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A CHARACTERIZATION OF MULTI-DIMENSIONAL PERRON INTEGRALS AND THE FUNDAMENTAL THEOREM

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ABSTRACT. In this paper weak Perron integrals are characterized as n-dimensional interval functions F which are additive, differentiable almost everywhere in the weak sense and which satisfy a new continuity condition concerning the singular set. Before, only one-dimensional Perron integrals were characterized by the theorem of Hake-Alexendrov-Looman, and analogous results for strong Perron integrals (which are best analyzed, but more restrictive) are not available in higher dimensions yet. In order to formulate our continuity condition we introduce an outer measure μ by means of a new weak variation of F which is required to vanish on all null sets. The same condition is also necessary and sufficient for the integral of the weak derivative to yield the original interval function. This "fundamental theorem" is split into two fundamental inequalities of very general nature which contain additional singular terms involving our variation. These inequalities are very useful also for Lebesgue integrals.

1. **Introduction.** Lebesgue has characterized his n-dimensional integral as a finite additive interval function which is absolutely continuous (AC). According to the theorem of Radon-Nikodym the L-integral can also be characterized as a finite measure which vanishes on null sets (null condition). In dimension one the restricted Denjoy integrals can be defined as finite point functions which are absolutely continuous in a generalized sense (ACG_*), see Saks [26, p. 241]. According to the theorem of Hake-Alexandrov-Looman as stated in Saks [26, p. 250–251] the one-dimensional Perron integral is equivalent to the restricted Denjoy integral, which implies a characterization of these Perron integrals. It is the purpose of this paper to characterize Perron integrals in all dimensions.

In higher dimensions we distinguish between *weak* concepts based on regular intervals and *strong* concepts based on unrestricted intervals. This is standard for derivatives, but can also be done for integrals and other concepts. In principle, the use of weak notions will lead to a more general theory. In generalizing AC it is important to define this property for more general sets than intervals. In generalizing the null condition we face the problem that in the case of non-absolutely convergent integrals the corresponding interval function cannot be extended to a measure. Both problems can be settled using weak notions, because we can associate with a weak integral an outer measure by means

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of a new weak variation. For this outer measure the null condition makes sense again and leads to our present characterization of weak integrals, which is given in Section 4. No corresponding characterization for strong integrals is known yet.

In Section 0 we give a definition of (weak) Perron integrals which includes most Perron-type integrals defined by other authors. In Section 1 we explain our new variational concepts which will be applied to null sets or the singular set where the derivative is not finite. With the help of our weak variations we are able to formulate continuity properties of the integral with respect to these sets which cannot be expressed by standard variations because they might be infinite. Such necessary conditions besides the additivity and differentiability of the integral are discussed in Section 2. And it is the object of Section 3 to show that these conditions also suffice to integrate the derivative and to recover the original function. As in Calculus, this result is called Fundamental Theorem and completes our characterization of the weak Perron integral. Furthermore we split the fundamental theorem into two fundamental inequalities of very general nature involving additional singular terms, cf. Theorem 3. Their consequences are even interesting for L-integrals.

0. **Basic concepts.** For clarity we list a number of definitions, which can be considered standard apart from slight modifications.

Given a fixed dimension $n \in \mathbb{N}$, an interval $I \subseteq \mathbb{R}^n$ is supposed to be compact and non-degenerate, i.e. I is given by $\{x \in \mathbb{R}^n \colon a_i \leq x_i \leq b_i\}$ with real $a_i < b_i$. We associate with I the number

$$r = \min(b_i - a_i)/\max(b_i - a_i)$$

and call I *regular* if $r \geq 1/2$. (Other positive constants < 1 in place of $1/2$ could be used as well and lead to equivalent theories.)

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ be the extended real number system, where we also consider infinite intervals like

$$[0, \infty] = \{y \in \bar{\mathbb{R}} : 0 \leq y \leq \infty\} \text{ or } (0, \infty] = \{y \in \bar{\mathbb{R}} : 0 < y \leq \infty).$$

Given an interval $I \subseteq \mathbb{R}^n$ we call F an interval function on I if F associates (at least) with every subinterval $J \subseteq I$ an element $F(J) \in \bar{\mathbb{R}}$. Such an F is called *subadditive* in I if for every subinterval J and any finite decomposition $J = \cup J_k$ into non-overlapping intervals J_k the inequality

$$F(J) \leq \sum_k F(J_k) \tag{0.1}$$

holds true whenever the sum is well-defined (i.e., $-\infty$ and ∞ do not occur simultaneously). We define superadditivity on I by reversing the inequality in (0.1) and additivity on I by requiring both, i.e. equality in (0.1).

