Compositio Mathematica **120:** 1–23, 2000.

Special Cubic Fourfolds

BRENDAN HASSETT*
University of Chicago, Department of Mathematics, 5734 University Avenue, Chicago, IL 60637-1546, U.S.A. e-mail: hassett@math.uchicago.edu.

(Received: 7 April 1997; accepted in final form: 28 September 1998)

Abstract. A cubic fourfold is a smooth cubic hypersurface of dimension four; it is *special* if it contains a surface not homologous to a complete intersection. Special cubic fourfolds form a countably infinite union of irreducible families $\mathcal{C}_d$, each a divisor in the moduli space $\mathcal{C}$ of cubic fourfolds. For an infinite number of these families, the Hodge structure on the nonspecial cohomology of the cubic fourfold is essentially the Hodge structure on the primitive cohomology of a K3 surface. We say that this K3 surface is *associated* to the special cubic fourfold. In these cases, $\mathcal{C}_d$ is related to the moduli space $\mathcal{N}_d$ of degree d K3 surfaces. In particular, $\mathcal{C}$ contains infinitely many moduli spaces of polarized K3 surfaces as closed subvarieties. We can often construct a correspondence of rational curves on the special cubic fourfold parametrized by the K3 surface which induces the isomorphism of Hodge structures. For infinitely many values of d, the Fano variety of lines on the generic cubic fourfold of $\mathcal{C}_d$ is isomorphic to the Hilbert scheme of length-two subschemes of an associated K3 surface.

Mathematics Subject Classifications (1991): 14C30, 14J28, 14J35.

Key words: K3 surfaces, Noether–Lefschetz loci, period domains.

1. Introduction

Let X be a special cubic fourfold, h its hyperplane class, and T the class of an algebraic surface not homologous to any multiple of h^2. The discriminant d is defined as the discriminant of the saturated lattice spanned by h^2 and T. Let $\mathcal{C}_d$ denote the special cubic fourfolds of discriminant d (Section 3.2).

THEOREM 1.0.1 (Classification of Special Cubic Fourfolds) (Theorems 3.1.2, 3.2.3, and 4.3.1). $\mathcal{C}_d \subset \mathcal{C}$ *is an* irreducible *divisor and is nonempty iff* $d > 6$ *and* $d \equiv 0, 2$ $(mod\ 6)$.

In Section 4, we give concrete descriptions of special cubic fourfolds with small discriminants and explain how certain Hodge structures at the boundary of the period domain arise from singular cubic fourfolds.

* This work was supported by Harvard University, a National Science Foundation Graduate Fellowship, and a Sloan Foundation Dissertation Fellowship. This paper was revised while the author was visiting the Institut Mittag–Leffler.

The *nonspecial cohomology* of a special cubic fourfold consists of the middle cohomology orthogonal to the distinguished classes h^2 and T. In many cases, it is essentially the primitive cohomology of a K3 surface of degree d, which is said to be *associated* to the special cubic fourfold. Furthermore, the varieties $\mathcal{C}_d$ are often closely related to moduli spaces of polarized K3 surfaces. Let $\mathcal{C}_d^{\mathrm{mar}}$ denote the marked special cubic fourfolds of discriminant d (Section 5.2). This is the normalization of $\mathcal{C}_d$ if $d \equiv 2 \pmod 6$ and is a double cover of the normalization otherwise.

THEOREM 1.0.2 (Associated K3 Surfaces and Maps of Moduli Spaces). (Theorems 5.1.3 and 5.2.4). *Special cubic fourfolds of discriminant d have associated K3 surfaces iff d is not divisible by four, nine, or any odd prime $p \equiv -1 \pmod 3$. In these cases, there is in open immersion of $\mathcal{C}_d^{\mathrm{mar}}$ into the moduli space of polarized K3 surfaces of degree d.*

In particular, an *infinite* number of moduli spaces of polarized K3 surfaces may be realized as moduli spaces of special cubic fourfolds.

We can explain geometrically the existence of certain associated K3 surfaces. The *Fano variety* of X parametrizes the lines contained in it. For certain special cubic fourfolds these Fano varieties are closely related to K3 surfaces.

THEOREM 1.0.3 (Geometry of Fano Varieties). (Theorem 6.1.4). *Assume that $d = 2(n^2 + n + 1)$ where n is an integer $\geqslant 2$, and let X be a generic special cubic fourfold of discriminant d. Then the Fano variety of X is isomorphic to the Hilbert scheme of length-two subschemes of a K3 surface associated to X.*

We should point out that the hypothesis on d is stronger than necessary, but simplifies the proof considerably. Combining this with the results on maps of moduli spaces, we obtain examples of distinct K3 surfaces with isomorphic Hilbert schemes of length-two subschemes (Proposition 6.2.2)

One motivation for this work is the rationality problem for cubic fourfolds. The Hodge structures on cubic fourfolds and their relevance to rationality questions have previously been studied by Zarhin [30]. Izadi [15] also studied Hodge structures on cubic hypersurfaces with a view toward rationality questions. All the examples of cubic fourfolds known to be rational ([10] [27] [5] [28]) are special and have associated K3 surfaces. Indeed, a birational model of the K3 surface is blown up in the birational map from $\mathbb{P}^4$ to the cubic fourfold. Is this the case for all rational cubic fourfolds? In a subsequent paper [14], we shall apply the methods of this paper to give new examples of rational cubic fourfolds. We show there is a countably infinite union of divisors in $\mathcal{C}_8$ parametrizing rational cubic fourfolds ($\mathcal{C}_8$ corresponds to the cubic fourfolds containing a plane).

Throughout this paper we work over $\mathbb{C}$. We use the term 'generic' to mean 'in the complement of some Zariski closed proper subset.' The term 'lattice' will denote a free abelian group equipped with a nondegenerate symmetric bilinear form.

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

2. Hodge Theory of Cubic Fourfolds

2.1. COHOMOLOGY AND THE ABEL–JACOBI MAP

Let X be a smooth cubic fourfold. The Hodge diamond of X has the form:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&0&&0&&& \\ &&0&&1&&0&& \\ &0&&0&&0&&0& \\ 0&&1&&21&&1&&0 \end{array}$$

Let L denote the cohomology group $H^4(X, \mathbb{Z})$, L^0 the primitive cohomology $H^4(X, \mathbb{Z})^0$, and $\langle , \rangle$ the symmetric nondegenerate intersection form on L. If h is the hyperplane class then $h^2 \in L$ and $L^0 = (h^2)^\perp$.

Our best tool for understanding the middle cohomology of X is the Abel–Jacobi mapping. Let F be the Fano variety of lines of X, the subvariety of the Grassmannian $\mathbb{G}(1, 5)$ parametrizing lines contained in X. This is a smooth fourfold ([1] Section 1). Let $Z \subset F \times X$ be the 'universal line', with projections p and q. The *Abel–Jacobi map* is defined as the map

$$\alpha = p_* q^* \colon H^4(X, \mathbb{Z}) \to H^2(F, \mathbb{Z}).$$

Let $M = H^2(F, \mathbb{Z})$, M^0 the primitive cohomology, and g the class of the hyperplane on F (induced from the Grassmannian). Recall that $\alpha(h^2)$ corresponds to the lines meeting a codimension-two subspace of $\mathbb{P}^5$, so $\alpha(h^2) = g$. Following [3] and [5], we define the *Beauville canonical form* $(,)$ on M so that g and M^0 are orthogonal, $(g, g) = 6$, and $(x, y) = \frac{1}{6} g^2 xy$ for $x, y \in M^0$. Extending by linearity we obtain an integral form on all of M.

PROPOSITION 2.1.1 (Beauville–Donagi [4] Proposition 6). *The Abel–Jacobi map induces an isomorphism between L^0 and M^0; moreover, for $x, y \in L^0$ we have $(\alpha(x), \alpha(y)) = -\langle x, y \rangle$.*

Indeed, we may interpret α is an isomorphism of Hodge structures $H^4(X, \mathbb{C})^0 \to H^2(F, \mathbb{C})^0(-1)$. The -1 means that the weight is shifted by two; this reverses the sign of the intersection form.

