Compositio Mathematica **106:** 61–70, 1997.

Stability of Frobenius pull-backs of tangent bundles and generic injectivity of Gauss maps in positive characteristic

ATSUSHI NOMA
Department of Mathematics, Faculty of Education, Yokohama National University, 79-2 Tokiwadai, Hodogaya-ku, Yokohama, 240, Japan; e-mail:nom@ms.ed.ynu.ac.jp

Received 4 July 1995; accepted in final form 22 December 1995

Abstract. For a smooth projective variety X of dimension n in a projective space $\mathbb{P}^N$ defined over an algebraically closed field k, the Gauss map is a morphism from X to the Grassmannian of n-plans in $\mathbb{P}^N$ sending $x \in X$ to the embedded tangent space $T_xX \subset \mathbb{P}^N$. The purpose of this paper is to prove the generic injectivity of Gauss maps in positive characteristic for two cases; (1) weighted complete intersections of dimension $n \geqslant 3$ of general type; (2) surfaces or 3-folds with μ-semistable tangent bundles; based on a criterion of Kaji by looking at the stability of Frobenius pull-backs of their tangent bundles. The first result implies that a conjecture of Kleiman–Piene is true in case X is of general type of dimension $n \geqslant 3$. The second result is a generalization of the injectivity for curves.

Mathematics Subject Classifications (1991): Primary 14N05, 14F10; Secondary 14M10, 14J29, 14F05, 14J60

Key words: Gauss map, tangent bundle, stable vector bundle

1. Introduction

Let X be a smooth projective variety of dimension n over an algebraically closed field k of characteristic $p > 0$. Let $\iota : X \to \mathbb{P}^N$ be an embedding whose image is not linear. Set $H = \iota^*\mathcal{O}_{\mathbb{P}^N}(1)$ and $L = \Omega_X^n \otimes H^{\otimes(n+1)}$, where Ω_X^q denotes the sheaf of differential q-forms. Recall that the Gauss map of ι is a morphism $\iota^{(1)} : X \to$ **Grass**$(\mathbb{P}^N, n)$ sending $x \in X$ to the n-dimensional embedded tangent space $T_xX \subset \mathbb{P}^N$.

A result of Zak [15, Ch. I, (2.8)] asserts the finiteness of the Gauss map $\iota^{(1)}$, equivalently the ampleness of L, in arbitrary characteristic, and the birationality of $\iota^{(1)}$ in characteristic 0. In positive characteristic, the birationality and even the generic injectivity of $\iota^{(1)}$ are no longer true in general. But several results suggest that the Gauss map would be generically injective for most cases; in other words, the field extension $K(X)$ over $K(\iota^{(1)}(X))$ would be purely inseparable ([6, 7, 8, 10, 12]).

The purpose of this paper is to prove the generic injectivity of Gauss maps in positive characteristic for two cases, by giving a criterion for the injectivity in terms of the stability of tangent bundles and by looking at the stability.

Our main results are the following:

THEOREM 1.1. *Let X be a smooth weighted complete intersection defined over k. Suppose that X is of* $\dim X = n \geqslant 3$ *and of general type. Then the Gauss maps $\iota^{(1)}$ are generically injective for any embeddings $\iota\colon X \to \mathbb{P}^N$.*

THEOREM 1.2. *Let X be a smooth projective variety of dimension $n = 2$ or 3 over k and $\iota\colon X \to \mathbb{P}^N$ an embedding. Set $H = \iota^*\mathcal{O}_{\mathbb{P}}(1)$ and $L = \Omega_X^n \otimes H^{\otimes(n+1)}$. Let $\mathcal{T}$ be the first piece of the Harder–Narasimhan filtration of the tangent bundle T_X with respect to L (see Section 3). Assume that $(c_1(\mathcal{T}), L^{n-1}) < 0$. When $n = 3$, we assume in addition that $\mathcal{T}$ is not of rank 2. Then the Gauss map $\iota^{(1)}$ is generically injective.*

Here ( ) denotes the intersection products of line bundles.

Theorem 1.1 implies that a conjecture of Kleiman–Piene [10], the generic injectivity of the Gauss map of a smooth complete intersection X for the natural embedding, is true when X is of dimension $n \geqslant 3$ and of general type. Our result is applicable not only to other projective variety than a 'usual' complete intersection but also to a complete intersection with any embedding in a projective space. Theorem 1.2 is one of a generalization of results for curves [6, 7] and [10]; roughly speaking, those assert that Gauss maps of smooth curves of genus $g \geqslant 2$ are always generically injective for any embeddings. In fact, Theorem 1.2 implies that the Gauss map $\iota^{(1)}$ of a smooth surface or 3-fold of general type with μ-stable tangent bundle with respect to L is generically injective.

To obtain these results, we essentially use Kaji's criterion for the generic injectivity of Gauss maps given in [8] (see (2.1)). By using Kaji's criterion, first we prove our key criterion via stability: Namely, if every eth Frobenius pull-back of the tangent bundle has no subsheaf of non-negative μ-slope with respect to L, then the Gauss map $\iota^{(1)}$ is generically injective (Proposition 3.1). Next we look at the stability of the Frobenius pull-backs of the tangent bundles and prove the main theorems. In Section 4, we show that every Frobenius pull-back of the tangent bundle of a smooth weighted complete intersection is μ-stable if the intersection is of general type and of dimension $\geqslant 3$ (Proposition 4.2). In Section 5, we show that if the tangent bundle of a smooth surface or 3-fold has no subsheaf of non-negative μ-slope, then the same is true for every eth Frobenius pull-back of the tangent bundle, based on Shepherd–Barron's argument [14, (9.1.3.3)]. Consequently, we obtain the main theorems by the above criterion.