Let F be an interval function on I and $x \in I$. We define the upper derivative of F at x relative to I by

$$D_I^+ F(x) = \sup \limsup_{k \to \infty} \frac{F(J_k)}{|J_k|} \in \bar{\mathbb{R}}, \tag{0.2}$$

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where the sup extends over all sequences of regular subintervals $J_k \subseteq I$ satisfying $x \in J_k$ and $|J_k| \longrightarrow 0$ with $|J_k|$ being the usual measure of J_k. This is a *weak* derivative in view of the imposed regularity condition. Consequently almost all intervals J_k lie in $B_\delta(x)$, the set of all points whose distance from x is less than δ, $\delta \in (0, \infty]$. Similarly we define the lower derivative $D_I^- f(x) = \inf \liminf F(J_k)/|J_k|$ and the derivative of F at x relative to I as the common value

$$D_I F(x) = D_I^+ F(x) = D_I^- F(x) \in \bar{\mathbb{R}}, \tag{0.3}$$

wherever equality prevails. If $D_I F(x)$ exists in this sense and is finite we call F *differentiable at x relative to I*. By an argument of Banach, cf. Saks [26, p. 112], the derivatives $D_I^+ F$, $D_I^- F$, and $D_I F$ are always L-measurable functions. The index I is only relevant for the boundary points x of I.

Our definition of multi-dimensional Perron integrals is based upon minor and major functions, which are characterized by properties of their weak derivatives. Several versions of such weak integrals are known in the literature, cf. Bauer [1], Kemisty [11, 12, 13], Mařik [16], Ridder [18, 19, 20, 21, 22], Romanowski [23, 24, 25], Trjitzinsky [30, 31, 32].

Our definition is a slight modification of Bauer's original definition and leads to a relatively general integral. In particular, it contains Henstock's variational integral, the Riemann complete integral of Kurzweil-Henstock, and the Perron-Ward integral as treated in Kurzweil [15]. All of these integrals can equivalently be defined using minor and major functions which are characterized by analogous properties of their strong derivatives. Hence they can be classified as strong integrals.

Given an interval $I \subseteq \mathbb{R}^n$, a point function f on I has values in $\bar{\mathbb{R}}$ and is defined at least on I. In that situation, a minor function for f on I is defined as a subadditve interval function V on I satisfying

$$D_I^+ V < \infty \quad \text{and} \quad D_I^+ V \leq f \text{ on } I. \tag{0.4}$$

This implies that $V(J) < \infty$ for all subintervals $J \subseteq I$. Similarly, a major function U is defined as superadditive satisfying $D_I^- U > -\infty$ and $D_I^- U \geq f$ on I. The constant functions $V = -\infty$, $U = +\infty$ have the required properties for any f. Now consider for fixed f all these V, U and form

$$\int_I^- f = \sup V(I), \quad \int_I^+ f = \inf U(I). \tag{0.5}$$

These elements of $\bar{\mathbb{R}}$ are called lower or upper integral respectively; more precisely they are our weak Perron integrals. Elementary considerations show that

$$\int_I^- f \leq \int_I^+ f \tag{0.6}$$

holds true in any case. In case of equality and finiteness we speak of the P-integral

$$\int_I f = \int_I^- f = \int_I^+ f \in \mathbb{R} \tag{0.7}$$

and we say that the point function f is P-integrable over I.

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1. **New variations for interval functions.** Suppose that an interval function F on I is given. Now consider all subintervals $J \subseteq I$, arbitrary subsets $A \subset \mathbb{R}^n$ and arbitrary positive functions $\delta : \mathbb{R}^n \longrightarrow (0, \infty]$. A family (J_k) is called *admissible* with respect to (J, A, δ) if it consists of finitely many non-overlapping regular subintervals of J such that for each k there is a point $a_k \in A \cap J_k$ satisfying $B_{\delta(a_k)}(a_k) \supseteq J_k$. The last condition requires the intervals J_k to intersect A in a locally controlled fashion. Note that the empty family is always admissible. We define a preliminary variation by

$$\underset{A,\delta}{\operatorname{var}}{}_J^+ F = \sup \sum_k F(J_k) \in [0, \infty], \tag{1.1}$$