PROPOSITION 2.1.2. *The middle integral cohomology lattice of a cubic fourfold is $L \cong (+1)^{\oplus 21} \oplus (-1)^{\oplus 2}$ i.e. the intersection form is diagonalizable over $\mathbb{Z}$ with entries ± 1 along the diagonal. The primitive cohomology is $L^0 \cong B \oplus H \oplus H \oplus E_8 \oplus E_8$, where*

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

is the hyperbolic plane, and E_8 is the positive definite quadratic form associated to the corresponding Dynkin diagram.

We first prove the statement on the full cohomology. L is unimodular by Poincaré duality and has signature $(21, 2)$ by the Riemann bilinear relations. L is odd because $\langle h^2, h^2\rangle = h^4 = 3$. Using the theory of indefinite quadratic forms (e.g. [25] chapter 5 Section 2.2) we conclude the result.

Now we turn to the primitive cohomology L^0. By Proposition 2.1.1 it suffices to compute M^0; we first compute M. In [5] Proposition 6, it is shown that F is a deformation of a variety $S^{[2]}$, where S is a degree-fourteen K3 surface and $S^{[2]}$ denotes the Hilbert scheme of length-two zero-dimensional subschemes of S (also called the *blown-up symmetric square* of S). By [3] Section 6 we have the canonical orthogonal decomposition $H^2(S^{[2]}, \mathbb{Z}) = H^2(S, \mathbb{Z}) \oplus_{\perp} \mathbb{Z}\delta$, where $(\delta, \delta) = -2$ and the restriction of $(,)$ to $H^2(S, \mathbb{Z})$ is the intersection form. Geometrically, the divisor 2δ corresponds to the nonreduced length-two subschemes of S. The cohomology lattice of a K3 surface is well-known (cf. [16] Proposition 1.2) $H^2(S, \mathbb{Z}) \cong \Lambda := H^{\oplus 3} \oplus (-E_8)^{\oplus 2}$, so $M \cong H^{\oplus 3} \oplus (-E_8)^{\oplus 2} \oplus (-2)$. Furthermore, the polarization $g = 2f - 5\delta$, where $f \in H^2(S, \mathbb{Z})$ satisfies $(f, f) = 14$ [5]. The automorphisms of $H^2(S, \mathbb{Z})$ act transitively on the primitive vectors of a given nonzero length ([16] Theorem 2.4). If v_1 and w_1 are elements of the first summand H with $(v_1, v_1) = (w_1, w_1) = 0$ and $(v_1, w_1) = 1$, then we may take $f = v_1 + 7w_1$ and $g = 2v_1 + 14w_1 - 5\delta$. Using $v_1 + 3w_1 - 2\delta$ and $\delta - 5w_1$ as the first two elements of a basis of M^0, we obtain the result. □

Remark. Note that our computation shows that L^0 is even.

2.2. HODGE THEORY AND THE TORELLI MAP

We review Hodge theory in the context of cubic fourfolds; a general introduction to Hodge theory is [12]. Recall that a complete marking of a polarized cubic fourfold is an isomorphism $\phi \colon H^4(X, \mathbb{Z}) \to L$ mapping the square of the hyperplane class to $h^2 \in L$. If we are given a complete marking, the complex structure on X determines a distinguished subspace $F^3(X) = H^{3,1}(X, \mathbb{C}) \subset L^0_{\mathbb{C}}$ satisfying the following properties:

(1) $F^3(X)$ is isotropic with respect to the intersection form $\langle , \rangle$;
(2) the Hermitian form $H(u, v) = -\langle u, \bar{v}\rangle$ on $F^3(X)$ is positive.

Let $Q \subset \mathbb{P}(L^0_{\mathbb{C}})$ be the quadric hypersurface defined by (1), and let $U \subset Q$ be the topologically open subset where (2) also holds. U is a homogeneous space for the real Lie group $\mathrm{SO}(L^0_{\mathbb{R}}) = \mathrm{SO}(20, 2)$. This group has two components; one of them reverses the orientation on the negative definite part of $L^0_{\mathbb{R}}$, which coincides with $(F^3 \oplus \overline{F^3}) \cap L^0_{\mathbb{R}}$. Changing the orientation corresponds to exchanging F^3 and $\overline{F^3}$ (see Section 6 of the appendix to [24] for details). Hence the two connected

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

components of U parametrize the subspaces F^3 and $\overline{F^3} = H^{1,3}(X)$ respectively; we denote them $\mathcal{D}'$ and $\overline{\mathcal{D}'}$. The component $\mathcal{D}'$ is a twenty-dimensional open complex manifold, called the *local period domain* for cubic fourfolds.

Let Γ denote the group of automorphisms of L preserving the intersection form and the distinguished class h^2, and $\Gamma^+ \subset \Gamma$ the subgroup stabilizing $\mathcal{D}'$. This is the index-two subgroup of Γ which preserves the orientation on the negative definite part of $L^0_{\mathbb{R}}$. Γ^+ acts holomorphically on $\mathcal{D}'$ from the left; for a point in $\mathcal{D}'$ corresponding to the marked cubic fourfold (X, ϕ) the action is $\gamma(X, \phi) = (X, \gamma \circ \phi)$. The orbit space $\mathcal{D} = \Gamma^+ \backslash \mathcal{D}'$ exists as an analytic space and is called the *global period domain*.

Two cubic fourfolds are isomorphic iff they are projectively equivalent. Let $\mathcal{C}$ denote the coarse moduli space for smooth cubic fourfolds, constructed as a Geometric Invariant Theory quotient [18] chapter 4 Section 2. Each cubic fourfold determines a point in $\mathcal{D}$, and the corresponding map $\tau \colon \mathcal{C} \longrightarrow \mathcal{D}$ is called the *period map*. By general results of Hodge theory, this is a holomorphic map of twenty-dimensional analytic spaces. For cubic fourfolds we can say much more. First, we have the following result due to Voisin.

THEOREM 2.2.1 (Torelli Theorem for Cubic Fourfolds[29]). *$\tau \colon \mathcal{C} \to \mathcal{D}$ is an open immersion of analytic spaces.*

In particular, if X_1 and X_2 are cubic fourfolds and there exists an isomorphism of Hodge structures $\psi \colon H^4(X_1, \mathbb{C}) \to H^4(X_2, \mathbb{C})$, then X_1 and X_2 are isomorphic. Second, τ is not just an analytic map.

PROPOSITION 2.2.2. *$\mathcal{D}$ is a quasi-projective variety of dimension twenty and $\tau \colon \mathcal{C} \to \mathcal{D}$ is an algebraic map.*

In Section 6 of the appendix to [24], it is shown that the manifold $\mathcal{D}'$ is a bounded symmetric domain of type IV. The group Γ^+ is arithmetically defined and acts holomorphically on $\mathcal{D}'$. In this situation we may introduce the Borel–Baily compactification ([2] Section 10): there exists a compactification of $\mathcal{D}'$, compatible with the action of Γ^+, so that the quotient is projective. Moreover, $\Gamma^+ \backslash \mathcal{D}'$ is a Zariski open subvariety of this quotient. To complete the proof, we use the following consequence of A. Borel's extension theorem [6]

Let D' be a bounded symmetric domain, and G an arithmetically defined torsion-free group of automorphisms. Let $D = G \backslash D'$ be the quasi-projective quotient space, and Z an algebraic variety. Then any holomorphic map $Z \to D$ is algebraically defined.

While Γ^+ has torsion, some normal subgroup H of finite index is torsion-free ([24] IV Lemma 7.2). Let $\Gamma^+(N)$ denote the subgroup of Γ^+ acting trivially on L/NL. For some large N, Γ^+/H acts faithfully on L/NL so $\Gamma^+(N) \subset H$ and is torsion-free. Let $\mathcal{C}(N)$ denote the moduli space of cubic fourfolds with marked $\mathbb{Z}/N\mathbb{Z}$

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

cohomology. This is a finite (and perhaps disconnected) cover of $\mathcal{C}$; we use $\mathcal{C}^0(N)$ to denote a connected component. Let $\mathcal{D}(N) = \Gamma^+(N)\backslash\mathcal{D}'$, which is also finite over $\mathcal{D}$. The period map lifts to a map $\tau_N\colon \mathcal{C}^0(N) \to \mathcal{D}(N)$. By Borel's theorem τ_N is algebraic, and a descent argument implies τ is also algebraic. □

Remark. It follows that $\mathcal{C}$ is a Zariski open subset of $\mathcal{D}$ and its complement is defined by algebraic equations.

