n) &= K(n) - \text{int}_X K(n) \\ &= [\text{cl}_X V(n) - V(n-1)] - [V(n) - \text{cl}_X V(n-1)] \\ &= [\text{cl}_X V(n) - V(n)] \cup [\text{cl}_X V(n-1) - V(n-1)] \\ &= [K(n) \cap K(n+1)] \cup [K(n) \cap K(n-1)] \end{aligned}$$

and (iv) is verified.

The following result appears as Lemma 2.1 of [**13**].

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2.3. LEMMA. *Let X be a locally compact, σ-compact, non-compact Hausdorff space and let $\{A(n): n \in N\}$ be a countable family of closed subsets of X. For each $n \in N$, define $k(n) \in N$ as follows:*

$$k(n) = \min \{j \in N: A(n) \cap V(j) \neq \emptyset\}$$

($V(j)$ is as defined at the beginning of the proof of 2.2). If $\lim_{n\to\infty} k(n) = \infty$, then $\bigcup_{n\in N} A(n)$ is closed in X.

2.4. LEMMA. *Let X be a locally compact, non-compact metric space without isolated points, and let Y be the free union of δX copies of the Cantor set. Then there exists an irreducible perfect map from Y onto X.*

Proof. First assume that X is σ-compact; then $\delta X = \aleph_0$. Write

$$X = \bigcup_{n\in N} K(n),$$

where the collection $\{K(n): n \in N\}$ has the properties described in 2.2. For each $n \in N$, we can, by 2.1 and 2.2 (ii), find a copy $C(n)$ of the Cantor set and an irreducible map f_n from $C(n)$ onto $K(n)$. Let Y be the free union of these $\aleph_0$ copies of the Cantor set, and define $f: Y \to X$ by requiring that $f|_{C(n)} = f_n$. Evidently f is a well-defined map from Y onto X, and as each f_n is continuous and each $C(n)$ is open in Y, f is continuous. Let A be closed in Y. Then

$$f[A] = f[\bigcup_{n\in N} (A \cap C(n))] = \bigcup_{n\in N} f[A \cap C(n)],$$

and $f[A \cap C(n)]$ is a compact subset of X contained in $X - V(n-1)$. Thus by 2.3, $f[A]$ is closed in X and f is a closed mapping. If $p \in X$, by 2.2 (iii) there exists $n \in N$ such that $n \neq k \neq n+1$ implies $p \notin K(k)$. Thus $f^{\leftarrow}(p) \subseteq C(n) \cup C(n+1)$, and hence $f^{\leftarrow}(p)$ is compact. Consequently f is a perfect mapping. To prove that f is irreducible, note that if A is a proper closed subset of Y, then there exists $n \in N$ such that $A \cap C(n)$ is a proper closed subset of $C(n)$. As f_n is irreducible, $f_n[A \cap C(n)]$ is a proper closed subset of $K(n)$. Thus there exists W open in X such that

$$W \cap K(n) = K(n) - f[A \cap C(n)] \neq \emptyset.$$

Hence $W \cap [V(n) - \text{cl}_X V(n-1)] \neq \emptyset$. If $k \neq n$, then

$$K(k) \cap [V(n) - \text{cl}_X V(n-1)] = \emptyset;$$

thus $X - f[A] \supseteq W \cap [V(n) - \text{cl}_X V(n-1)] \neq \emptyset$ and f is irreducible.

Now suppose that X is not σ-compact. By 1.1, X is the free union of δX locally compact, σ-compact non-compact spaces-say $X = \dot{\bigcup}_{\alpha\in\Sigma} X(\alpha)$, where $|\Sigma| = \delta X$ and each $X(\alpha)$ is locally compact, σ-compact, and non-compact. For each $\alpha \in \Sigma$, let $Y(\alpha)$ be the free union of $\aleph_0$ copies of the Cantor set. The preceding argument shows that there exists an irreducible perfect map f_α from $Y(\alpha)$ onto $X(\alpha)$. Let $Y = \dot{\bigcup}_{\alpha\in\Sigma} Y(\alpha)$ and define $f: Y \to X$ by $f|_{Y(\alpha)} = f_\alpha$.