where the sup extends over all admissible families (J_k) for which the occurring sum is well-defined. If there is an instance where the sum is not well-defined the sup is automatically ∞. In the variation above only regular subintervals of J are used which belong to a "δ-neighborhood" of A, where δ is allowed to vary with the points of A. Such a δ-neighborhood was first introduced in Kurzweil's definition of the Riemann complete integral, cf. Kurzweil [14]. In Henstock [3, 4, 5, 6, 7, 8, 9, 10] and Thomson [27, 28] this idea was also used to define variations without regularity. On the other hand, in Kempisty [11, 12, 13] regular intervals are used to define a variation, in which the size of the intervals is uniformly restricted. It is a combination of both ideas, δ and regularity, which leads to a variation that remains finite in our applications. Nevertheless (1.1) is a technical quantity which is used to define our upper variation of F over A relative to J

$$\underset{A}{\operatorname{var}}{}_J^+ F = \inf \underset{A,\delta}{\operatorname{var}}{}_J^+ F \in [0, \infty], \tag{1.2}$$

where the inf extends over all positive δ-functions. Note that we need $\delta(\cdot)$ only on A and that the points of A outside of J play no role. Furthermore, the variation (1.2) is independent of J as long as A lies in the interior of J.

We also define the lower variation and the variation of F over A relative to J

$$\underset{A}{\operatorname{var}}{}_J^- F = \underset{A}{\operatorname{var}}{}_J^+ (-F) \in [0, \infty], \tag{1.3}$$

$$\underset{A}{\operatorname{var}}{}_J F = \underset{A}{\operatorname{var}}{}_J^+ F + \underset{A}{\operatorname{var}}{}_J^- F. \tag{1.4}$$

In particular, we have associated with each (F, J) two set functions

$$\mu^+(A) = \underset{A}{\operatorname{var}}{}_J^+ F, \;\; \mu^-(A) = \underset{A}{\operatorname{var}}{}_J^+ F, \tag{1.5}$$

which are defined for arbitrary subsets $A \subseteq \mathbb{R}^n$. In case $A = J$ and provided that F is finite and additive on J, we remark that our preliminary variation is independent of δ and our upper variation is the usual one. We are now going to analyze the dependency of our upper variation with respect to A, J and F.

PROPOSITION 1. *Suppose that an interval function F on I and a subinterval $J \subseteq I$ are given. Then the set functions μ^+, μ^- of (1.5) are (metric) outer measures in the sense*

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of Carathéodory, i.e., they are defined for arbitrary sets in $\mathbb{R}^n$ *with values in* $[0, \infty]$ *and satisfy the following conditions* ($\mu = \mu^{\pm}$):

(1.6) $$\mu(A_1) \leq \mu(A_2) \text{ if } A_1 \subseteq A_2,$$

(1.7) $$\mu\left(\bigcup_1^\infty A_i\right) \leq \sum_1^\infty \mu(A_i),$$

(1.8) $$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2)$$

if the distance between A_1 *and* A_2 *is positive .*

PROOF. It suffices to consider $\mu = \mu^+$. Since (1.6) and (1.8) are straightforward we only discuss (1.7). We may suppose, by (1.6), that the A_i are disjoint. We may also assume that $\sum \mu(A_i) < \infty$. Now we seleect functions δ_i on A_i (only) to approximate $\mu(A_i)$ and define δ on $A = \cup A_i$ as δ_i on A_i. Any admissible family (J_k) with respect to (J, A, δ) can be split into admissible families with respect to (J, A_i, δ_i) so that

$$\sum_k F(J_k) \leq \sum_{i=1}^\infty \operatorname*{var}_{A_i,\delta_i}{}_J^+ F \leq \sum_1^\infty \mu(A_i) + \varepsilon,$$

which implies (1.7) in the limit.

For general properties of outer measures we refer to Saks [26] and Wheeden-Zygmund [34]; in particular, we note that $\mu^{\pm}$ are Borel measures. Next we discuss how our upper variation depends on J.

LEMMA 1. *Suppose that an interval function F on I, a subset* $A \subseteq \mathbb{R}^n$, *and a positive function* δ *are given. Then the interval functions*

$$G_\delta(J) = \operatorname*{var}_{A,\delta}{}_J^+ F \text{ and } G(J) = \operatorname*{var}_{A}{}_J^+ F$$

are superadditive on I.