1.1. NOTATION

Unless otherwise mentioned, we work over an algebraically closed field k of characteristic $p > 0$ throughout. By a variety, we mean an irreducible and reduced algebraic scheme over k. By the eth Frobenius morphism of a variety X defined over k, we mean the induced morphism from the p^eth power map of the structure sheaf. For a torsion-free $\mathcal{O}_X$-module $\mathcal{E}$ on a variety X over k, by $\mathcal{E}^\vee$ we denote the dual $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$, and by $c_1(\mathcal{E})$ we mean $\det(\mathcal{E})^{\vee\vee}$.

2. Kaji's criterion for generic injectivity of Gauss maps

PROPOSITION 2.1 (Kaji [8], [9]). *Let X be a smooth projective variety of* $\dim X = n$ *over k and $\iota: X \to \mathbb{P}^N$ an embedding whose image is not linear. Set* $H = \iota^*\mathcal{O}_{\mathbb{P}}(1)$ *and* $L = \Omega_X^n \otimes H^{\otimes(n+1)}$*. Consider the following condition for X and ι, called* Kaji's condition (K):

$$H^0(Y, f^*(T_X \otimes H^\vee) \otimes \sigma^* f^* H) = 0$$

holds for any finite surjective morphism $f: Y \to X$ from a normal projective variety Y with a decomposition $f = g \circ h$, $h: Y \to Y'$ a finite, separable morphism to a normal projective variety Y' and $g: Y' \to X$ a finite, purely inseparable morphism, and for any k-automorphism $\sigma: Y \to Y$ of finite order with $\sigma^* f^* L = f^* L$.

If Kaji's condition (K) holds for X and ι, then the Gauss map $\iota^{(1)}$ is generically injective.

Remark 2.2. Kaji's criterion above is a key step in his proof of the main theorem in [8] and hence it is not stated explicitly in [8]. The criterion in the form above will be given in the forthcoming paper [9]. Historically, the prototype of the criterion and its proof were already announced in Kaji's seminar talk at Waseda University in July 1989.

3. Criterion for generic injectivity of Gauss maps via stability

First we recall notation and results about stability (see, for example, [13]). In general, let X be a normal projective variety of dimension n over k with an ample line bundle L. For a torsion-free $\mathcal{O}_X$-module $\mathcal{E}$ of rank r, we set $\mu_L(\mathcal{E}) = (c_1(\mathcal{E}), L^{n-1})/r$. We say that $\mathcal{E}$ is *μ-stable* (resp. *μ-semistable*) with respect to L if for every $\mathcal{O}_X$-submodule $\mathcal{F}(0 < \operatorname{rank} \mathcal{F} < r)$, we have $\mu_L(\mathcal{F}) <$ (resp. $\leqslant$) $\mu_L(\mathcal{E})$. For the Harder–Narasimhan filtration (or H.-N. filtration, for short) $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$ of a torsion-free $\mathcal{O}_X$-module $\mathcal{E}$ with respect to L (w.r.t. L) (i.e., $\mathcal{E}_i/\mathcal{E}_{i-1}$ are torsion-free μ-semistable with $\mu_L(\mathcal{E}_1/\mathcal{E}_0) > \cdots > \mu_L(\mathcal{E}_l/\mathcal{E}_{l-1})$), we set

$$\mu_{L-\max}(\mathcal{E}) = \mu_L(\mathcal{E}_1/\mathcal{E}_0) \quad \text{and} \quad \mu_{L-\min}(\mathcal{E}) = \mu_L(\mathcal{E}_l/\mathcal{E}_{l-1})$$

and by the *type* of $\mathcal{E}$ we mean a sequence of numbers $(\operatorname{rank}(\mathcal{E}_1/\mathcal{E}_0), \ldots, \operatorname{rank}(\mathcal{E}_l/\mathcal{E}_{l-1}))$. We sometimes call $\mathcal{E}_1$ the first piece of the H.-N. filtration of $\mathcal{E}$. With this notation, for torsion-free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{E}'$ with $\mu_{L-\min}(\mathcal{E}') > \mu_{L-\max}(\mathcal{E})$, we have $\mathrm{Hom}_X(\mathcal{E}', \mathcal{E}) = 0$.

Let X be a smooth projective variety of dim $X = n$ over k and $\iota: X \to \mathbb{P}^N$ an embedding whose image is not linear. Set $H = \iota^*\mathcal{O}_{\mathbb{P}}(1)$ and $L = \Omega_X^n \otimes H^{\otimes(n+1)}$. Note that L is ample by a corollary of Zak's finiteness theorem of $\iota^{(1)}$ [15, Ch. I, (2.14)].

