The reduction after substituting for A, B, C, D is somewhat tedious, but we get, if $u > v$, for the value of the integral

$$4\pi e^{-mu}\frac{\cosh mv \sin m\phi \sin 2\phi(\cosh 2u - \cosh 2v) + \sinh mv \sinh 2v \cos m\phi}{c^2(\cosh 2u - \cosh 2v)(\cosh 2u - \cos 2\phi)}$$
$$\frac{(\cosh 2u - \cos 2\phi) + \cosh mv \sinh 2u \cos m\phi(\cosh 2v - \cos 2\phi)}{(\cosh 2v - \cos 2\phi)}$$

If $u < v$, then u and v must be interchanged.

If we suppose $a = b$, and $\alpha = \beta = \sqrt{x^2 + y^2} = f$ as before, then

$$\int_0^{2\pi}\frac{\cos m\theta d\theta}{f^2 - 2af\cos(\theta - \phi) + a^2} = \frac{2\pi f^m \cos m\phi}{a^m(a^2 - f^2)} \text{ or } \frac{2\pi a^m \cos m\phi}{f^m(f^2 - a^2)}$$

according as $f <$ or $> a$.

If we had taken $\phi = 0$, the integral would have been much more easily evaluated, since

$$\int_0^{2\pi}\cos m\theta . d\theta/(\mathrm{A} - \cos\theta)$$

is very easily evaluated.

Matthew Stewart's Theorem.

By J. S. Mackay, M.A., LL.D.

In 1746, when he was a candidate for the Chair of Mathematics in the University of Edinburgh rendered vacant by the death of Maclaurin, Matthew Stewart published his first work, *Some General Theorems of considerable use in the higher parts of Mathematics.* In the preface to it he states that "the theorems contained in the following sheets . . . . are entirely new, save one or two at most," but he does not specify the two. They are*

Theorem 1.

If in triangle ABC any straight line AD be drawn to BC, and DE, DF be drawn parallel to AC, AB, and meeting AB, AC in E, F, then

$$AD^2 + BD \cdot CD = AB \cdot AE + AC \cdot AF.$$

---

* The enunciations of these theorems have been modernised.

Theorem 2.

If in triangle ABC any straight line AD be drawn to BC, then

$$AB^2 \cdot CD + AC^2 \cdot BD = BC \cdot BD \cdot CD + AD^2 \cdot BC.$$

It is the second of these that it is usual to call Stewart's theorem, and Stewart employs the first one to prove it. There is good reason, however, for believing that the theorem, though it was first published by Stewart, was discovered by Robert Simson. In the second book of Simson's *Loci Plani*, which was published in 1749, the tenth lemma (p. 156) is to the following effect.*

If ABC be a triangle, and from the vertex to the base BC there be drawn any straight line AD, and DE be any straight line, then

$$\frac{AB^2 \cdot CD}{DE} + \frac{AC^2 \cdot BD}{DE} = \frac{BD^2 \cdot CD}{DE} + \frac{CD^2 \cdot BD}{DE} + \frac{AD^2 \cdot BC}{DE}.$$

After proving this, Simson adds that if BD, CD, DE are all equal, the lemma is the same as Prop. 122 of Book VII. of Pappus's *Mathematical Collection*; if CD, DE be equal, the lemma would be enunciated thus:

If from the vertex A of triangle ABC there be drawn AD to the base, then

$$AB^2 + \frac{AC^2 \cdot BD}{CD} = BD^2 + \frac{CD^2 \cdot BD}{CD} + \frac{AD^2 \cdot BC}{CD}$$

$$= BC \cdot BD \qquad + \frac{AD^2 \cdot BC}{CD}.$$

This is Stewart's theorem exactly.

Now in the appendix to the *Loci Plani* (p. 221) Simson states and proves another lemma.

Let AB be a straight line, C and D two points in it, C lying between A and B, and let CE be any straight line, then

$$\frac{AD^2 \cdot BC}{CE} + \frac{BD^2 \cdot AC}{CE} = \frac{AC^2 \cdot BC}{CE} + \frac{BC^2 \cdot AC}{CE} + \frac{CD^2 \cdot AB}{CE}$$

He then adds:†

---

* Simson's mode of statement has been somewhat modernised.

† It may be well to quote Simson's own words:

Lemmate hoc usus sum . . . antequam inveneram Lem. 10. *Alias autem ejus demonstrationes, olim, hortatu meo, excogitarunt discipuli quondam mei Dominus Jacobus Moor et Dominus Matthaeus Steuart . . . Et demonstratio quidem Dom.*

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"I made use of this lemma before I had discovered lemma 10. Other demonstrations of it were made out years ago by my former pupils Mr James Moor and Mr Matthew Stewart, whom I had exhorted to consider it. . . . Mr James Moor's demonstration is exactly similar to those of the 9th and 10th propositions of the second book of Euclid's *Elements*, and he has rightly remarked that these propositions are very simple cases of this lemma. Mr Matthew Stewart gave also another proof of the last case of lemma 10 in the first and second propositions of his book, *Some General Theorems, &c.*, and pointed out its use in the demonstration of some remarkable propositions."

This extract from Simson's *Loci Plani* and the extract from Stewart's preface are conclusive.

It may be added that Simson had probably discovered his theorem before 1741, because in the early part of that year he informed Matthew Stewart that he intended soon to publish his *Loci Plani.* Those who are curious to know why Simson did not carry out his intention may be referred to William Trail's *Account of the Life and Writings of Robert Simson*, pp. 24–26 (1812), and to a paper by T. S. Davies in the *Philosophical Magazine*, 3rd series, Vol. 33, pp. 201–6 (1848).