PROOF. It suffices to consider G_δ. Consider a finite decomposition $J = \cup J_i$ and arbitrary admissible families with respect to (J_i, A, δ). Since they can be joined to form an admissible family with respect to (J, A, δ) we obtain lower bounds for $G_\delta(J)$ which imply the result.

Finally we discuss our upper variation as functional of F. Like a norm, it satisfies

(1.9) $$\operatorname*{var}_{A}{}_J^+(F_1 + F_2) \leq \operatorname*{var}_{A}{}_J^+ F_1 + \operatorname*{var}_{A}{}_J^+ F_2$$

provided that $F_1 + F_2$ is a well-defined interval function on J. It is particularly important to understand the relation between the variation of F and the derivative of F. Here we use $|A|$ for the outer Lebesgue measure of $A \subseteq \mathbb{R}^n$.

THEOREM 1. *Suppose that an interval function F on I, a subinterval $J \subseteq I$, and a subset $A \subseteq \mathbb{R}^n$ are given. Then*

$$\operatorname*{var}_A{}^+_J F \geq a|A \cap J| \text{ if } D^+_I F \geq a \text{ on } A \cap J \tag{1.10}$$

with $a \in [0, \infty]$ and the understanding that $0 \cdot \infty = 0$. Furthermore for $b \in [0, \infty)$ we have

$$\operatorname*{var}_A{}^+_J F \leq b|A \cap J| \text{ if } D^+_I F \leq b \text{ on } A \cap J. \tag{1.11}$$

The estimate (1.10) is based on the regularity condition and standard arguments for set functions, cf. Saks [26, p. 114], while the estimate (1.11) makes essential use of the δ-function. Thus it appears that our variation is very natural in this context.

PROOF OF (1.10). We may assume $a > 0$ and $\operatorname*{var}_A{}^+_J F < \infty$. Given $\varepsilon > 0$ there is a positive function δ such that for all admissible families (J_k) with respect to (J, A, δ)

$$\sum F(J_k) \leq \varepsilon + \operatorname*{var}_A{}^+_J F \tag{1.12}$$

holds true. Now select c such that $0 < c < a$ and consider all regular subintervals $K \subseteq J$ for which there is a point $x \in A \cap K$ satisfying $B_{\delta(x)}(x) \supseteq K$ and which have the additional property

$$F(K) \geq c|K|. \tag{1.13}$$

It is clear that these intervals K form a Vitali covering of $A^* = A \cap J^\circ$, J° being the interior of J. By the Vitali covering theorem there exist finitely many disjoint intervals K_i from the collection above such that

$$|A^*| \leq \sum_i |K_i| + \varepsilon / c. \tag{1.14}$$

Since the family (K_i) is admissible with respect to (J, A, δ) we may combine the three inequalities above and obtain

$$c|A^*| \leq \varepsilon + \sum F(K_i) \leq 2\varepsilon + \operatorname*{var}_A{}^+_J F,$$

which implies the result in view of $|A^*| = |A \cap J|$.

PROOF OF (1.11). Given $\varepsilon > 0$, there is an open set $M \supseteq A \cap J$ such that $|M| \leq |A \cap J| + \varepsilon$ and a positive function δ with the following property: for any regular subinterval $K \subseteq J$ for which there is a point $x \in A \cap K (\subseteq A \cap J)$ satisfying $B_{\delta(x)}(x) \supseteq K$ we have

$$F(K) \leq (b + \varepsilon)|K| \text{ and } K \subseteq M. \tag{1.15}$$

This can be applied to any admissible family (K_i) with respect to (J, A, δ) resulting in

$$\operatorname*{var}_{A,\delta}{}^+_J F \leq \sup \sum (b + \varepsilon)|K_i| \leq (b + \varepsilon)|M|,$$

which implies the result.

COROLLARY 1′. *Let $F(K) = |K|$ for all intervals K. Then*

$$\operatorname*{var}_A{}^{+}_{J} F = |A \cap J| \tag{1.16}$$

holds for any subset $A \subset \mathbb{R}^n$ and any interval J.

PROOF. Select $I \supseteq J$ and observe that $D_I^+ F = 1$ on $A \cap J$.

COROLLARY 1″. *Under the assumptions of Theorem 1 we have*

$$\operatorname*{var}_A{}^{+}_{J} F = 0 \text{ if } D_I^+ F < \infty \text{ on } A \cap J \text{ and } |A \cap J| = 0. \tag{1.17}$$

PROOF. Decompose $A \cap J$ into countably many parts to which (1.11) applies, and use (1.7).

