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THE ARENS IRREGULARITY OF AN EXTREMAL ALGEBRA

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A class of extremal Banach algebras has Arens irregular multiplication.

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For any compact convex set $K \subset \mathbb{C}$ there is a unital Banach algebra $Ea(K)$ generated by an element h in which every polynomial in h attains its maximum norm over all Banach algebras subject to its numerical range $V(h)$ being contained in K, [**1**, **2**]. In [**3**] we showed that $Ea(K)$ does not have Arens regular multiplication when K is a line segment. Here we extend this to any other case, where a different argument is required.

Proposition. *If K has non-empty interior, then $Ea(K)$ is Arens irregular.*

Proof. We use Pym's criterion [**4**] that it is enough to find bounded sequences a_n, b_n in $Ea(K)$ and a bounded linear functional ϕ such that the two repeated limits of $\phi(a_m b_n)$ exist and differ.

First assume that $K = \bar{D}(0,\tau)$, where $\tau = 4/e$. As in [**2**], any entire function f such that $f(z)e^{-\tau|z|}$ is bounded on $\mathbb{C}$ gives a ϕ in $Ea(K)'$ by $\phi(e^{zh}) = f(z)$ $(z \in \mathbb{C})$. Define an entire function f by

$$f(z) = 4z^2 \prod_{p=1}^{\infty} \left[1 - \left(\frac{z}{2^p}\right)^{2^p}\right],$$

and put

$$g_n(z) = 4z^2 \prod_{p=1}^{n} \left(\frac{z}{2^p}\right)^{2^p} = \left(\frac{z}{2^{n-1}}\right)^{2^{n+1}} \quad (n \in \mathbb{N}).$$

Then

$$(-1)^n \frac{f(z)}{g_n(z)} = \prod_1^{n-1}\left[1-\left(\frac{2^p}{z}\right)^{2^p}\right]\cdot\left(1-\left(\frac{2^n}{z}\right)^{2^n}\right)$$

$$\times\left(1-\left(\frac{z}{2^{n+1}}\right)^{2^{n+1}}\right)\cdot\prod_{n+2}^{\infty}\left[1-\left(\frac{z}{2^p}\right)^{2^p}\right]$$

$$= A_n(z)B_n(z)C_n(z), \text{ say.} \tag{1}$$

Put $D_n = \{z \in \mathbb{C}: 2^n \leqq |z| \leqq 2^{n+1}\}$. Then $A_n(z) \to 1$ as $n \to \infty$ uniformly on D_n, since

$$\sum_1^{n-1}\left|\frac{2^p}{z}\right|^{2^p} \leqq \sum_1^{n-1} 2^{(p-n)2^p} \leqq (n-1)2^{2(1-n)} \to 0.$$

Further, $C_n(z) \to 1$ uniformly on D_n, since

$$\sum_{n+2}^{\infty} 2^{(n+1-p)2^p} \to 0.$$

Since also $|B_n(z)| \leqq 4(z \in D_n)$, we have $|f/g_n| < 5$ on D_n for all large enough n. Together with the fact that $g_n(r)e^{-\tau r} \leqq 1(r>0)$, this gives that $f(z)e^{-\tau|z|}$ is bounded on $\mathbb{C}$. Thus f defines ϕ in $Ea(K)'$ as described.

With some fixed $\alpha \in \mathbb{R}$, put $r_n = 2^{n-1}e + \alpha (n \in \mathbb{N})$. Then $r_n \in D_n$ for all large n, $B_n(r_n) \to 1$, and (1) gives that $(-1)^n f(r_n)/g_n(r_n) \to 1$.

Also, $\log[g_n(r_n)e^{-\tau r_n}] = 2^{n+1}\log(e+2^{1-n}\alpha) - 4r_n/e = 2^{n+1}\log(1+2^{1-n}\alpha/e) - 4\alpha/e \to 0$ as $n \to \infty$. Hence $g_n(r_n)e^{-\tau r_n} \to 1$, and

$$(-1)^n f(r_n)e^{-\tau r_n} \to 1 \text{ as } n \to \infty. \tag{2}$$

Now for $n \in \mathbb{N}$ put $\alpha_n = 2^{2n}e$, $\beta_n = 2^{2n-1}e$, and $a_n = e^{\alpha_n(h-\tau)}$, $b_n = e^{\beta_n(h-\tau)}$. Then a_n, $b_n \in Ea(K)$ with $\|a_n\| = \|b_n\| = 1$, and $\phi(a_m b_n) = f(\alpha_m + \beta_n)\ e^{-\tau(\alpha_m+\beta_n)}$. Since $\alpha_m + \beta_n = 2^{2n-1}e + \alpha_m$ can be taken as r_{2n} in (2), we have $\lim_{n\to\infty}\phi(a_m b_n) = 1$. With n fixed, $\alpha_m + \beta_n$ can be taken as r_{2m+1}, and (2) gives $\lim_{m\to\infty}\phi(a_m b_n) = -1$. Thus the repeated limits of $\phi(a_m b_n)$ differ, and $Ea(K)$ is Arens irregular.

Given any compact convex $K \subset \mathbb{C}$, by replacing K by $\alpha K + \beta$ for suitable α, $\beta \in \mathbb{C}$ we can assume that $\bar{D}(0,\tau) \subseteq K$ and $\operatorname{Re} K \leqq \tau$. Then we can construct ϕ, a_n, b_n in exactly the same way to complete the proof; for since $\|e^{zh}\| = \max\{|e^{\lambda z}| : \lambda \in K\}$ we still have $\|a_n\| = \|b_n\| = 1$ and $\phi \in Ea(K)'$.

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