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Then Y is a free union of δX copies of the Cantor set, and f is an irreducible perfect map because each f_α is.

2.5. LEMMA. *Let X be a locally compact, non-compact metric space without isolated points, let Y be the free union of δX copies of the Cantor set, and let $f: Y \to X$ be the irreducible perfect map constructed in* 2.4. *If $f^\beta: \beta Y \to \beta X$ is the Stone extension of f* (see [**4**, 6.5]), *then* $f^\beta[\mathrm{T}(\beta Y)] = \mathrm{T}(\beta X)$.

Proof. Suppose that $p \in \beta Y$ and $f^\beta(p) \notin \mathrm{T}(\beta X)$. By 1.5 there exists a closed subset A of X such that $\mathrm{int}_X A = \emptyset$ and $f^\beta(p) \in \mathrm{cl}_{\beta X} A$. As $X - A$ is dense in X and f is irreducible, by 1.6, $f^\leftarrow[X - A] = Y - f^\leftarrow[A]$ is dense in Y. Thus $\mathrm{int}_Y f^\leftarrow[A] = \emptyset$. Evidently $p \in (f^\beta)^\leftarrow[\mathrm{cl}_{\beta X} A]$. It follows that $p \in \mathrm{cl}_{\beta Y} f^\leftarrow[A]$; to prove this we adapt the argument used by Isiwata in Lemma 1.2 of [**7**]. Suppose that $p \notin \mathrm{cl}_{\beta Y} f^\leftarrow[A]$. Then there exists $g \in C(\beta Y)$ such that $g(p) = 0$ and $g[\mathrm{cl}_{\beta Y} f^\leftarrow[A]] = \{1\}$. Put $M = Y \cap g^\leftarrow[-\frac{1}{2}, \frac{1}{2}]$; this is a zero-set of Y. Obviously $M \cap f^\leftarrow[A] = \emptyset$, so $f[M] \cap A = \emptyset$. Since f is a closed map, $f[M]$ is closed in X. As X is metric, it follows that

$$\mathrm{cl}_{\beta X} f[M] \cap \mathrm{cl}_{\beta X} A = \emptyset$$

(see [**4**, 6.5 IV]). Now $g(p) = 0$, so $p \in \mathrm{cl}_{\beta Y} M$. Thus

$$f^\beta(p) \in f^\beta[\mathrm{cl}_{\beta Y} M] = \mathrm{cl}_{\beta X} f^\beta[M] = \mathrm{cl}_{\beta X} f[M].$$

Hence $f^\beta(p) \notin \mathrm{cl}_{\beta X} A$, which contradicts the hypothesis. We conclude that $p \in \mathrm{cl}_{\beta Y} f^\leftarrow[A]$. Since Y, being a free union of compact metric spaces, is itself a metric space, it follows from 1.5 and the fact that $\mathrm{int}_Y f^\leftarrow[A] = \emptyset$ that $p \notin \mathrm{T}(\beta Y)$. Thus $p \in \mathrm{T}(\beta Y)$ implies $f^\beta(p) \in \mathrm{T}(\beta X)$ and $f^\beta[\mathrm{T}(\beta Y)] \subseteq \mathrm{T}(\beta X)$.