Further consequences will be discussed in a subsequent paper.

2. **Characteristic properties of the weak Perron integral.** First we list various standard results including the differentiation theorem for which we give a simple proof based on Theorem 1. Then we discuss our new null condition thereby completing the necessary conditions occurring in our characterization of integrals.

Suppose that a point function f on an interval $I \subseteq \mathbb{R}^n$ is given. If V is a minor function for f on I then V is also a minor function for f on every subinterval $J \subseteq I$ since $D_J^+ V \leq D_I^+ V$ on J. In particular, we have

$$V(J) \leq \int_J^- f \text{ for all } J \subseteq I,$$

and it follows that the interval function $\int_J^- f (J \subseteq I)$ is always subadditive on I. Similarly the interval function $\int_J^+ f$ is always superadditive on I. The function f is P-integrable over I iff for any $\varepsilon > 0$ there exist a (finite) minor function V and a (finite) major function U on I such that

$$0 \leq U(I) - V(I) < \varepsilon.$$

Since $U - V$ is superadditive we also have

$$0 \leq U(J) - V(J) < \varepsilon \text{ for all } J \subseteq I.$$

Hence P-integrability over I implies P-integrability over all subintervals $J \subseteq I$, and the resulting finite interval function $\int_J f (J \subseteq I)$ is additive on I.

Suppose that two point functions f, g on I are given such that $f + g$ is well-defined on I. Adding the minor functions for f and g we see that

$$\int_I^- (f+g) \geq \int_I^- f + \int_I^- g \tag{2.1}$$

holds provided that the sum on the right is well-defined.

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By adding the minor functions of $f + g$ and $-g$ we also obtain

$$\int_I^- (f+g) \leq \int_I^- f + \int_I^+ g \tag{2.2}$$

provided that the sum on the right is well-defined. In particular, we have equality in case that g is P-integrable over I, and if f is also P-integrable over I so is $f + g$.

Suppose that a point function f on I is given which is ≥ 0 and L-measurable on I. By the theorem of Vitali-Carathéodory one can approximate f from below (and above, if ${}^L\int_I f < \infty$) by semi-continuous functions whose L-integrals can be used as minor (and major) functions for f on I, cf. Saks [26, p. 191]. It follows that in this case the Perron integral over I exists and coincides with the L-integral over I. As a consequence, any point function g on I which is L-integrable over I will also be P-integrable over I with both integrals coinciding. In particular, if g is a null function, i.e. zero almost everywhere its Perron integral exists and is zero. In connection with (2.1), (2.2) we see that the lower (or upper) P-integral does not change if we change the integrand only on a null set (set of L-measure 0). Hence it even suffices if the integrand is defined almost everywhere on I, and we may correspondingly relax the second inequality in (0.4) for minor functions without changing the lower P-integral, since that inequality can always be corrected by changing f on a null set.

Of particular importance is the connection between integration and differentiation.

DIFFERENTIATION THEOREM. *Suppose that the point function f is P-integrable over an interval $I \subseteq \mathbb{R}^n$. Then the interval function $F(J) = \int_J f\,(J \subseteq I)$ is differentiable relative to I at almost all points of I, and we have*

$$D_I F = f \text{ almost everywhere on } I. \tag{2.3}$$

In particular, the function f is necessarily finite almost everywhere on I and L-measurable on I.

PROOF. Given $\varepsilon_1 > 0$ there is a (finite) minor function V on I such that $F(I) - V(I) < \varepsilon_1$. We consider the interval function $G(J) = F(J) - V(J)$, $J \subseteq I$, which is clearly ≥ 0 and superadditive on I. Any finite family (I_k) of non-overlapping subintervals of I is part of a decomposition of I, so that

$$\sum G(I_k) \leq G(I) < \varepsilon_1 \tag{2.4}$$

follows. Taking the sup over all $(I < A, \delta)$-admissible families followed by the inf over all δ we conclude

$$\operatorname*{var}_A{}_I^+ G \leq G(I) < \varepsilon_1 \text{ for any } A \subseteq \mathbb{R}^n. \tag{2.5}$$

Given, independently, a second $\varepsilon_2 > 0$ we consider the set $A = \{x \in I : D_I^+ G \geq \varepsilon_2\}$ and infer from (1.10) that

$$\operatorname*{var}_A{}_I^+ G \geq \varepsilon_2 |A|. \tag{2.6}$$

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Combining (2.6) with (2.5) we obtain