PROPOSITION 3.1. *Let X, ι, H and L be as above. If $\mu_{L-\max}(F^{e*}T_X) < 0$ for every eth Frobenius morphism $F^e: X \to X$ $(e \geqslant 0)$, then the Gauss map $\iota^{(1)}$ is generically injective.*

Proof. By (2.1), we have only to show that the condition (K) holds. Assume to the contrary that the vanishing in (K) does not holds for some $f: Y \to X$ and $\sigma: Y \to Y$. Hence f^*T_X has a submodule $f^*H \otimes \sigma^* f^* H^\vee$ with $((f^*L)^{n-1}, f^*H \otimes \sigma^* f^* H^\vee) = 0$, and hence $\mu_{f^*L-\max}(f^*T_X) \geqslant 0$. Since h is finite and separable, by Gieseker [3, (1.1)], we have $\mu_{g^*L-\max}(g^*T_X) \geqslant 0$. By noting that g is purely inseparable, let $\pi: X \to Y'$ a morphism with $g \circ \pi = F^e$ for some $e \geqslant 0$. Since $F^{e*}L = L^{\otimes p^e}$ and since π is flat in codimension 1, we have

$$\begin{aligned} p^{e(n-1)}\mu_{L-\max}(F^{e*}T_X) &= \mu_{F^{e*}L-\max}(\pi^* g^* T_X) \\ &\geqslant (\deg \pi)\mu_{g^*L-\max}(g^*T_X) \geqslant 0. \end{aligned}$$

This contradicts to the assumption. □

From Proposition 3.1, we recover an improved version of Kaji [8] (see [9]). Recall that a vector bundle $\mathcal{F}$ on a normal projective variety X of dimension n with an ample and globally generated line bundle L is *generically ample* with respect to $L^{\otimes m_1}, \ldots, L^{\otimes m_{n-1}} (m_i > 0)$ if its restriction $\mathcal{F}|C$ is ample on C for a complete intersection scheme $C = D_1 \cap \cdots \cap D_{n-1}$ with $D_i \in |L^{\otimes m_i}|$ (see [9]).

COROLLARY 3.2 ([9]). *Let X, ι, H and L be as in* (3.1). *If the tangent bundle T_X is an $\mathcal{O}_X$-submodule of a vector bundle $\mathcal{F}$ whose dual $\mathcal{F}^\vee$ is generically ample with respect to $L^{\otimes m_1}, \ldots, L^{\otimes m_{n-1}}$ for some $m_i > 0$, then the Gauss map $\iota^{(1)}$ is generically injective.*

Proof. Let $\mathcal{E}$ be the first piece of the H.-N. filtration of $F^{e*}\mathcal{F}$. If $D_i \in |L^{\otimes m_i}|$ are general, then $\mathcal{F}^\vee|C$ is ample on $C = D_1 \cap \cdots \cap D_{n-1}$ by the open property of ampleness ([4, Sect. 4, (4.4)]), and $\mathcal{E}|C$ is a subbundle of $F^{e*}\mathcal{F}|C$. Since $F^e|C: C \to C$ is finite, $(F^{e*}\mathcal{F}|C)^\vee$ is ample, and hence $\deg \mathcal{E}|C < 0$. Since F^e is flat, we have $\mu_{L-\max}(F^{e*}T_X) \leqslant \mu_{L-\max}(F^{e*}\mathcal{F}) = \mu_L(\mathcal{E}) < 0$ as required. □

4. Stability of $F^{e*}T_X$ for weighted complete intersections

In this section, we fix the following notation (see [11]). A weak projective space $\mathbb{P}$ is an open subscheme $\cap_{\nu>1} D_+(\{T_\beta; \nu \nmid e_\beta\})$ of a weighted projective space

Proj $k[T_0, \ldots, T_{n+m}]$, where the grading of $R := k[T_0, \ldots, T_{n+m}]$ is defined by $\deg T_\beta = e_\beta > 0 (0 \leqslant \beta \leqslant n+m)$ and $\deg a = 0 (a \in k)$. A weighted complete intersection X of $\mathbb{P}$ is a complete subscheme of $\mathbb{P}$ isomorphic to $\mathrm{Proj}(R/(F_1, \ldots, F_m))$ for some homogeneous regular sequence $F_1, \ldots, F_m$ of R with $\deg F_\alpha = d_\alpha$. If $e_\beta = 1$ for every β, then X is a complete intersection in the usual sense.

LEMMA 4.1. *Let X be a smooth weighted complete intersection of dimension n. For every eth Frobenius morphism $F^e\colon X \to X (e \geqslant 0)$,*

$$H^t(X, (F^{e*}\Omega_X^q) \otimes \mathcal{O}_X(\ell)) = 0 \tag{4.1.0}$$

holds for $\ell < 0$, $0 \leqslant t + q \leqslant n-1$, and $1 \leqslant q \leqslant n-1$.