Conversely, suppose that $p \notin \mathrm{T}(\beta Y)$. First let us assume that X is σ-compact, and write $X = \bigcup_{n \in N} K(n)$ as in 2.2. Write $Y = \bigcup_{n \in N} C(n)$, where each $C(n)$ is a copy of the Cantor set. There exists a discrete subspace D of Y such that $p \in \mathrm{cl}_{\beta Y} D$. Put $D(n) = D \cap C(n)$. It follows from 2.1 and 2.4 that for each $n \in N$, there exists a discrete subset $E(n)$ of $K(n)$ such that $f[D(n)] \subseteq \mathrm{cl}_{K(n)} E(n)$. Using 2.2 (iii) and 2.2 (iv), we see that

$$\bigcup_{n \in N} [E(n) \cap \mathrm{int}_X K(n)]$$

is a discrete subspace F of X. Now 2.2 (iv), 2.3, and the Baire category theorem imply that $G = \bigcup_{n \in N} [K(n) \cap K(n+1)]$ is a closed nowhere dense subset of X. It follows from 2.2 (iv) that $\bigcup_{n \in N} E(n) \subseteq F \cup G$. Hence

$$f[D] \subseteq \bigcup_{n \in N} \mathrm{cl}_{K(n)} E(n) \subseteq \mathrm{cl}_X[\bigcup_{n \in N} E(n)] \subseteq G \cup \mathrm{cl}_X F.$$

Thus

$$f^\beta(p) \in f^\beta[\mathrm{cl}_{\beta Y} D] = \mathrm{cl}_{\beta X} f[D] \subseteq \mathrm{cl}_{\beta X} G \cup \mathrm{cl}_{\beta X} F \subseteq \beta X - \mathrm{T}(\beta X).$$

This, combined with our previous result, shows that $f^\beta[\mathrm{T}(\beta Y)] = \mathrm{T}(\beta X)$.

If X is not σ-compact, then it is the free union of δX locally compact, σ-compact subspaces. It follows from the preceding paragraph that if D is a

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discrete subspace of Y, then $f[D]$ is contained in a free union of discrete subspaces of X, together with a free union of closed nowhere dense subspaces of X. This free union of discrete (closed nowhere dense) subspaces of X will be discrete (closed nowhere dense), and our result follows.

2.6. THEOREM. *Let X and Y be two locally compact, non-compact metric spaces without isolated points. If $\delta X = \delta Y$ then $\mathrm{T}(\beta X)$ and $\mathrm{T}(\beta Y)$ are homeomorphic.*

Proof. It clearly will suffice to show that, for any locally compact non-compact metric space X without isolated points, $\mathrm{T}(\beta X)$ and $\mathrm{T}(\beta Y)$ are homeomorphic, where Y is the free union of δX copies of the Cantor set. Consider the map $f^\beta : \beta Y \to \beta X$ constructed in 2.4. Since, by 2.5, $f^\beta[\mathrm{T}(\beta Y)] = \mathrm{T}(\beta X)$, all we need to show is that the restriction of f^β to $\mathrm{T}(\beta Y)$ is one-to-one and closed. If A is closed in βY, by 2.5, $f^\beta[A \cap \mathrm{T}(\beta Y)] = f^\beta[A] \cap \mathrm{T}(\beta X)$, which is closed in $\mathrm{T}(\beta X)$ as f^β is a closed map. Hence $f^\beta|_{T(\beta Y)}$ is closed. To show that f^β is one-to-one on $\mathrm{T}(\beta Y)$, suppose that $p \in \beta X$ and that q and s are distinct points of βY such that $f^\beta(s) = f^\beta(q) = p$. As Y is a free union of compact spaces with bases of open-and-closed sets, it follows from 16.17 of [**4**] that βY has a basis of open-and-closed sets. Hence we can find an open-and-closed subset A of βY such that $q \in A$ and $s \in \beta Y - A$. Put $B = A \cap Y$. Then $q \in \mathrm{cl}_{\beta Y} B$ and $s \in \mathrm{cl}_{\beta Y}(Y - B)$. Hence

$$\begin{aligned} p \in f^\beta[\mathrm{cl}_{\beta Y} B] \cap f^\beta[\mathrm{cl}_{\beta Y}(Y - B)] &= \mathrm{cl}_{\beta X} f[B] \cap \mathrm{cl}_{\beta X} f[Y - B] \\ &= \mathrm{cl}_{\beta X}[f[B] \cap f[Y - B]]; \end{aligned}$$