3. Special Cubic Fourfolds

3.1. BASIC DEFINITIONS

DEFINITION 3.1.1. *A cubic fourfold X is* special *if it contains an algebraic surface T not homologous to a complete intersection.*

Let $A(X) = H^{2,2}(X) \cap H^4(X,\mathbb{Z})$, which is positive definite by the Riemann bilinear relations. The Hodge conjecture is true for cubic fourfolds [31], so $A(X)$ is generated (over $\mathbb{Q}$) by the classes of algebraic cycles; X is special if and only if the rank of $A(X)$ is at least two. This is equivalent to saying that the rank of $L \cap F^3(X)^\perp$ is at least two, or that $L^0 \cap F^3(X)^\perp \neq 0$. A Hodge structure $x \in \mathcal{D}'$ is *special* if $L^0 \cap F^3(x)^\perp$ is nonzero.

THEOREM 3.1.2 (Structure of Special Cubic Fourfolds). *Let $K \subset L$ be a positive definite rank-two saturated sublattice containing h^2, $[K]$ the Γ^+ orbit of K, and $\mathcal{C}_{[K]}$ the cubic fourfolds X such that $A(X) \supset K'$ for some $K' \in [K]$. Every special cubic fourfold is contained in some $\mathcal{C}_{[K]}$, which is an irreducible algebraic divisor of $\mathcal{C}$, and is nonempty for all but a finite number of $[K]$.*

Given such a lattice K, we set $K^0 = K \cap L^0$. Let $\mathcal{D}'_K$ be the $x \in \mathcal{D}'$ such that $K^0 \subset x^\perp$; this is a hyperplane section of $\mathcal{D}' \subset \mathbb{P}(L^0_{\mathbb{C}})$. Each special Hodge structure is contained in some $\mathcal{D}'_K$. Let $K^\perp$ denote the orthogonal complement to K in L. We see that $\mathcal{D}'_K$ is a topologically open subset of a quadric hypersurface in $\mathbb{P}(K^\perp_{\mathbb{C}})$, has dimension nineteen, and classifies Hodge structures structures on the lattice $K^\perp$. As in the previous section, we can prove that $\mathcal{D}'_K$ is a bounded symmetric domain of type IV. Let $\Gamma^+_K = \{\gamma \in \Gamma^+ \colon \gamma(K) \subset K\}$. As before, the quotient $\Gamma^+_K\backslash\mathcal{D}'_K$ is quasi-projective. Furthermore, the induced holomorphic map $\Gamma^+_K\backslash\mathcal{D}'_K \to \Gamma^+\backslash\mathcal{D}' = \mathcal{D}$ is algebraically defined, so its image is an irreducible algebraic divisor.

We enumerate the divisors parametrizing special Hodge structures in $\mathcal{D}$. Each one corresponds to $\Gamma^+_K\backslash\mathcal{D}'_K$ for some $K \subset L$ as above, but K is not uniquely determined. K_1 and K_2 give rise to the same divisor if and only if $K_1 = \gamma(K_2)$ for some $\gamma \in \Gamma^+$, i.e. $\Gamma^+_{K_1}$ and $\Gamma^+_{K_2}$ are conjugate in Γ^+. Let $\mathcal{D}_{[K]}$ denote the corresponding irreducible divisor in $\mathcal{D}$. Since $\mathcal{C}$ is *Zariski* open in $\mathcal{D}$ (Proposition 2.2.2), $\mathcal{C}_{[K]} = \mathcal{C} \cap \mathcal{D}_{[K]}$ is an irreducible algebraic divisor in $\mathcal{C}$, and $\mathcal{D}_{[K]} \subset (\mathcal{D} - \mathcal{C})$ for finitely many $[K]$. □

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

DEFINITION 3.1.3. Let $(K, \langle , \rangle)$ be a positive definite rank-two lattice containing a distinguished element h^2 with $\langle h^2, h^2 \rangle = 3$. A *marked* (resp. *labelled*) special cubic fourfold is a cubic fourfold X with the data of a primitive imbedding of lattices $K \hookrightarrow A(X)$ preserving h^2 (resp. the image of such an imbedding.) A special cubic fourfold is *typical* if it has a unique labelling.

We write $\mathcal{D}^{\mathrm{lab}}_{[K]}$ for $\Gamma^+_K \backslash \mathcal{D}'_K$. The morphism $\mathcal{D}^{\mathrm{lab}}_{[K]} \to \mathcal{D}_{[K]}$ is birational (indeed $\mathcal{D}^{\mathrm{lab}}_{[K]}$ is the normalization of $\mathcal{D}_{[K]}$), so a general point in $\mathcal{D}_{[K]}$ has a unique labelling. The fiber product $\mathcal{D}^{\mathrm{lab}}_{[K]} \times_{\mathcal{D}} \mathcal{C}$ will be denoted $\mathcal{C}^{\mathrm{lab}}_{[K]}$.

3.2. DISCRIMINANTS AND SPECIAL CUBIC FOURFOLDS

DEFINITION 3.2.1. The *discriminant* of a labelled special cubic fourfold (X, K) is the determinant of the intersection matrix of K.

PROPOSITION 3.2.2. *Let (X, K) be a labelled special cubic fourfold of discriminant d and let v be a generator of K^0.*

(1) $d > 0$ *and* $d \equiv 0, -1 \pmod 3$,

(2) $d' := \langle v, v \rangle = \begin{cases} 3d & \text{if } d \equiv -1 \pmod 3, \\ \frac{d}{3} & \text{if } d \equiv 0 \pmod 3, \end{cases}$

(3) $\langle v, L^0 \rangle = \begin{cases} 3\mathbb{Z} & \text{if } d \equiv -1 \pmod 3, \\ \mathbb{Z} & \text{if } d \equiv 0 \pmod 3, \end{cases}$

(4) d *is even.*

The first three statements are straightforward computations, so we omit their proofs. The fourth follows from the remark after Proposition 2.1.2. □

We refine the results of the previous section by classifying the orbits of the rank-two sublattices under the action of Γ^+. The following theorem is a consequence of Theorem 3.1.2 and Proposition 3.2.4.

THEOREM 3.2.3 (Irreducibility Theorem). *The special cubic fourfolds possessing a labelling of discriminant d form an irreducible (possibly empty) algebraic divisor $\mathcal{C}_d \subset \mathcal{C}$.*

Elements of $\mathcal{C}_d$ are called *special cubic fourfolds of discriminant d*; the corresponding rank-two lattice is denoted K_d. We write $\mathcal{D}_d$ for $\mathcal{D}_{[K_d]}$, $\mathcal{D}^{\mathrm{lab}}_d$ for $\mathcal{D}^{\mathrm{lab}}_{[K_d]}$, $\mathcal{C}_d$ for $\mathcal{C}_{[K_d]}$, and $\mathcal{C}^{\mathrm{lab}}_d$ for $\mathcal{C}^{\mathrm{lab}}_{[K_d]}$.

PROPOSITION 3.2.4. *Let K and K' be saturated rank-two nondegenerate sublattices of L containing h^2. Then $K = \gamma(K')$ for some $\gamma \in \Gamma^+$ if and only if K and K' have the same discriminant.*

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

We claim it suffices to prove the result for Γ. We find some $g \in \Gamma - \Gamma^{+}$ stabilizing sublattices with every possible discriminant. Take g to be the identity except on the second hyperbolic plane in the orthogonal decomposition for L^0; on this component set g equal to multiplication by -1. (We refer to the computation of L^0 in Proposition 2.1.2)

Now we analyze the action of Γ on our rank-two sublattices, or equivalently, on saturated nondegenerate rank-one sublattices $K^0 \subset L^0$. We apply the results of Nikulin on discriminant groups and quadratic forms; see [22] or [9] for basic definitions and proofs. The elements of Γ fix h^2, so they act trivially on the discriminant groups $d(\mathbb{Z}h^2)$ and $d(L^0)$ [22] Section 1.5. Conversely, any automorphism of L^0 that acts trivially on $d(L^0)$ extends to an element of Γ [22] 1.5.1.