Proof. For an $\mathcal{O}_X$-module $\mathcal{G}$ on X and for $\ell \in \mathbb{Z}$, we set $\mathcal{G}(\ell) = \mathcal{G} \otimes \mathcal{O}_X(\ell)$. First we claim that

$$H^t(X, F^{e*}(\Omega_{\mathbb{P}}^q \otimes \mathcal{O}_X)(\ell)) = 0 \tag{4.1.1}$$

for $0 \leqslant t \leqslant n-1$, $1 \leqslant q \leqslant n-1$, and $\ell < 0$. To prove this, we consider the Frobenius pull-back of the restriction of the Euler sequence on $\mathbb{P}(e)$ to X (see [11, Remark 2.4]),

$$0 \to F^{e*}(\Omega_{\mathbb{P}}^1 \otimes \mathcal{O}_X) \to \oplus_{i=0}^{n+m} \mathcal{O}_X(-p^e e_i) \to \mathcal{O}_X \to 0.$$

By taking the exterior product $\wedge^q$ and the twist by $\mathcal{O}_X(\ell)$, for each $q (1 \leqslant q \leqslant n-1)$, we have an exact sequence

$$\begin{aligned} 0 &\to F^{e*}(\Omega_{\mathbb{P}}^q \otimes \mathcal{O}_X)(\ell) \to (\wedge^q \oplus_{i=0}^{n+m} \mathcal{O}_X(-e_i p^e))(\ell) \\ &\to F^{e*}(\Omega_{\mathbb{P}}^{q-1} \otimes \mathcal{O}_X)(\ell) \to 0. \end{aligned} \tag{4.1.2}$$

We note that

$$H^t(X, (\wedge^q \oplus_{i=0}^{n+m} \mathcal{O}_X(-e_i p^e))(\ell)) = 0$$

for every $t (0 \leqslant t \leqslant n-1)$ and $q (0 \leqslant q \leqslant n-1)$ (see [11, Proposition 3.3]). So the claim (4.1.1) follows from (4.1.2) by induction on q.

Now we prove the vanishing (4.1.0) by induction on q. By pulling back the conormal-to-cotangents sequence of X to $\mathbb{P}$ by F^e, we have an exact sequence

$$0 \to \oplus_{\alpha=1}^m \mathcal{O}_X(-p^e d_\alpha) \to F^{e*}(\Omega_{\mathbb{P}}^1 \otimes \mathcal{O}_X) \to F^{e*}\Omega_X^1 \to 0. \tag{4.1.3}$$

When $q = 1$, by using (4.1.3), the vanishing (4.1.0) follows from $H^t(X, \mathcal{O}_X(\ell - p^e d_\alpha)) = 0 (0 \leqslant t \leqslant n-1, 1 \leqslant \alpha \leqslant m)$ and from (4.1.1) for $q = 1$.

When $1 < q \leqslant n-1$, the exact sequence (4.1.3) induces a filtration of $F^{e*}(\Omega^q_{\mathbb{P}} \otimes \mathcal{O}_X)$,

$$0 = \mathcal{F}_{q+1} \subseteq \mathcal{F}_q \subseteq \cdots \subseteq \mathcal{F}_j \subseteq \cdots \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_0 = F^{e*}(\Omega^q_{\mathbb{P}} \otimes \mathcal{O}_X)$$

such that

$$0 \to \mathcal{F}_{j+1} \to \mathcal{F}_j \to (F^{e*}\Omega^{q-j}_X) \otimes (\wedge^j \oplus^m_{\alpha=1} \mathcal{O}_X(-p^e d_\alpha)) \to 0 \tag{4.1.4}$$

are exact for j $(j = 0, \ldots, q)$ (see, for example, [5, Ch. II Ex. 5.16]). Before proving the vanishing (4.1.0) for this q, we prove that

$$H^t(X, \mathcal{F}_j(\ell)) = 0 \tag{4.1.5}$$

for every $j(1 \leqslant j \leqslant q)$ and $t(0 \leqslant t \leqslant n-q-1+j)$, by descending induction on j. If $1 < q \leqslant m$, we have only to start this induction from $j = q$ with

$$\mathcal{F}_q(\ell) = \bigoplus_{1 \leqslant \alpha_1 < \cdots < \alpha_q \leqslant m} \mathcal{O}_X(\ell - p^e(d_{\alpha_1} + \cdots + d_{\alpha_q})),$$

and hence our claim (4.1.5) for $j = q$ follows. If $q > m$, we have only to start from $j = m$ with

$$\mathcal{F}_m(\ell) = (F^{e*}\Omega^{q-m}_X)(\ell - p^e \sum_{\alpha=1}^{m} d_\alpha),$$

and hence our claim (4.1.5) for $j = m$ follows from the inductive hypothesis (4.1.0) on q. For general $j \geqslant 1$, by using (4.1.4), our claim (4.1.5) follows from the inductive hypothesis (4.1.5) on j and the inductive hypothesis (4.1.0) on q. Thus, in particular, we have $H^t(X, \mathcal{F}_1(\ell)) = 0$ for $0 \leqslant t \leqslant n-q$. Therefore, (4.1.0) for $q(1 < q \leqslant n-1)$ follows from the vanishing above and (4.1.1), by using the exact sequence (4.1.4) for $j = 0$. □

PROPOSITION 4.2. *Let X be a smooth weighted complete intersection of dimension $n \geqslant 3$. The eth Frobenius pull-backs $F^{e*}\Omega^1_X (e \geqslant 0)$ are μ-stable (resp. μ-semistable) with respect to $\mathcal{O}_X(1)$ (and hence with respect to every ample line bundle on X) if X is of general type (resp. of Kodaira dimension 0).*