With regard to this celebrated theorem, which must now be attributed to Simson, a few historical remarks may be useful.

In 1752 Thomas Simpson proves* it in his *Select Exercises for Young Proficients in the Mathematicks.* In 1780 Euler demonstrates † it as a lemma in order to inscribe in a circle a triangle whose three sides pass through three given points.

In 1803 Carnot proves it and says ‡

"This theorem is very useful for finding the relations of the linear quantities of a figure without introducing into the calculation lineo-angular quantities. We shall often have occasion to make

---

Jac. Moor similis omnino est iis quae habentur in Prop. 9 *et Prop.* 10, *Lib.* 2 *Element. Euclidis, quas Lemmatis hujus casus esse simplicissimos recte observavit. Dom. Matth. Steuart aliam etiam ultimi casus Lem.* 10 *demonstrationem dedit in Prop.* 1 *et* 2 *libri sui de quibusdam Theorematibus generalibus, &c. . . . ejusque usum in eximiis quibusdam propositionibus demonstrandis ostendit.*

* See after Problem 31.

† *Acta Academiae . . . Petropolitanae*, Part I., pp. 92-3.

‡ *Géométrie de Position*, pp. 262-3.

use of it, and it should be regarded as fundamental." Chasles states* that the eight lemmas of Pappus on the *Loci Plani* of Apollonius can be derived from it as corollaries or easy consequences, and he gives a generalisation of a particular case of it. Numerous and interesting applications of the theorem will be found in M. Clément Thiry's *Applications remarquables du Théorème de Stewart et Théorie du Barycentre* (1891).

The following is Simson's demonstration of the first proposition in Stewart's *General Theorems.* It is not so simple as Stewart's, but it takes account of the case when D is in the base produced. It is extracted from an unpublished letter (in my possession) of Robert Simson to Matthew Stewart. The letter is dated 13th March 1741, and is written in English, but the demonstration is in Latin.

If from the vertex A of triangle ABC there be drawn AD to the base, and DE, DF be drawn from D parallel to AC, AB; then

$$BA \cdot AE \pm CA \cdot AF = BD \cdot DC \pm AD^2$$

FIGURES 32, 33.

Draw AG making $\angle$ GAF equal to $\angle$ ABC or $\angle$ FDG;
from D draw DH making $\angle$ ADH equal to $\angle$ AED;
then $\angle$ AHD = $\angle$ ADE = $\angle$ DAF.

Join GF.

Since $\angle$ GAF = $\angle$ FDG,
therefore the points G, D, A, F are concyclic.
Hence (fig. 32) $\angle$ FGC = $\angle$ DAF = $\angle$ AHD;
therefore $\angle$ DGF = $\angle$ DHB;
and (fig. 3) $\angle$ FGD = $\angle$ FAD = $\angle$ AHD;
therefore $\angle$ FGC = $\angle$ AHD.

Now in triangles DGF, BHD
$\angle$ FDG = $\angle$ HBD;
therefore BD : BH = DF : DG;
therefore BD·DG = BH·DF,
= BH·AE.

Again because $\angle$ ABC = $\angle$ GAC,
and $\angle$ BDE = $\angle$ GCA;
therefore BD : DE = AC : CG;

* *Aperçu Historique,* 2nd ed., pp. 175-6 (1875).

therefore $BD \cdot CG = AC \cdot DE,$
$= CA \cdot AF.$

Now $BD \cdot DG = BH \cdot AE;$

therefore $BH \cdot AE + CA \cdot AF = BD \cdot DG + BD \cdot GC$
$= BD \cdot DC.$

In triangles AHD, ADE

since $\angle ADH = \angle AED$, and $\angle DAH$ is common;

therefore (both figures) $HA \cdot AE = AD^2$;

therefore $BH \cdot AE + CA \cdot AF + HA \cdot AE = BD \cdot DC + AD^2;$

therefore $BA \cdot AE + CA \cdot AF = BD \cdot DC + AD^2.$

FIGURE 33.

$$BA \cdot AE + HA \cdot AE = BH \cdot AE + BD \cdot GC,$$
$$= BD \cdot DG + BD \cdot GC,$$
$$= BD \cdot DC + BD \cdot GC;$$

therefore $BA \cdot AE + AD^2 = BD \cdot DC + CA \cdot AF;$

therefore $BA \cdot AE - CA \cdot AF = BD \cdot DC - AD^2.$

It may also be pointed out that the lemma which Simson employed before he had discovered Lemma 10 of the *Loci Plani*, namely,

If AB be a straight line, C and D two points in it, C lying between A *and* B, *then*

$$AD^2 \cdot BC + BD^2 \cdot AC = AC^2 \cdot BC + BC^2 . AC + CD^2 \cdot AB$$

contains a theorem given by Euler in the *Novi Commentarii Academiae . . . Petropolitanae*, vol. i., p. 49 (1747).

For $(AD^2 - AC^2)BC + (BD^2 - BC^2)AC = CD^2 \cdot AB.$

But $AD - AC \quad = BD + BC \quad = CD;$

therefore $(AD + AC)BC + (BD - BC)AC = CD \cdot AB;$

therefore $AD \; BC + \quad BD \cdot AC = CD \cdot AB;$

which is Euler's theorem.

Note on a property of a quadrilateral.

By Professor J. E. A. STEGGALL.

The property is that if any quadrilateral, ABCD, skew or otherwise, have its sides AB, DC divided in E, F so that

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