Let K^0 denote a lattice generated by an element v with $\langle v, v\rangle = d'$, q_{K^0} the quadratic form on $d(K^0)$, and q the quadratic form on $d(L^0)$. The lattice L^0 is the unique even lattice of signature $(20, 2)$ with discriminant quadratic form q [22] 1.14.3. Any saturated codimension-one sublattice $K^{\perp} \subset L^0$ is the orthogonal complement in L of a rank-two sublattice K, so there is an induced isomorphism $d(K^{\perp}) \cong d(K)$ [22] 1.6.1, and $d(K^{\perp})$ is generated by at most two elements. This implies the isomorphism class of $K^{\perp}$ is determined by its signature and discriminant form, and any isomorphism of $d(K^{\perp})$ preserving the discriminant quadratic form is induced by an automorphism of $K^{\perp}$ [22] 1.14.3.

Two primitive imbeddings of $i\colon K^0 \to L^0$ differing only by an element of Γ are said to be *congruent*. Applying the results of [22] Section 1.15 in our situation, we find the primitive imbeddings $i\colon K^0 \to L^0$ correspond to the following data:

(1) a subgroup $H_q \subset d(L^0)$,
(2) a subgroup $H_{K^0} \subset d(K^0)$,
(3) an isomorphism $\phi : H_{K^0} \to H_q$ preserving the restrictions of the quadratic forms to these subgroups, with graph $\Gamma_{\phi} = \{(h, \phi(h))\colon h \in H_{K^0}\} \subset d(K^0) \oplus d(L^0)$,
(4) an even lattice $K^{\perp}$ with complementary signature and discriminant form $q_{K^{\perp}}$, and an isomorphism $\phi_{K^{\perp}}\colon q_{K^{\perp}} \to -\delta$, where $\delta = ((q_{K^0} \oplus -q)|\Gamma_{\phi}^{\perp})/\Gamma_{\phi}$ (and $\Gamma_{\phi}^{\perp}$ is the orthogonal complement to Γ_{ϕ} with respect to $q_{K^0} \oplus q$).

Another imbedding i' with data $(H'_q, H'_{K^0}, \phi', (K')^{\perp}, \phi_{(K')^{\perp}})$ is congruent to i if and only if $H_{K^0} = H'_{K^0}$ and $\phi = \phi'$.

Our proof now divides into two cases. In the first case $H_q = \{0\}$, or equivalently, $\langle i(K^0), L^0)\rangle = \mathbb{Z}$ (i.e. $3|d$). By the characterization above, all primitive imbeddings of K^0 are congruent. In the second case $H_q = d(L^0) \cong \mathbb{Z}/3\mathbb{Z}$, or equivalently, $\langle i(K^0), L^0)\rangle = 3\mathbb{Z}$. In this case, $d(K^0)$ has a subgroup H_{K^0} of order three and $3|d'$. There are two possible isomorphisms between $d(L^0)$ and H_{K^0}, thus two congruence classes of imbeddings of K^0 into L^0. □

Using [22] Section 1.15 and Proposition 3.2.2, we can compute the discriminant quadratic forms of the lattices $K_d^{\perp}$:

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

PROPOSITION 3.2.5. *If $d \equiv 0 \pmod 6$ then $d(K_d^{\perp}) \cong \mathbb{Z}/\frac{d}{3}\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, which is cyclic unless $9|d$. We may choose this isomorphism so that $q_{K_d^{\perp}}(0,1) = \frac{2}{3} \pmod{2\mathbb{Z}}$ and $q_{K_d^{\perp}}(1,0) = -\frac{3}{d} \pmod{2\mathbb{Z}}$. If $d \equiv 2 \pmod 6$ then $d(K_d^{\perp}) \cong \mathbb{Z}/d\mathbb{Z}$. We may choose a generator u so that $q_{K_d^{\perp}}(u) = \frac{2d-1}{3d} \pmod{2\mathbb{Z}}$.*

4. Examples

4.1. SPECIAL CUBIC FOURFOLDS WITH SMALL DISCRIMINANTS

The examples here are discussed in more detail in [13]. If T is a smooth surface contained in a cubic fourfold X then $\langle T, T\rangle = c_2(\mathcal{N}_{T/X}) = 6h_T^2 + 3h_T K_T + K_T^2 - \chi_T$ where χ_T is the topological Euler characteristic and h_T is the hyperplane class restricted to T.

4.1.1. $d = 8$: *Cubic fourfolds containing a plane* (see [29])

X contains a plane P, so that $\langle P, P\rangle = 3$ and our marking is

$$K_8 = \begin{array}{c|cc} & h^2 & P \\ \hline h^2 & 3 & 1 \\ P & 1 & 3 \end{array}.$$

The cubic fourfolds in $\mathcal{C}_8$ generally contain other surfaces, like quadric surfaces and quartic del Pezzo surfaces.

4.1.2. $d = 12$: *Cubic fourfolds containing a cubic scroll*

X contains a rational normal cubic scroll T, so that $\langle T, T\rangle = 7$ and our marking is

$$K_{12} = \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 3 \\ T & 3 & 7 \end{array}.$$

4.1.3. $d = 14$: *Cubic fourfolds containing a quartic scroll/Pfaffian cubic fourfolds*

X is a cubic fourfold containing a rational normal quartic scroll T, so that $\langle T, T\rangle = 10$ and our marking is

$$K_{14} = \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 4 \\ T & 4 & 10 \end{array}.$$

Special cubic fourfolds of discriminant 14 generally also contain quintic del Pezzo surfaces and quintic rational scrolls. One can show that the quartic scrolls, quintic

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

scrolls, and quintic del Pezzos on X form families of dimensions two, two, and five respectively. Note that Morin [17] uses a spurious parameter count to deduce that the quartic scrolls form a *one* dimensional family. From this, he concludes incorrectly that *every* cubic fourfold contains a quartic scroll.

Another description of an open subset of $\mathcal{C}_{14}$ is the Pfaffian construction of Beauville and Donagi [5]. The dimension counts above follow easily from their results. They also show that the Pfaffian cubic fourfolds are rational. Finally, we should point out that the cubic fourfolds containing two disjoint planes possess a marking with discriminant 14, and thus are also contained in $\mathcal{C}_{14}$. (See [10] and [27] for more discussion of these examples.)

4.1.4. $d = 20$: *Cubic fourfolds containing a Veronese*

X contains a Veronese surface V, so that $\langle V, V\rangle = 12$ and our marking is

$$K_{20} = \begin{array}{c|cc} & h^2 & V \\ \hline h^2 & 3 & 4 \\ V & 4 & 12 \end{array}.$$

4.2. $d = 6$: CUBIC FOURFOLDS WITH DOUBLE POINTS

A double point is *ordinary* if its projectivized tangent cone is smooth. Cubic hypersurfaces in $\mathbb{P}^5$ with an ordinary double point are stable in the sense of Geometric Invariant Theory. This is proved using Mumford's numerical criterion for stability ([18] Section 2.1) and the methods of ([18] Section 4.2). Let $\tilde{\mathcal{C}}$ denote the quasi-projective variety parametrizing cubic fourfolds with (at worst) a single ordinary double point.

Let X_0 be a cubic fourfold with a single ordinary double point p. Projection from p gives a birational map $\pi_p\colon X_0 \to \mathbb{P}^4$ which can be factored

$$\begin{array}{ccc} \overline{X_0} = \mathrm{Bl}_S(\mathbb{P}^4) & \xrightarrow{q_1} & X_0 \\ \downarrow{\scriptstyle q_2} & & \\ \mathbb{P}^4 & & \end{array}$$

where q_1 is the blow-up of the double point p and q_2 is the blow-down of the lines contained in X_0 passing through p. These lines are parametrized by a surface $S \subset \mathbb{P}^4$, which is the complete intersection of a quadric and a cubic. The quadric is nonsingular because p is ordinary; the complete intersection is smooth because p is the only singularity of X_0. In particular, S is a sextic K3 surface. The inverse map π_p^{-1} is given by the linear system of cubic polynomials through this K3 surface.