Proof. Let $\mathcal{F}$ be a submodule of $F^{e*}\Omega^1_X$ of rank $r(1 \leqslant r \leqslant n-1)$. So there is an injection $(\wedge^r\mathcal{F})^{\vee\vee} \to F^{e*}\Omega^r_X$. Since $\operatorname{Pic} X \cong \mathcal{O}_X(1) \cdot \mathbb{Z}$ by Grothendieck–Lefschetz theorem for $\dim X \geqslant 3$ [11, Theorem 3.7], we may assume that $c_1(\mathcal{F}) = (\wedge^r\mathcal{F})^{\vee\vee} = \mathcal{O}_X(\ell)$ for some $\ell \in \mathbb{Z}$. Hence $H^0(X, (F^{e*}\Omega^r_X) \otimes \mathcal{O}_X(-\ell)) \neq 0$. By (4.1), we have $\ell \leqslant 0$. Therefore

$$\mu_{\mathcal{O}_X(1)}(\mathcal{F}) = (\deg X/r)\ell \leqslant 0 < (\text{resp. } \leqslant)\ \mu_{\mathcal{O}_X(1)}(F^{e*}\Omega^1_X).$$

Thus $F^{e*}\Omega^1_X$ is μ-stable (resp. μ-semistable) w.r.t. $\mathcal{O}_X(1)$. Since Pic $X \cong \mathcal{O}_X(1) \cdot \mathbb{Z}$, the same is true for the μ-stability w.r.t. any ample line bundle on X. □

Proof of Theorem 1.1. By (4.2), for every eth Frobenius morphism $F^e : X \to X$ $(e \geqslant 0)$, $F^{e*}\Omega^1_X$ is μ-stable with respect to L, and hence the dual $F^{e*}T_X$ is μ-stable of $\mu_L(F^{e*}T_X) < 0$. Therefore the theorem follows from (3.1). □

Remark 4.3. In (4.2), the assumption $n = 3$ is used only when we deduce Pic $X \cong \mathcal{O}_X(1) \cdot \mathbb{Z}$. Thus the same results as in (4.2) and hence (1.1) hold for a weighted complete intersection surface with Pic $X \cong \mathcal{O}_X(1) \cdot \mathbb{Z}$.

5. Stability of $F^{e*}T_X$ for surfaces and 3-folds

First we slightly generalize a lemma of Shepherd–Barron. To this purpose, we recall a result from foliation theory in positive characteristic due to Ekedahl ([1], see also [14, (9.1.2.1)]). A smooth 1-foliation $\mathcal{F}$ on a smooth variety X is a subbundle of T_X closed under the bracket and pth power operation of derivations. Then there exist a smooth variety, denoted by $X/\mathcal{F}$, and a k-morphism $\pi : X \to X/\mathcal{F}$ with the following properties: $X/\mathcal{F}$ is homeomorphic to X via π; $\mathcal{O}_{X/\mathcal{F}}$ consists of those elements of $\mathcal{O}_X$ killed by the derivations of $\mathcal{F}$; and π is purely inseparable of $\deg \pi = p^{\operatorname{rank} \mathcal{F}}$ factoring through the k-Frobenius morphism $F_X : X \to X^{(-1)}$ as $F_X = \lambda \circ \pi$ for some $\lambda : X/\mathcal{F} \to X^{(-1)}$. Here $X^{(e)}$ denotes the base change of X by the p^eth power map of k. Conversely, a factorization $X \xrightarrow{\pi} Y \xrightarrow{\lambda} X^{(-1)}$ with a smooth variety Y and a finite surjective $\pi : X \to Y$ is recovered by a smooth 1-foliation $\mathcal{F} := \operatorname{Ker}(d\pi)$ in this way.

LEMMA 5.1 (Shepherd-Barron, cf. [14, (9.1.3.3)]). *Let X be a normal projective variety of* $\dim X = n$*, and L an ample line bundle on X. Let $\mathcal{E}$ be a torsion-free $\mathcal{O}_X$-module that is μ-semistable with respect to L but the Frobenius pull-back $\widetilde{\mathcal{E}} := F^*\mathcal{E}$ is not. Let $\mathcal{A}$ be a piece of the Harder–Narasimhan filtration of $\widetilde{\mathcal{E}}$, and set $\mathcal{B} = \widetilde{\mathcal{E}}/\mathcal{A}$. Then there exists a nonzero map $T_X \to (\mathcal{A}^\vee \otimes \mathcal{B})^{\vee\vee}$, and hence $\mu_{L-\min}(T_X) \leqslant \mu_{L-\max}((\mathcal{A}^\vee \otimes \mathcal{B})^{\vee\vee})$.*

Proof. Let $\mathcal{E}^{(-1)}$ be the pull-back of $\mathcal{E}$ to $X^{(-1)}$, and hence $F^*\mathcal{E} = F_X^*\mathcal{E}^{(-1)}$. We consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{P}(\mathcal{B}) & \hookrightarrow & \mathbb{P}(\widetilde{\mathcal{E}}) & \xrightarrow{\widetilde{F}} & \mathbb{P}(\mathcal{E}^{(-1)}) \\
 & \overset{\sigma}{\searrow} & \downarrow{\scriptstyle \widetilde{\pi}} & \square & \downarrow{\scriptstyle \pi} \\
 & & X & \xrightarrow[F_X]{} & X^{(-1)}.
\end{array}$$

https://doi.org/10.1023/A:1000192211465 Published online by Cambridge University Press