(2.7) $$|A| < \varepsilon_1/\varepsilon_2 = \varepsilon$$

Now observe for $x \in I$

(2.8) $$D_I^+F \leq D_I^+G + D_I^+V < \varepsilon_2 + f \quad \text{if } x \notin A,$$

since $D_I^+V < \infty$. Setting $\varepsilon_1 \to 0$ we learn that $D_I^+F < \varepsilon_2 + f$ holds almost everywhere on I which implies

(2.9) $$D_I^+F < \infty \text{ and } D_I^+F \leq f \text{ a.e. on } I.$$

Replacing f by $-f$ we also obtain $D_I^-F > -\infty$ and $D_I^-F \geq f$ almost every where on I. This completes the proof.

In order to formulate further necessary conditions for P-integrability we give the following definitions: an outer measure μ on $\mathbb{R}^n$ satisfies the null condition if it vanishes on all null sets. An interval function F on the interval $I \subseteq \mathbb{R}^n$ belongs to the class n_I^+ resp. n_I^- if for all null sets A

$$\mu^+(A) = \underset{A}{\text{var}}_I^+ F = 0 \text{ resp. } \mu^-(A) = \underset{A}{\text{var}}_I^- F = 0.$$

We write $F \in n_I$ if both conditions are satisfied.

THEOREM 2. *If the point function f is P-integrable over an interval $I \subseteq \mathbb{R}^n$, the interval function $F(J) = \int_J f (J \subseteq I)$ belongs to the class n_I.*

PROOF. As in the previous proof we have (2.4) which can be rewritten as

(2.10) $$\sum F(I_k) \leq G(I) + \sum V(I_k), \ G(I) < \varepsilon_1.$$

As before we infer for any $A \subseteq \mathbb{R}^n$

(2.11) $$\underset{A}{\text{var}}_I^+ F \leq \varepsilon_1 + \underset{A}{\text{var}}_I^+ V.$$

By (1.17) applied to V we deduce

(2.12) $$\underset{A}{\text{var}}_I^+ F \leq \varepsilon_1 \text{ for any null set } A.$$

This implies $F \in n_I^+$ and replacing f by $-f$ we also obtain $F \in n_I^-$.

We add a few remarks concerning $n_I^{\pm}$: let

(2.13) $$S^+ = \{x \in I : D_I^+F = \infty\}, \quad S^- = \{x \in I : D_I^-F = -\infty\}$$

denote the singular sets associated with F on I. In view of (1.17) we have $F \in n_I^+$ iff $\mu^+(A) = 0$ for all null sets $A \subseteq S^+$, and we only need to consider $A = S^+$ in case that S^+ itself is a null set. Corresponding statements are true for n_I^-. If F is a subadditive interval function on I which satisfies $F \in n_I^+$ it follows that $F(J) < \infty$ for all subintervals $J \subseteq I$, since in case $F(J) = \infty$ we can construct a contracting sequence J_k of regular subintervals with limit point x such that always $F(J_k) = \infty$ which implies $\underset{A}{\text{var}}_I^+ F = \infty$ whenever $x \in A$.

3. **The fundamental inequality.** The characterization of integrals is connected to the possibility of integral representations for interval functions in the form

$$F(J) = \int_J f \qquad (J \subseteq I), \tag{3.1}$$

which necessarily leads to the problem of integrating derivatives. If the derivative of F exists everywhere it is a classical result for Perron integration that the derivative can be integrated and leads to the representation (3.1). However, in general the derivative might only exist almost everywhere, and even if the derivative can be integrated a further condition concerning the singular sets is needed for the validity of (3.1). For the integrability it is important to consider integrals which are sufficiently general like, e.g., the weak Perron integral. We are going to show that the null condition introduced before is exactly the relevant condition for this integral to guarantee integrability and the identity (3.1). Besides (3.1) we are also interested in the corresponding inequality

$$F(J) \leq \int_J f \qquad (J \subseteq I), \tag{3.2}$$

for which our one-sided null condition is relevant. It is even possible to identify the occurring error term (singular term) in the case that this null condition is not satisfied. The resulting fundamental inequality seems to be central in many applications, even for L-integrals, since general exceptional sets can now be handled.

The fundamental inequality is based upon an explicit construction of special minor functions.