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

Conversely, given any sextic K3 surface contained in a smooth quadric, the image of $\mathbb{P}^4$ under this linear system is a cubic fourfold with an ordinary double point. Note that the sextic K3 surfaces contained in a singular quadric hypersurface are precisely those containing a cubic plane curve.

This construction suggests that we associate a sextic K3 surface to any element of $\tilde{\mathcal{C}} - \mathcal{C}$.

PROPOSITION 4.2.1. *The Torelli map extends to an open immersion* $\tilde{\tau}\colon \tilde{\mathcal{C}} \to \mathcal{D}$. *The closed set* $\tilde{\mathcal{C}}_6 := \tilde{\mathcal{C}} - \mathcal{C}$ *is mapped into* $\mathcal{D}_6$.

In Section 5.2 we shall see that $\mathcal{D}_6$ coincides with the period domain for sextic K3 surfaces. A detailed proof of the proposition is given in Section 4 of [29], so we merely explain some details needed for our calculations. (It also follows from the delicate analysis of singular cubic fourfolds in Section 6.3.) Let X_0 be a cubic fourfold with an ordinary double point and let S be the associated K3 surface. Smoothings of ordinary double points of even codimension have monodromy satisfying $T^2 = I$, so any smoothing of X_0 yields a pure limiting mixed Hodge structure H^4_{lim}. The corresponding point of the period domain is denoted $\tilde{\tau}(X_0)$. The limiting Hodge structure may be computed with the Clemens–Schmid exact sequence [7], which implies there is a natural imbedding of the primitive cohomology $H^2(S, \mathbb{C})^0(-1)$ into H^4_{lim}. The orthogonal complement to the image consists of a rank-two lattice of integral $(2, 2)$ classes

$$K_6 = \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 0 \\ T & 0 & 2 \end{array}$$

so $\tilde{\tau}(X_0) \in \mathcal{D}_6$.

4.3. EXISTENCE OF SPECIAL CUBIC FOURFOLDS

$\mathcal{D}_d \subset \mathcal{D}$ is nonempty if and only if d is positive and congruent to $0, 2 \pmod 6$ (Proposition 3.2.2), so we restrict to these values of d.

THEOREM 4.3.1 (Existence of Special Cubic Fourfolds). *Let* $d > 6$ *be an integer with* $d \equiv 0, 2$ *(mod* 6*). Then the divisor* $\mathcal{C}_d$ *is nonempty.*

We saw in the last section why there are no smooth cubic fourfolds of discriminant six: $\mathcal{D}_6$ corresponds to the limiting Hodge structures arising from cubic fourfolds with double points. In the next section we shall explain why there are no cubic fourfolds of discriminant two: $\mathcal{D}_2$ corresponds to the limiting Hodge structures arising from another class of singular cubic fourfolds. Is the complement $\mathcal{D} - \mathcal{C}$ equal to $\mathcal{D}_2 \cup \mathcal{D}_6$?

To prove the theorem, we need the following lemmas.

LEMMA 4.3.2. *Let P be an indefinite even rank-two lattice representing six. Assume that P is not isomorphic to any of the following:*

$$\begin{pmatrix} 6 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 6 & 2 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix} \quad \begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}.$$

Then there exists a smooth sextic K3 surface S lying on a smooth quadric with $\mathrm{Pic}(S) \cong P$.

LEMMA 4.3.3. *Let P be a rank-two indefinite even lattice, $f \in P$ a primitive element with $d := f^2 > 0$, and assume there is no $E \in P$ with $E^2 = -2$ and $fE = 0$. Then there exists a K3 surface S with* $\mathrm{Pic}(S) = P$ *and f a polarization on S. Moreover, f is very ample unless there exists an elliptic curve C on S with $C^2 = 0$ and $fC = 1$ or 2.*

Recall that Λ denotes the lattice isomorphic to the middle cohomology of a K3 surface. Using the results of Section 2 of [16], there exists an imbedding $P \hookrightarrow \Lambda$. So for some elements of the period domain P equals the lattice of $(1, 1)$-classes. The surjectivity of the period map for K3 surfaces implies the existence of a K3 surface S with Picard group P so that f contained in the Kähler cone of S (see pp. 127 of [4]). This implies f is a polarization of S. To complete the proof, we apply Saint Donat's results for linear systems on K3 surfaces [23]. Specifically, we use Theorems 3.1, 5.2, and 6.1, along with the analysis of fixed components in Section 2.7. □

To prove Lemma 4.3.2, we note that the image under $|f|$ is not contained in a singular quadric because $P \not\cong \begin{pmatrix} 6 & 3 \\ 3 & 0 \end{pmatrix}$ (i.e. S does not contain a plane cubic). □

Now we prove the theorem. Let S be one of the K3 surfaces constructed in Lemma 4.3.2 and X_0 the corresponding singular cubic fourfold. Let $v \in P$ be primitive with respect to the sextic polarization. Recall that $H^2(S, \mathbb{C})^0(-1)$ is naturally imbedded into the limiting Hodge structure H^4_{lim} arising from X_0. The image of v is an integral class of type $(2, 2)$ in H^4_{lim}, denoted v'. Relabel H^4_{lim} by letting K_d denote the saturation of the lattice $\mathbb{Z}h^2 + \mathbb{Z}v'$. By Proposition 3.2.2, $d = -\frac{1}{2}\mathrm{disc}(P)$. For each $d \equiv 0, 2 \pmod 6$, $d > 6$, there exist lattices P satisfying the hypotheses of Lemma 4.3.2 with discriminant $-2d$. If $d = 6n$ (resp. $d = 6n + 2$) we may take

$$P = \begin{pmatrix} 6 & 0 \\ 0 & -2n \end{pmatrix} \quad \left(\text{resp.} \begin{pmatrix} 6 & 2 \\ 2 & -2n \end{pmatrix}\right).$$

Set $x_0 = \tilde{\tau}(X_0)$ so that $x_0 \in \mathcal{D}_6 \cap \mathcal{D}_d$. We construct a smoothing $\phi\colon \mathcal{X} \to \Delta'$ where X_t is smooth for $t \neq 0$, and $\tau(X_t) \in \mathcal{D}_d$. Let $\gamma\colon \Delta \to \mathcal{D}$ be a holomorphic map such that $\gamma(0) = x_0$ and $\gamma(u) \in \mathcal{D}_d - \mathcal{D}_6$ for $u \neq 0$. The existence of such a curve follows from the construction of $\mathcal{D}$ as the quotient $\Gamma^+\backslash\mathcal{D}'$. Because $\tilde{\tau}$ is an

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

open immersion, we may shrink Δ so that γ lifts through $\tilde{\tau}$, giving a map $\mu\colon \Delta \to \tilde{\mathcal{C}}$. Consequently, there exists a ramified base change $b\colon \Delta' \to \Delta$ and a family $\mathcal{X} \to \Delta'$ so that $X_t = \mu(b(t))$. By construction we have $X_t \in \mathcal{C} \cap \tau^{-1}(\mathcal{D}_d) = \mathcal{C}_d$ for $t \neq 0$, so $\mathcal{C}_d \neq \emptyset$. □

4.4. $d = 2$: THE DETERMINANTAL CUBIC FOURFOLD

The *determinantal cubic fourfold* X_0 is defined by the homogeneous equation:

$$R := \begin{vmatrix} a & b & c \\ b & d & e \\ c & e & f \end{vmatrix} = 0.$$

It is singular where the 2×2 minors of the determinant are simultaneously zero, *i.e.* along a Veronese surface V. We shall consider deformations $\mathcal{X} \to \Delta$ of X_0 with equations $R + tG$, where G is the equation of a smooth cubic fourfold, and the curve $C \subset V$ defined by the equation $G|_V = 0$ is also smooth. Let S be the double cover of V branched over C, a degree-two K3 surface.