Here π, $\widetilde{\pi}$, and σ are natural projections and $\widetilde{F}$ is the base change morphism of F_X by π. Let $U \subseteq X$ be the largest open subset of points x where $\mathcal{O}_{x,X}$ is regular, and $\widetilde{\mathcal{E}}_x$ and $\mathcal{B}_x$ are free. Set $\mathbb{P} = \pi^{-1}(U^{(-1)})$, $\widetilde{\mathbb{P}} = \widetilde{\pi}^{-1}(U)$, and $\widetilde{\mathbb{P}}' = \sigma^{-1}(U)$. Since $\widetilde{F}$ factors through $F_{\mathbb{P}(\widetilde{\mathcal{E}})}$, by Ekedahl [1], $\mathcal{H} := T_{\widetilde{\mathbb{P}}/\mathbb{P}} = \mathrm{Ker}(d\widetilde{F})$ is a smooth 1-foliation with $\widetilde{\mathbb{P}}/\mathcal{H} \cong \mathbb{P}$. Since $\widetilde{\pi}$ is the base change of π by F_X, a natural map $\mathcal{H} \to \widetilde{\pi}^* T_U$ is isomorphism. Let $\tau\colon \mathcal{H}' := \mathcal{H}|\widetilde{\mathbb{P}}' \to \mathcal{N}_{\widetilde{\mathbb{P}}'/\widetilde{\mathbb{P}}}$ be the composition of the inclusion and a natural map $T_{\widetilde{\mathbb{P}}} \otimes \mathcal{O}_{\widetilde{\mathbb{P}}'} \to \mathcal{N}_{\widetilde{\mathbb{P}}'/\widetilde{\mathbb{P}}}$ to the normal bundle of $\widetilde{\mathbb{P}}'$ to $\widetilde{\mathbb{P}}$.

Then we claim that τ is nonzero. Indeed, if not, $\mathcal{H}'$ is a smooth 1-foliation of $\widetilde{\mathbb{P}}'$. Set $\mathbb{P}' = \widetilde{\mathbb{P}}'/\mathcal{H}'$. Then the induced inclusion $\mathbb{P}' \hookrightarrow \mathbb{P}$ is a bundle homomorphism over $U^{(-1)}$, since $\widetilde{\mathbb{P}}' \to \widetilde{\mathbb{P}}$ is a bundle homomorphism and $\mathbb{P} = \widetilde{\mathbb{P}}/\mathcal{H}$. So a torsion-free quotient $\mathcal{O}_X$-module $\mathcal{B}'$ of $\mathcal{E}^{(-1)}$ such that $\mathcal{B}'|U^{(-1)}$ corresponds to $\mathbb{P}'$ destroys the stability of $\mathcal{E}$, since $\mu_L(\mathcal{B}) < \mu_L(\widetilde{\mathcal{E}})$ and since $p(c_1(\mathcal{B}'), L^{n-1}) = (c_1(\mathcal{B}), L^{n-1})$. Thus τ is nonzero. By pushing τ out by σ, we have a nonzero map $T_X \to (\mathcal{A}^\vee \otimes \mathcal{B})^{\vee\vee}$. By stability, we have $\mu_{L-\min}(T_X) \leqslant \mu_{L-\max}((\mathcal{A}^\vee \otimes \mathcal{B})^{\vee\vee})$. □

COROLLARY 5.2. *Let X, $\mathcal{E}$, $\widetilde{\mathcal{E}}$, and L be as in (5.1). Let* $0 = \widetilde{\mathcal{E}}_0 \subset \widetilde{\mathcal{E}}_1 \subset \cdots \subset \widetilde{\mathcal{E}}_l = \widetilde{\mathcal{E}}$ $(l \geqslant 2)$ *be the Harder–Narasimhan filtration $\widetilde{\mathcal{E}}$ with respect to L. Assume that* $\mathrm{rank}\, \widetilde{\mathcal{E}} \leqslant 3$. *Set* $\rho(1,1) = \frac{1}{2}$, $\rho(2,1) = \frac{1}{3}$, $\rho(1,2) = \frac{2}{3}$, *and* $\rho(1,1,1) = 1$. *Then we have*

$$\mu_L(\widetilde{\mathcal{E}}_1) \leqslant p\mu_L(\mathcal{E}) - \rho(\mathrm{rank}\, \widetilde{\mathcal{E}}_1/\widetilde{\mathcal{E}}_0, \ldots, \mathrm{rank}\, \widetilde{\mathcal{E}}_l/\widetilde{\mathcal{E}}_{l-1}) \cdot \mu_{L-\min}(T_X).$$

Proof. Set $\mathcal{G}_i = \widetilde{\mathcal{E}}_i/\widetilde{\mathcal{E}}_{i-1}$. Since $\widetilde{\mathcal{E}}_i$ or $\widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i$ is of rank 1 and hence $\widetilde{\mathcal{E}}_i = \mathcal{G}_i$ or $\widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i = \mathcal{G}_{i+1}$, for each i, we have

$$\mu_{L-\max}((\widetilde{\mathcal{E}}_i^\vee \otimes \widetilde{\mathcal{E}}/\widetilde{\mathcal{E}}_i)^{\vee\vee}) = \mu_L((\mathcal{G}_i^\vee \otimes \mathcal{G}_{i+1})^{\vee\vee}) = \mu_L(\mathcal{G}_{i+1}) - \mu_L(\mathcal{G}_i).$$