CONSTRUCTION PRINCIPLE. *Suppose that F is a subadditive interval function on $I \subseteq \mathbb{R}^n$ with singular set S^+ as in (2.13) and assume*

$$\underset{S^+,\delta}{\operatorname{var}_I^+} F < \infty \text{ for some positive function } \delta. \tag{3.3}$$

With that δ the corresponding interval function

$$V_\delta(J) = F(J) - \underset{S^+,\delta}{\operatorname{var}_J^+} F \qquad (J \subseteq I) \tag{3.4}$$

is well-defined and subadditive on I; moreover it satisfies the following conditions

$$V_\delta \leq F \text{ and } D_I^+ V_\delta \leq D_I^+ F \text{ on } I, \tag{3.5}$$

$$D_I^+ V_\delta \leq 0 \text{ on } S^+, \tag{3.6}$$

so that V_δ is a minor function for $D_I^+ F$ on I.

PROOF. By Lemma 1 the interval function

$$G_\delta(J) = \underset{S^+,\delta}{\operatorname{var}_J^+} F \qquad (J \subseteq I)$$

is superadditive on I and satisfies $0 \leq G_\delta < \infty$ on I in view of (3.3). Hence (3.4) is well-defined, subadditive, and (3.5) is trivially satisfied. If $x \in S^+$ and if J is any regular subinterval of I satisfying $x \in J \subseteq B_{\delta(x)}(x)$ then the family consisting only of J is admissible with respect to (J, S^+, δ) which implies $F(J) \leq G_\delta(J)$, i.e. $V_\delta(J) \leq 0$. Hence $D_I^+ V_\delta(x) \leq 0$ follows.

Now we formulate the *fundamental inequality*.

THEOREM 3. *Suppose that F is a subadditive interval function on $I \subseteq \mathbb{R}^n$ with singular set S^+ as in (2.13). Then we have for each subinterval $J \subseteq I$*

$$F(J) \leq \int_J^- D_I^+ F + \operatorname*{var}_{S^+} {}_J^+ F, \tag{3.7}$$

whenever the sum on the right is well-defined.

PROOF. First we consider the case $J = I$, and we may assume $\operatorname*{var}_{S^+} {}_I^+ F < \infty$. Hence we may select δ as in (3.3) and by the construction principle we infer

$$F(I) - \operatorname*{var}_{S^+,\delta} {}_I^+ F = V_\delta(I) \leq \int_J^- D_I^+ F.$$

Taking the inf over these δ the inequality (3.7) follows. For general J we consider F as interval function on J and note that the corresponding singular $S_J^+ = \{x \in J : D_J^+ F = \infty\}$ is contained in S^+. We may assume

$$\operatorname*{var}_{S^+} {}_J^+ F < \infty, \qquad \text{hence } \operatorname*{var}_{S_J^+} {}_J^+ F < \infty,$$

and observe that the inequality already shown, i.e.

$$F(J) \leq \int_J^- D_J^+ F + \operatorname*{var}_{S_J^+} {}_J^+ F,$$

is stronger than (3.7). One can even show

$$\int_J^- D_J^+ F = \int_J^- D_I^+ F \text{ and } \operatorname*{var}_{S_J^+} {}_J^+ F = \operatorname*{var}_{S^+} {}_J^+ F$$

by applying Corollary 1″ with $I = J$.

If we apply Theorem 3 to $-F$ we obtain for any superadditive interval function F on I the inequality

$$F(J) \geq \int_J^+ D_I^- F - \operatorname*{var}_{S^-} {}_J^- F \tag{3.8}$$

for each subinterval $J \subseteq I$ for which the difference on the right is well-defined. Both inequalities, (3.7) and (3.8) are new. If we want the variational error in (3.7) to vanish all the time we should require

$$\operatorname*{var}_{S^+} {}_I^+ F = 0. \tag{3.9}$$

This condition implies $|S^+| = 0$ by (1.10). Hence (3.9) is equivalent to

$$D_I^+ F < \infty \text{ a.e. on } I \text{ and } F \in n_I^+. \tag{3.10}$$

Therefore we have the following

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COROLLARY 3′. *Suppose that F is a subadditive interval function on I satisfying $F \in n_I^+$. If $D_I^+F < \infty$ almost everywhere on I we have*

$$F(J) \leq \int_J^- D_I^+ F \text{ for all subintervals } J \subseteq I, \tag{3.11}$$

and $D_I^+F \leq 0$ almost everywhere on I implies $F \leq 0$ on I.