THEOREM 4.4.1. *The limiting mixed Hodge structure arising from $\mathcal{X} \to \Delta$ is pure and special of discriminant two. The orthogonal complement to K_2 is isomorphic to the primitive Hodge structure $H^2(S, \mathbb{C})^0(-1)$.*

This result will not be used elsewhere in this paper. Its proof is essentially a calculation on the semistable reduction for $\mathcal{X}$ using the Clemens–Schmid exact sequence [7] (see [13] for details). Geometrically, X_0 is contained in the indeterminacy locus of the Torelli map, but after blowing up the map is well-defined at the generic point of the exceptional divisor. Moreover, this exceptional divisor maps birationally to $\mathcal{D}_2 \subset \mathcal{D}$.

5. Associated K3 Surfaces

5.1. NONSPECIAL COHOMOLOGY

DEFINITION 5.1.1. The *nonspecial cohomology lattice* of a labelled special cubic fourfold (X, K_d) is defined as the orthogonal complement $K_d^{\perp}$. The *nonspecial cohomology*, denoted W_{X,K_d}, is the polarized Hodge structure induced on $K_d^{\perp}$ by the Hodge structure on $H^4(X, \mathbb{C})^0$.

PROPOSITION 5.1.2. *Let (X, K_{14}) be a generic Pfaffian cubic fourfold. Then there exists a degree-fourteen K3 surface S and an isomorphism of Hodge structures $W_{X,K_{14}} = H^2(S, \mathbb{C})^0(-1)$.*

This is a consequence of [5] Proposition 6 (cf. Section 2.1) and Proposition 6.1.1. However, it is best explained by observing that the birational map $\mathbb{P}^4 \to X$ blows

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

up a surface birational to S, which therefore parametrizes a correspondence of rational curves on X.

Motivated by this example, we determine the special cubic fourfolds whose nonspecial cohomology is isomorphic to the primitive cohomology of a polarized K3 surface.

THEOREM 5.1.3 (Existence of Associated K3 Surfaces). *Let* (X, K_d) *be a labelled special cubic fourfold of discriminant* d*, with nonspecial cohomology* W_{X,K_d}*. There exists a polarized K3 surface* (S, f) *such that* $W_{X,K_d} \cong H^2(S, \mathbb{C})^0(-1)$ *if and only if the following conditions are satisfied:*

(1) $4 \not| d$ *and* $9 \not| d$*,*
(2) $p \not| d$ *if* p *is an odd prime,* $p \equiv -1 (\mathrm{mod}\ 3)$.

We say that the pair (S, f) *is* associated *to* (X, K_d).

We first show the theorem boils down to a computation of lattices (Proposition 5.1.4). Recall that a *pseudo-polarization* is a divisor f contained in the closure of the Kähler cone with $(f, f) > 0$; the primitive cohomology of a pseudo-polarized K3 surface (S, f) is the orthogonal complement to f in $H^2(S, \mathbb{Z})$. Let Λ_d^0 be a lattice isomorphic to the primitive middle cohomology of a degree d K3 surface. The isomorphism asserted in the theorem implies an isomorphism of lattices $K_d^\perp \cong -\Lambda_d^0$. On the other hand, given a labelled special cubic fourfold (X, K_d) and an isomorphism of lattices $K_d^\perp \cong -\Lambda_d^0$, $W_{X,K_d}(+1)$ has the form of the primitive cohomology of a pseudo-polarized K3 surface. Indeed, since the Torelli map for K3 surfaces is surjective [4] [26], there exists a pseudo-polarized K3 surface (S, f) such that $H^2(S, \mathbb{C})^0(-1) \cong W_{X,K_d}$. Moreover, X is smooth so $H^4(X, \mathbb{Z})^0 \cap H^{2,2}(X)$ does not contain any classes with self-intersection $+2$ ([29] Section 4 Proposition 1). Therefore there are no (-2)-curves on S orthogonal to f, and f is actually a polarization.

PROPOSITION 5.1.4. *Retain the notation above.* $K_d^\perp \cong -\Lambda_d^0$ *if and only if the conditions of Theorem* 5.1.3 *are satisfied.*

The automorphisms of $\Lambda = H^2(S, \mathbb{Z})$ act transitively on the primitive vectors with $(v, v) = d \neq 0$ ([16] Theorem 2.4), so $\Lambda_d^0 \cong (-d) \oplus H^{\oplus 2} \oplus (-E_8)^{\oplus 2}$, let y denote the distinguished element with $(y, y) = -d$. The discriminant group $d(\Lambda_d^0)$ and quadratic form $q_{\Lambda_d^0}$ are equal to $\mathbb{Z}(y/d)/\mathbb{Z}y$, with $q_{\Lambda_d^0}(y/d) = -1/d \pmod{2\mathbb{Z}}$.We determine when $d(K_d^\perp)$ and $d(-\Lambda_d^0)$ are isomorphic as groups with a $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form. We first consider the case $d \equiv 2 \pmod 6$. Here both discriminant groups are isomorphic to $\mathbb{Z}/d\mathbb{Z}$, so we just need to check when the quadratic forms are conjugate by an automorphism of $\mathbb{Z}/d\mathbb{Z}$. Let u and w be generators of $d(K_d^\perp)$ and $d(-\Lambda_d^0)$ such that $q_{K_d^\perp}(u) = 2d - 1/3d \pmod{2\mathbb{Z}}$ and $q_{-\Lambda_d^0}(w) = 1/d \pmod{2\mathbb{Z}}$ (see Proposition 3.2.5). The quadratic forms are conjugate if and only if the integer $(2d-1)/3$ is a square modulo $2d$, or equivalently, -3 is a square

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

modulo $2d$. By quadratic reciprocity this is the case if and only if d is not divisible by four and any odd prime $p|d$ satisfies $p \not\equiv -1 \pmod 3$. A similar argument holds in the case $d \equiv 0 \pmod 6$.

We have seen that the conditions on d are necessary for $K_d^{\perp}$ to be isomorphic to $-\Lambda_d^0$. On the other hand, $K_d^{\perp}$ is the *unique* even lattice of signature $(19, 2)$ with discriminant form $(d(K_d^{\perp}), q_{K_d^{\perp}})$ [22] 1.14.3. Hence if the discriminant forms of $K_d^{\perp}$ and $-\Lambda_d^0$ agree then $K_d^{\perp} \cong -\Lambda_d^0$. □

5.2. ISOMORPHISMS OF PERIOD DOMAINS

We retain the notation of Sections 2.1 and 5.1. Let Σ denote the automorphisms of Λ, and Σ_d the automorphisms fixing some primitive $v \in \Lambda$ with $(v, v) = d$, which yield automorphisms of $\Lambda_d^0 = v^{\perp}$. As in Section 2.2, let $\mathcal{N}_d'$ be the *local period domain* for degree d K3 surfaces, an open 19-dimensional complex manifold. Let $\Sigma_d^+ \subset \Sigma_d$ denote the subgroup stabilizing $\mathcal{N}_d'$. As before, $\mathcal{N}_d'$ is a bounded symmetric domain of type IV, Σ_d^+ is an arithmetic group acting holomorphically on $\mathcal{N}_d'$, and the quotient $\mathcal{N}_d := \Sigma_d^+/\mathcal{N}_d'$ is therefore a quasi-projective variety, the *global period domain* for degree d K3 surfaces.

We introduce a bit more notation for special cubic fourfolds as well. Let $G_d^+ \subset \Gamma_d^+$ be the subgroup acting trivially on K_d and let $\mathcal{D}_d^{\mathrm{mar}}$ denote the marked special Hodge structures of discriminant d, modulo the action of G_d^+. The fiber product $\mathcal{D}_d^{\mathrm{mar}} \times_{\mathcal{D}} \mathcal{C}$ is written $\mathcal{C}_d^{\mathrm{mar}}$, the *marked special cubic fourfolds of discriminant* d. We have natural forgetting maps $\mathcal{D}_d^{\mathrm{mar}} \to \mathcal{D}_d^{\mathrm{lab}}$ and $\mathcal{C}_d^{\mathrm{mar}} \to \mathcal{C}_d^{\mathrm{lab}}$.

