Some New Properties of the Triangle.

By J. S. MACKAY, M.A., LL.D.

[The substance of this communication will be included in Dr Mackay's paper on *The Triangle* in the first volume of the *Proceedings*, which is about to be printed.]

Proofs of some optical theorems.

By WILLIAM PEDDIE, D.Sc.

[The results of this paper will be contained in Dr Peddie's book on Physics, which will appear in a short time.]

Second Meeting, December 12th, 1890.

R. E. ALLARDICE, Esq., President, in the Chair.

On the condition that the straight line $lx + my + nz = 0$ should be a normal to the conic $(a, b, c, f, g, h)(x, y, z)^2 = 0$ the co-ordinates being trilinear.

By R. H. PINKERTON, M.A.

1. The condition in question may be found by using the following theorem:—

If the equation in trilinear co-ordinates

$$F(x, y, z) \equiv (u, v, w, u', v', w')(x, y, z)^2 = 0 \quad .. \quad (A)$$

represents a pair of straight lines, then the line whose equation is

$$lx + my + nz = 0 \quad \dots \quad \dots \quad \dots \quad (B)$$

will be perpendicular to one of those lines if

$$F(l - m\cos C - n\cos B,\ m - n\cos A - l\cos C,\ n - l\cos B - m\cos C) = 0 \quad (C)$$

where A, B, C are the angles of the fundamental triangle.

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To prove this, transform the equations (A) and (B) to Cartesian co-ordinates by writing, as usual,

$x,\ y,\ z = x\cos\alpha + y\sin\alpha - p_1,\quad x\cos\beta + y\sin\beta - p_2,\quad x\cos\gamma + y\sin\gamma - p_3,$
where $\beta - \gamma = 180° - A$, etc.

The equation (A) thus becomes in Cartesian co-ordinates

$$F(x\cos\alpha + y\sin\alpha - p_1,\ x\cos\beta + y\sin\beta - p_2,\ x\cos\gamma + y\sin\gamma - p_3) = 0,$$

and the equation to the pair of straight lines through the origin of co-ordinates parallel to the lines (A), is

$$F(x\cos\alpha + y\sin\alpha,\ x\cos\beta + y\sin\beta,\ x\cos\gamma + y\sin\gamma) = 0 \quad \ldots \quad (A').$$

The equation in Cartesian co-ordinates to the straight line through the origin parallel to the (B) is similarly

$$\lambda x + \mu y = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (B')$$

where $\lambda,\ \mu = l\cos\alpha + m\cos\beta + n\cos\gamma,\ l\sin\alpha + m\sin\beta + n\sin\gamma$.

Now the line (B) will be perpendicular to one of the lines (A) if the line (B′) is perpendicular to one of the lines (A′). The condition that (B′) should be perpendicular to one of the lines (A′) is found by substituting in the equation (A′) λ, μ for x, y. The line (B) will therefore be perpendicular to one of the lines (A) if

$$F(\lambda\cos\alpha + \mu\sin\alpha,\ \lambda\cos\beta + \mu\sin\beta,\ \lambda\cos\gamma + \mu\cos\gamma) = 0.$$

Replacing λ, μ by their values in terms of l, m, n, we get

$$\begin{aligned}\lambda\cos\alpha + \mu\cos\beta &= \cos\alpha(l\cos\alpha + m\cos\beta + n\cos\gamma)\\ &\quad + \sin\alpha(l\sin\alpha + m\sin\beta + n\sin\gamma)\\ &= l + m\cos(\alpha \sim \beta) + n\cos(\gamma \sim \alpha)\\ &= l - m\cos C - n\cos B,\end{aligned}$$

with similar values $\lambda\cos\beta + \mu\sin\beta$ and $\lambda\cos\gamma + \mu\sin\gamma$. Hence the theorem follows.

2. Taking now the conic

$$S \equiv (a,\ b,\ c,\ f,\ g,\ h)(x,\ y,\ z)^2 = 0$$

and the straight line

$$lx + my + nz = 0 \quad \ldots \quad \ldots \quad \ldots \quad (P),$$

we write down the equation to the pair of tangents to the conic at the points where the straight line cuts the conic. This equation is

$$S\Sigma = \Delta(lx + my + nz)^2 \quad \ldots \quad \ldots \quad (T),$$

where Σ is written for $(A,\ B,\ C,\ F,\ G,\ H)(l,\ m,\ n)^2$, and Δ, A, B, C, F, G, H have their usual meanings.

The line (P) will be a normal to the conic S if it is perpendicular to one of the lines (T). The condition for this is, by (C), found by substituting $l - m\cos C - n\cos B$, etc., for x, y, z in (T). The result is $(a, b, c, f, g, h)(l - m\cos C - n\cos B,\ m - n\cos A - l\cos C,\ n - l\cos B - m\cos C)^2 \times \Sigma = \Delta(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C)^2$, the condition sought for.

The triangle and its escribed parabolas.

By A. J. Pressland, M.A.

§ 1. The problem "to inflect a straight line between two sides of a triangle so that the intercepted portion is equal to the segments cut off" has been discussed in the third volume of the *Proceedings*.

If we discuss the same analytically; taking CB and CA as axes of x and y (Fig. 1) and calling each segment k, the equation of the line considered is

$$x/(a-k) + y/(b-k) = 1, \quad \ldots \quad \ldots \quad (\alpha)$$

where $$k^2 = (a-k)^2 + (b-k)^2 - 2(a-k)(b-k)\cos C \quad \ldots \quad (\beta)$$

The envelope of (α) considering k unrestricted by (β) is

$$(x+y)^2 - 2(a-b)(x-y) + (a-b)^2 = 0 \quad \ldots \quad (\gamma)$$

a parabola touching the axis of x at $(a-b, 0)$

and the axis of y at $(0, b-a)$

and which can be shown to touch AB

at the point $\left(\dfrac{a^2}{a-b}, -\dfrac{b^2}{a-b}\right)$.

Its axis is $x + y = 0$

and tangent at vertex $x - y = \dfrac{a-b}{2}$.

§ 2. If we consider $x/(a-k) + y/(b+k) = 1$ which cuts off equal portions from BC and CA produced, the envelope is

$$(x-y)^2 - 2(a+b)(x+y) - (a+b)^2 = 0.$$

which touches CB at $(a+b, 0)$ the point l,

CA at $(0, a+b)$ the point k,

AB at $\left(\dfrac{a^2}{a+b}, \dfrac{b^2}{a+b}\right)$ the point t,

the axis being $x - y = 0$

and tangent at vertex $x + y = \dfrac{a+b}{2}$.