Applying (5.1) to $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}_i$, we have $\mu_L(\mathcal{G}_i) - \mu_L(\mathcal{G}_{i+1}) \leqslant -\mu_{L-\min}(T_X)$. By the definition of μ_L, we have $\sum_i \mathrm{rank}\, \mathcal{G}_i \cdot \mu_L(\mathcal{G}_i) = \mathrm{rank}\, \widetilde{\mathcal{E}} \cdot \mu_L(\widetilde{\mathcal{E}})$. Thus we get the required inequalities. □

Proof of Theorem 1.2. By (3.1), we have only to show that $\mu_{L-\max}(F^{e*}T_X) < 0$ for every $e \geqslant 0$. By stability, we have only to check that for every $e \geqslant 0$, $F^{e*}T_X$ has a (possibly trivial) filtration each of whose graded piece is a torsion-free μ-semistable $\mathcal{O}_X$-module of negative μ-slope.

First we consider the case when T_X is μ-semistable. Hence $\mu_{L-\min}(T_X) = \mu_L(T_X)$. When $F^{e*}T_X$ is also μ-semistable for $e > 0$, then there is nothing to prove. Otherwise, let $e_0 (< e)$ be the least non-negative integer such that $F^{e_0*}T_X$ is μ-semistable but $F^{e_0+1*}T_X$ is not. Let $0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_l = F^{e_0+1*}T_X$ be the H.-N. filtration and set $e_1 = e - (e_0 + 1)$.

When $F^{e_0+1*}T_X$ is of type (1, 1) (resp. (1, 1, 1)), by (5.2) for $\mathcal{E} = F^{e_0*}T_X$ and our assumption, we have

$$\begin{aligned} &\mu_L(\mathcal{T}_2/\mathcal{T}_1) < \mu_L(\mathcal{T}_1) \leqslant p\mu_L(F^{e_0*}T_X) - \tfrac{1}{2}\mu_L(T_X) \\ &\quad = (p^{e_0+1} - \tfrac{1}{2})\mu_L(T_X) < 0 \\ &(\text{resp. } \mu_L(\mathcal{T}_3/\mathcal{T}_2) < \mu_L(\mathcal{T}_2/\mathcal{T}_1) < \mu_L(\mathcal{T}_1) \leqslant (p^{e_0+1} - 1)\mu_L(T_X) < 0.) \end{aligned}$$

Thus for $e_1 := e - (e_0 + 1)$, $F^{e_1*}(\mathcal{T}_{i+1}/\mathcal{T}_i)$ are torsion-free sheaves of rank 1 of negative μ-slopes, as required.

When $F^{e_0+1*}T_X$ is of type (2, 1) (resp. (1, 2)), by (5.2) and our assumption, we have

$$\mu_L(\mathcal{T}_2/\mathcal{T}_1) < \mu_L(\mathcal{T}_1) \leqslant (p^{e_0+1} - \tfrac{1}{3})\mu_L(T_X) < 0$$

$$(\text{resp. } \mu_L(\mathcal{T}_2/\mathcal{T}_1) < \mu_L(\mathcal{T}_1) \leqslant (p^{e_0+1} - \tfrac{2}{3})\mu_L(T_X) < 0.)$$

Set $e_1 = e - (e_0 + 1)$ and $\mathcal{G} = \mathcal{T}_1$ (resp. $\mathcal{G} = \mathcal{T}_2/\mathcal{T}_1$). If $F^{e_1*}\mathcal{G}$ is μ-semistable, then there is nothing to prove. If $F^{e_1*}\mathcal{G}$ is not μ-semistable, let e_2 be the least integer with $e - (e_0 + 1) > e_2 \geqslant 0$ such that $F^{e_2*}\mathcal{G}$ is μ-semistable but $F^{e_2+1*}\mathcal{G}$ is not. For the H.-N. filtration $0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 = F^{e_2+1*}\mathcal{G}$ of $F^{e_2+1*}\mathcal{G}$, we have

$$\begin{aligned} \mu_L(\mathcal{S}_2/\mathcal{S}_1) < \mu_L(\mathcal{S}_1) &\leqslant p\mu_L(F^{e_2*}\mathcal{G}) - \tfrac{1}{2}\mu_L(T_X) \\ &\leqslant \{p^{e_2+1}(p^{e_0+1} - \tfrac{1}{3}) - \tfrac{1}{2}\}\mu_L(T_X) < 0 \\ (\text{resp.} \qquad &\leqslant \{p^{e_2+1}(p^{e_0+1} - \tfrac{2}{3}) - \tfrac{1}{2}\}\mu_L(T_X) < 0) \end{aligned}$$

by (5.2) and our assumption. Thus for $e_3 := e - (e_0 + e_2 + 2)$, we have $\mu_L(F^{e_3*}(\mathcal{S}_2/\mathcal{S}_1)) < \mu_L(F^{e_3*}\mathcal{S}_1) < 0$ and $\mu_L(F^{e_1*}(\mathcal{T}_2/\mathcal{T}_1)) < 0$ (resp. $\mu_L(F^{e_1*}\mathcal{T}_1) < 0$), as required.