The inequality (3.11) is classical if $D_I^+F < \infty$ everywhere on I since in this case F itself is a minor function for D_I^+F on I. In this situation $S^+ = \emptyset$ so that $F \in n_I^+$ follows. In the literature there also is a discussion of certain cases where $D_I^+F < \infty$ is not satisfied throughout I, cf. Ridder [17], Besicovitch [2], Trjitzinsky [29]. If the exceptional set S^+ is countable we consider the following semi-continuity condition for each $x \in S^+$: for every sequence (J_k) of regular subintervals $J_k \subseteq I$ satisfying $x \in J_k$ and $|J_k| \longrightarrow 0$ we require $\limsup F(J_k) \leq 0$. This condition is equivalent to $F \in n_I^+$ in the present situation. Ridder uses a somewaht stronger condition since his subintervals need not be regular.

As an application we consider the inequality (3.2).

COROLLARY 3″. *Suppose that F is a subadditive interval function on I and that f is a P-integrable point function over I. Then we have*

$$F(J) \leq \int_J f \text{ for all subintervals } J \subseteq I \tag{3.12}$$

iff $F \in n_I^+$ and $D_I^+F \leq f$ almost everywhere on I.

PROOF. The interval function $G(J) = \int_J f (J \subseteq I)$ is differentiable almost everywhere on I relative to I and satisfies $G \in n_I^+$. Since $F \leq G$, the conditions in the corollary are clearly necessary. The sufficiency follows from Corollary 3′.

We like to emphasize the necessity of $F \in n_I^+$ and remark that the Corollary 3″ is already interesting for L-integrals since general exceptional sets S^+ are permitted.

Finally we ask under which conditions the integral of the derivative represents the original interval function. This leads to the

FUNDAMENTAL THEOREM. *Suppose that F is an additive interval function on I belonging to n_I which is differentiable a.e. on I, relative to I. Then D_IF is P-integrable over I, and we have*

$$F(J) = \int_J D_I F \text{ for all subintervals } J \subseteq I. \tag{3.13}$$

REMARK. By a theorem of Ward [33, Theorem 4], see e.g. Saks [26], the differentiability can be replaced by the weaker condition that $S^+ \cap S^-$ be a null set, where $S^\pm$ are defined by (2.13).

PROOF. By the last remark of Section 2 our F is finite on I. Now apply Corollary 3′ to F and $-F$ and infer

$$\int_J^+ D_I F \leq F(J) \leq \int_J^- D_I F \tag{3.14}$$

for all subintervals $J \subseteq I$. Hence the integral of $f = D_IF$ exists over each J, and (3.13) follows including the finiteness of the integrals.

4. **Characterization of integrals.** Let $P(I)$ denote the class of all point functions which are defined at least on I and which are P-integrable over I. An interval function F on I of the form

$$F(J) = \int_J f \text{ for all subintervals } J \subseteq I \tag{4.1}$$

with $f \in P(I)$ is called a weak Perron integral on I. Combining the results of Section 2 with the Fundamental Theorem we obtain the following characterization.

THEOREM 4. *An interval function F on I is a weak Perron integral on I iff F is additive, belongs to n_I and is differentiable a.e. on I relative to I.*

Further characterizations of the weak Perron integral will be given in a subsequent paper. Theorem 4 can also be used to characterize any process of integration which is stronger in the following sense. Consider a notion *Q-integration*, which assigns to each interval I a class $Q(I)$ of point functions, defined at least on I, called *Q-integrable onI*, and real numbers ${}^Q\int_I f$ for $f \in Q(I)$ such that each $f \in Q(I)$ is P-integrable over I and ${}^Q\int_I f = \int_I f$ holds. Examples of stronger integrations are: strong Perron integrations, Lebesgue integration etc. It is clear that Theorem 4 implies the following result.

THEOREM 4′. *An interval function F on I is of the form $F(J) = {}^Q\int_J f$ for all subintervals $J \subseteq I$ iff F is additive, belongs to n_I, and $D_I F$ exists a.e. on I and coincides a.e. on I with a function F on I which belongs to all $Q(J)$, $J \subseteq I$.*

In dimension one the usual Perron integration as discussed in Saks [26] is stronger than our P-integration. Hence the well-known condition (ACG_*) must imply our condition $F \in n_I$.

We owe to the referee the remark that in dimension one an analogous result occurs in the paper by J. Jarnik and J. Kurzweil. A general form of the product integral and ordinary differential equations, Czech. Math. J. 37 (112)1987, 642–659, see (3.19).

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