the last equality follows since f is a closed map and X is metric. Again let us momentarily assume that X is σ-compact, and employ the notation used in the proof of 2.5. By 2.4, 2.1, and 1.4, f takes complementary open-and-closed subsets of $C(n)$ onto complementary regular closed sets of $K(n)$. This implies that for each $n \in N$, $f[B \cap C(n)] \cap f[C(n) - B]$ is a closed nowhere dense subset of $K(n)$. It follows from 2.2 that $f[B] \cap f[Y - B]$ is contained in

$$\bigcup_{n \in N} ([K(n) \cap K(n+1)] \cup [f[B \cap C(n)] \cap f[C(n) - B]]),$$

which by 2.3 and the Baire category theorem is a closed nowhere dense subset of X. Thus $p \notin \mathrm{T}(\beta X)$, and consequently $f^\beta|_{\mathrm{T}(\beta Y)}$ is a one-to-one, closed, continuous map from $\mathrm{T}(\beta Y)$ onto $\mathrm{T}(\beta X)$. Hence $\mathrm{T}(\beta Y)$ and $\mathrm{T}(\beta X)$ are homeomorphic.

If X is not σ-compact, we can in the usual way write X as a free union of locally compact, σ-compact spaces and employ the results of the preceding paragraph to obtain the desired result.

2.7. COROLLARY. *Assume the continuum hypothesis. If X and Y are two locally compact, non-compact metric spaces without isolated points, and if $\delta X = \delta Y$, then $\mathrm{R}(\beta X - X)$ and $\mathrm{R}(\beta Y - Y)$ are homeomorphic.*

Proof. According to Robinson's results quoted at the end of **1**, and using 2.6, we see that $\beta X - X$ and $\beta Y - Y$ contain homeomorphic dense subsets.

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The corollary now follows from the fact that if S is a dense subspace of the (completely regular Hausdorff) space T, then the map $A \to A \cap S$ is a Boolean algebra isomorphism from $\mathrm{R}(T)$ onto $\mathrm{R}(S)$.

The next corollary appears as Theorem 4.3 of [**13**].

2.8. COROLLARY. *Assume the continuum hypothesis. If X is a locally compact, separable, non-compact metric space without isolated points, then* $\mathrm{T}(\beta X)$ *is homeomorphic to a dense subset of* $\beta\mathbf{N} - \mathbf{N}$ ($\mathbf{N}$ *is the countable discrete space*).

Proof. By 2.6, $\mathrm{T}(\beta X)$ and $\mathrm{T}(\beta Y)$ are homeomorphic, where Y is the free union of $\aleph_0$ copies of the Cantor set. By 1.5, $\mathrm{T}(\beta Y)$ is dense in $\beta Y - Y$. But, by 14.27 of [**4**], $\beta Y - Y$ is a compact F-space, and, by 3.1 of [**2**], the zero-sets of $\beta Y - Y$ are regular closed. Evidently $\beta Y - Y$ is totally disconnected (see [**4**, 16.11 and 16.17]) and has $2^{\aleph_0}$ open-and-closed subsets. According to a theorem due to Rudin [**11**] and Parovičenko [**8**], on the assumption of the continuum hypothesis this implies that $\beta Y - Y$ is homeomorphic to $\beta\mathbf{N} - \mathbf{N}$.

We conclude with a question. Is it possible to characterize $\mathrm{T}(\beta X)$ (where X is as in 2.8) "internally" as a subset of $\beta\mathbf{N} - \mathbf{N}$, i.e., in terms of the topology of $\beta\mathbf{N} - \mathbf{N}$ and without reference to other spaces?

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https://doi.org/10.4153/CJM-1971-052-1 Published online by Cambridge University Press