PROPOSITION 5.2.1. *$G_d^+ = \Gamma_d^+$ if $d \equiv 2 \ (mod\ 6)$ and $G_d^+ \subset \Gamma_d^+$ is an index-two subgroup if $d \equiv 0 \ (mod\ 6)$. The natural map $\mathcal{D}_d^{\mathrm{mar}} \to \mathcal{D}_d^{\mathrm{lab}}$ is an isomorphism if $d \equiv 2 \ (mod\ 6)$ and a double cover if $d \equiv 0 \ (mod\ 6)$. Furthermore, $\mathcal{D}_d^{\mathrm{mar}} = G_d^+ \backslash \mathcal{D}_d'$ and thus is connected for all $d \neq 6$.*

We begin with the first statement. The lattice K_d has no automorphisms preserving h^2 if $d \equiv 2 \ (mod\ 6)$, so $G_d^+ = \Gamma_d^+$. If $d \equiv 0 \ (mod\ 6)$ then K_d has an involution, which acts on K_d^0 as multiplication by -1. We claim it extends to an element $\gamma \in \Gamma_d^+$. By Proposition 3.2.4 we may assume $K_d^0 = \mathbb{Z}(v_1 + \frac{d}{6} w_1)$. We use the notation of Section 2.1, so v_1 and w_1 form a basis for a hyperbolic summand $H \subset L^0$. Choose γ equal to multiplication by -1 on both hyperbolic summands of L^0 and equal to the identity elsewhere. We have that $\gamma \in \Gamma_d^+$ but $\gamma \notin G_d^+$, so G_d^+ is a proper subgroup of Γ_d^+.

The second statement follows immediately from the first. As for the third statement, recall that $\mathcal{D}_d^{\mathrm{lab}} = \Gamma_d^+ \backslash \mathcal{D}_d'$. Hence for $d \equiv 2 \ (mod\ 6)$ the result is immediate. For $d \equiv 0 \ (mod\ 6)$, we must check that any $\gamma \in \Gamma_d^+$ acting nontrivially on K_d also acts nontrivially on $\mathcal{D}_d'$. For $d \neq 6$, if γ acts nontrivially on K_d then the induced action on $d(K_d)$ is not equal to ± 1. However, the groups $d(K_d)$ and $d(K_d^{\perp})$ are isomorphic, so the induced action on $d(K_d^{\perp})$ is not ± 1. Now $\mathcal{D}_d'$ is a topologically

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

open subset of a quadric hypersurface in $\mathbb{P}(K_d^{\perp} \otimes \mathbb{C})$, so only scalar multiplications act trivially on $\mathcal{D}'_d$. In particular, γ necessarily acts nontrivially. □

Remark. There exists an element $\gamma \in \Gamma_6^+ - G_6^+$ acting trivially on $K_6^{\perp}$. It follows that $\mathcal{D}_6^{\text{mar}} \neq G_6^+ \backslash \mathcal{D}'_6$ but rather that $\mathcal{D}_6^{\text{lab}} = G_6^+ \backslash \mathcal{D}'_6$.

THEOREM 5.2.2. *Let d be a positive integer such that there exists an isomorphism $j_d \colon K_d^{\perp} \to -\Lambda_d^0$ (see Proposition 5.1.4.) Choose orientations on the negative definite parts of $K_d^{\perp}$ and $-\Lambda_d^0$ compatible with j_d, so there is an induced isomorphism of local period domains $\mathcal{D}'_d$ and $\mathcal{N}'_d$. If $d \neq 6$ then there is an induced isomorphism $i_d \colon \mathcal{D}_d^{\text{mar}} \to \mathcal{N}_d$; we also have $\mathcal{D}_6^{\text{lab}} \cong \mathcal{N}_6$.*

The isomorphism of period domains depends on the choice of j_d. Each j_d induces an isomorphism of discriminant groups $j'_d \colon d(K_d^{\perp}) \to d(-\Lambda_d^0)$ preserving the $\mathbb{Q}/2\mathbb{Z}$-valued quadratic forms on these groups [22] Section 1.3. We denote the set of such isomorphisms $\text{Isom}(d(K_d^{\perp}), d(-\Lambda_d^0))$; the group $\{n \in \mathbb{Z}/d\mathbb{Z} : n^2 = 1\}$ acts faithfully and transitively on this set.

THEOREM 5.2.3. *For $d \neq 6$, the various isomorphisms $i_d \colon \mathcal{D}_d^{\text{mar}} \to \mathcal{N}_d$ correspond to elements of $\text{Isom}(d(K_d^{\perp}), d(-\Lambda_d^0))/(\pm 1)$. The isomorphism $i_6 \colon \mathcal{D}_6^{\text{lab}} \to \mathcal{N}_6$ is unique.*

These two theorems have the following corollary.

COROLLARY 5.2.4 (Immersions into Moduli Spaces of K3 Surfaces). *Let $d \neq 6$ be a positive integer such that there exists an isomorphism $j_d \colon K_d^{\perp} \to -\Lambda_d^0$. Then there is an imbedding $i_d \colon \mathcal{C}_d^{\text{mar}} \hookrightarrow \mathcal{N}_d$, unique up to the choice of an element of* $\text{Isom}(d(K_d^{\perp}), d(-\Lambda_d^0))/(\pm 1)$. *Moreover, there is a unique imbedding $i_6 \colon \tilde{\mathcal{C}}_6^{\text{lab}} \hookrightarrow \mathcal{N}_6$.*

As we shall see in Section 6, geometrical considerations will sometimes mandate specific choices of i_d (e.g. in the case $d = 14$).

We prove the first theorem. First, we compare the action of Σ_d^+ on Λ_d^0 to the action of G_d^+ on $K_d^{\perp}$. We claim that Σ_d^+ is the group of automorphisms of Λ_d^0 preserving the orientation on the positive definite part of $\Lambda_d^0 \otimes \mathbb{R}$ and acting trivially on the discriminant group $d(\Lambda_d^0)$. This follows from the results of [22] Section 1.4, which imply that any such automorphism extends uniquely to an element of Σ_d^+. Similarly, G_d^+ is the group of automorphisms of $K_d^{\perp}$ preserving the orientation on the negative definite part of $K_d^{\perp} \otimes \mathbb{R}$ and acting trivially on the discriminant group $d(K_d^{\perp})$.

Now suppose we are given an isomorphism $j_d \colon K_d^{\perp} \to -\Lambda_d^0$. This induces isomorphisms $\mathcal{D}'_d \to \mathcal{N}'_d$, $G_d^+ \to \Sigma_d^+$, and $i_d \colon G_d^+ \backslash \mathcal{D}'_d \to \Sigma_d^+ \backslash \mathcal{N}'_d$. Applying Proposition 5.2.1, we obtain an isomorphism $i_d \colon \mathcal{D}_d^{\text{mar}} \to \mathcal{N}_d$ for $d \neq 6$. The remark after the proposition also yields an isomorphism $i_6 \colon \mathcal{D}_6^{\text{lab}} \to \mathcal{N}_6$. □

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

We turn to the proof of the second theorem. We must determine when two different isomorphisms $j_d^1\colon K_d^{\perp} \to -\Lambda_d^0$ and $j_d^2\colon K_d^{\perp} \to -\Lambda_d^0$ induce the same isomorphism $i_d\colon G_d^+\backslash\mathcal{D}_d' \to \Sigma_d^+\backslash\mathcal{N}_d'$. If $j_d^2 = \sigma \circ j_d^1$ for some $\sigma \in \Sigma_d^+$ then j_d^1 and j_d^2 induce the same isomorphisms of period domains. Also, if $j_d^1 = -j_d^2$ then j_d^1 and j_d^2 induce the same isomorphism between $\mathcal{D}_d'$ and $\mathcal{N}_d'$, because these manifolds lie in the projective spaces $\mathbb{P}(K_d^{\perp} \otimes \mathbb{C})$ and $\mathbb{P}(\Lambda_d^0 \otimes \mathbb{C})$.