Second we consider the case when T_X is not μ-semistable with H.-N. filtration $0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_l = T_X$. By assumption, the type of T_X is (1, 1), (1, 1, 1), or (1, 2).

When T_X is of type (1, 1) (resp. (1, 1, 1)), by assumption, we have $0 > \mu_L(F^{e*}(\mathcal{T}_1/\mathcal{T}_0)) > \cdots > \mu_L(F^{e*}(\mathcal{T}_l/\mathcal{T}_{l-1}))$, as required.

When T_X is of type (1, 2) with H.-N. filtration $0 \subset \mathcal{T}_1 \subset \mathcal{T}_2 = T_X$, we have $0 > \mu_L(\mathcal{T}_1) > \mu_L(T_X) > \mu(T_X/\mathcal{T}_1) = \mu_{L-\min}(T_X)$. By the similar argument as in the case of $F^{e_0+1*}T_X$ of type (1, 2), we have $\mu_{L-\max}(F^{e*}T_X) < 0$ for $e \geqslant 0$. □

Remark 5.3. (1) When X is a smooth 3-fold such that the first piece $\mathcal{T}$ of the H.-N. filtration of T_X is of rank 2, by the same argument as above, it turns out that the Gauss map $\iota^{(1)}$ is generically injective if $(2(p+1)/3)\mu_L(\mathcal{T}) < \mu_L(T_X) < \mu_L(\mathcal{T}) < 0$.

(2) In the proof of (1.2) above, by using only the ampleness of L but without using the form of $L = \Omega_X^n \otimes H^{\otimes n+1}$, we show that for a smooth projective surface or 3-fold, if $\mu_{L-\max}(T_X) < 0$, then $\mu_{L-\max}(F^{e*}T_X) < 0$ for every eth Frobenius morphism F^e, with the exceptional case for $n = 3$.

(3) A result of Ekedahl [2, Theorem 2.4] tells us the structure of surfaces of non-μ-stable tangent bundles in case $\mu_{L-\max}(T_X) \geqslant (\Omega_X^2, L)/(p-1)$.

Acknowledgements

I would like to thank Professor Hajime Kaji for his several seminar talks on the Gauss map at Waseda University and for giving me his preprint [8]. I am also grateful to Professor Noboru Nakayama for informing me Shepherd–Barron's paper [14] before this work.

References

1. Ekedahl, T.: Foliations and inseparable morphisms, In Algebraic Geometry, Bowdoin 1985, (Proc. of Symp. Pure Math., Vol. 46-2) S. Bloch (ed.), *Amer. Math. Soc.*, 1987, pp. 139–150.
2. Ekedahl, T.: Canonical models of surfaces of general type in positive characteristic, *Publ. Math. IHES* 67 (1988), 98–144.
3. Gieseker, D.: On a theorem of Bogomolov on Chern classes of stable bundles, *Amer. J. Math.* 101 (1979), 77–85.
4. Hartshorne, R.: Ample vector bundles, *Publ. Math. IHES* 29 (1966), 63–94
5. Hartshorne, R.: *Algebraic Geometry, (Graduate Texts in Math. Vol. 52)*, Springer, Berlin Heidelberg New York, 1977.
6. Kaji, H.: On the Gauss maps of space curves in characteristic p, *Compositio Math.* 70 (1989), 177–197.
7. Kaji, H.: On the Gauss maps of space curves in characteristic p, II, *Compositio Math.* 78 (1991), 261–269.
8. Kaji, H.: On the pure inseparability of the Gauss map, preprint 1992 (unpublished).
9. Kaji, H. and Noma, A.: On the generic injectivity of the Gauss map in positive characteristic, to appear in *J. Reine Angew. Math.* 482 (1997).
10. Kleiman, S. and Piene, R.: On the inseparability of the Gauss map, In Enumerative Algebraic Geometry, Proceedings of the 1989 Zeuthen Symposium (Contemp. Math., Vol. 123), *Amer. Math. Soc.*, 1991, pp. 107–129.
11. Mori, S.: On a generalization of complete intersections, *J. Math. Kyoto Univ.* 15-3 (1975), 619–646.
12. Piene, R.: Projective algebraic geometry in positive characteristic, In Analysis, Algebra and Computer in Mathematical Research, Proc. Twenty-First Nordic Congress of Math., M. Gyllenberg and L. Persson, (eds.) Dekker, New York-Basel-Hong Kong, 1994, pp. 263–273
13. Shatz, S.: The decomposition and specialization of algebraic families of vector bundles, *Compositio Math.* 35 (1977), 163–187.
14. Shepherd-Barron, N.I.: Miyaoka's theorems on the generic semi-negativity of T_X and on the Kodaira dimension of minimal regular threefolds, In Flips and abundance for algebraic threefolds, A summer seminar at the university of Utah, Salt Lake City, 1991 (Astérisque Vol. 211), J. Kollár (ed.), *Soc. Math. France*, 1992, pp. 103–114.
15. Zak, F.L.: Tangents and secants of algebraic varieties, Transl. Math. Monographs, *Amer. Math. Soc.* 127 (1993).