On the other hand, assume that j_d^1 and j_d^2 induce the same isomorphism between $G_d^+\backslash\mathcal{D}_d'$ and $\Sigma_d^+\backslash\mathcal{N}_d'$. Then there exist $\gamma \in G_d^+$ and $\sigma \in \Sigma_d^+$ such that $j_d^1 \circ \gamma$ and $\sigma \circ j_d^2$ induce the same isomorphism between $\mathcal{D}_d'$ and $\mathcal{N}_d'$, so $j_d^1 \circ \gamma = \pm\sigma \circ j_d^2$. We conclude that the isomorphisms between $G_d^+\backslash\mathcal{D}_d'$ and $\Sigma_d^+\backslash\mathcal{N}_d'$ correspond to certain elements of $\mathrm{Isom}(d(K_d^{\perp}), d(-\Lambda_d^0))/(\pm 1)$.

It remains to check that each element of $\mathrm{Isom}(d(K_d^{\perp}), d(-\Lambda_d^0))/(\pm 1)$ actually arises from an isomorphism between $K_d^{\perp}$ and $-\Lambda_d^0$ respecting the orientations on the negative definite parts. Now $K_d^{\perp}$ has an automorphism g reversing the orientation on the negative part and acting trivially on $d(K_d)$. Take g to be the identity except on a hyperbolic summand of the orthogonal decomposition for $K_d^{\perp}$; on the hyperbolic summand set g equal to multiplication by -1. Hence it suffices to show that the automorphisms of $K_d^{\perp}$ induce all the automorphisms of $d(K_d^{\perp})$, which is proved in [22], Theorem 1.14.2 and Remark 1.14.3. □

6. Fano Varieties of Special Cubic Fourfolds

6.1. INTRODUCTION AND NECESSARY CONDITIONS

Here we provide a geometric explanation for the K3 surfaces associated to some special cubic fourfolds. The general philosophy underlying our approach is due to Mukai [19–21]. Let S be a polarized K3 surface and let $\mathcal{M}_S$ be a moduli space of simple sheaves on S. Quite generally, $\mathcal{M}_S$ is smooth and possesses a natural nondegenerate holomorphic two-form ([19] Theorem 0.1). Furthermore, the Chern classes of the 'quasi-universal sheaf' on $S \times \mathcal{M}_S$ induce correspondences between S and $\mathcal{M}_S$. If $\mathcal{M}_S$ is compact of dimension two then it is a K3 surface isogenous to S; the Hodge structure of $\mathcal{M}_S$ can be read off from the Hodge structure of S and the numerical invariants of the sheaves ([20] Theorem 1.5). Conversely, given a variety F with a nondegenerate holomorphic two-form and an isogeny $H^2(S, \mathbb{Q}) \to H^2(F, \mathbb{Q})$, one can try to interpret F as a moduli space of sheaves on S. In the case where F is a K3 surface, we often have such interpretations ([20] Theorem 1.9). Note that $F \cong S^{[n]}$ can be interpretted as the moduli space of ideal sheaves on S of colength n; such sheaves are simple.

PROPOSITION 6.1.1. *Let X be a cubic fourfold with Fano variety F. Assume there is an isomorphism between F and $S^{[2]}$ for some K3 surface S. Then X has a labelling K_d such that S is associated to (X, K_d); $i_d\colon \mathcal{C}_d^{\mathrm{mar}} \hookrightarrow \mathcal{N}_d$ may be chosen so that $i_d(X, K_d) = S$. If (X_1, K_d) is a generic element of $\mathcal{C}_d^{\mathrm{mar}}$ and $S_1 = i_d(X_1, K_d)$, then the Fano variety F_1 is isomorphic to $S_1^{[2]}$.*

For nongeneric X_1 the isomorphism between F_1 and $S_1^{[2]}$ can break down. Let X_1 contain two disjoint planes π_1 and π_2, so that $X_1 \in \mathcal{C}_{14}$. The proposition holds for $d = 14$, but the (birational) map between F_1 and $S_1^{[2]}$ acquires indeterminacy at the lines supported in the π_i (see [13] for details).

We prove the proposition. As in Section 2.1, there is an isomorphism $H^2(F, \mathbb{Z}) \cong H^2(S, \mathbb{Z}) \oplus_\perp \mathbb{Z}\delta$ and the hyperplane class $g = af - b\delta$ where f is some polarization of S with $d := (f, f)$. Let $K_d^\perp$ equal $\alpha^{-1}(H^2(S, \mathbb{Z})^0(-1))$ where α is the Abel–Jacobi map, and set $K_d = (K_d^\perp)^\perp$. Applying Theorem 5.2.2 with $j_d = \alpha | K_d^\perp$, we obtain a map i_d with the desired properties. To explain i_d geometrically, we need the following result.

THEOREM 6.1.2 (Deformation Spaces of $S^{[2]}$ [3]). *Let S be a K3 surface and $2\delta \subset S^{[2]}$ be the elements supported at a single point. The deformation space of $S^{[2]}$ is smooth and has dimension twenty-one. Deformations of the form $S_1^{[2]}$ correspond to a divisor in this space which may be characterized as the deformations for which δ remains a divisor.*

We define $\mathcal{C}_d$ as the deformations of F for which δ remains algebraic. Applying Theorem 6.1.2, there is some small analytic neighborhood in $\mathcal{C}_d$ where the deformations are isomorphic to $S_1^{[2]}$ for some deformation S_1 of S. This isomorphism holds in an open étale neighborhood of X in $\mathcal{C}_d$, so a generic cubic fourfold in $\mathcal{C}_d$ has Fano variety of the form $S_1^{[2]}$. □

For which values of d are the conclusions of Proposition 6.1.1 valid? Theorem 5.1.3 gives sufficient conditions for the existence of a K3 surface associated to (X, K_d), but these do not guarantee that $F \cong S^{[2]}$.

PROPOSITION 6.1.3. *Assume that the Fano variety of a generic special cubic fourfold of discriminant d is isomorphic to $S^{[2]}$ for some K3 surface S. Then there exist positive integers n and a such that $d = 2(n^2 + n + 1)/a^2$.*

This is equivalent to the existence of a line bundle on $S^{[2]}$ of degree 108, the degree of the Fano variety. For instance, Fano varieties of special cubic fourfolds of discriminant 74 are not generally of the form $S^{[2]}$, because $74a^2 = 2(n^2 + n + 1)$ has no integral solutions (see [11]).

We can produce infinitely many examples of special cubic fourfolds with Fano variety isomorphic to the symmetric square of a K3 surface.

THEOREM 6.1.4. *Assume that $d = 2(n^2 + n + 1)$ where n is an integer $\geqslant 2$. Then the Fano variety of a generic special cubic fourfold X of discriminant d is isomorphic to $S^{[2]}$, where S is a K3 surface associated to (X, K_d).*

This is proved in the next two sections. The condition on d corresponds to setting $a = 1$ in Proposition 6.1.3. The proof of the theorem suggests that the condition of the proposition is the correct sufficient condition.

6.2. AMBIGUOUS SYMPLECTIC VARIETIES

DEFINITION 6.2.1. Let F be an irreducible symplectic Kähler manifold, and assume that there exist K3 surfaces S_1 and S_2 and isomorphisms $r_1\colon F \to S_1^{[2]}$ and $r_2\colon F \to S_2^{[2]}$ such that $r_1^*\delta_1 \neq r_2^*\delta_2$. Then we say that F is *ambiguous*.

Our first example is a special case of a construction of Beauville and Debarre [8]. Let S be a smooth quartic surface in $\mathbb{P}^3$, $p_1 + p_2$ a generic point in $S^{[2]}$, and $\ell(p_1 + p_2)$ the line containing p_1 and p_2. By Bezout's theorem $\ell(p_1 + p_2) \cap S = p_1 + p_2 + q_1 + q_2$. Setting $j(p_1 + p_2) = q_1 + q_2$ for each $p_1 + p_2$, we obtain a birational involution $j\colon S^{[2]} \to S^{[2]}$. If S contains no lines then j extends to a biregular morphism. Let f_4 be the degree-four polarization on S and the corresponding class on $S^{[2]}$. Following [8], one may compute $j^*(x) = -x + (x, f_4 - \delta)(f_4 - \delta)$ on $H^2(S^{[2]}, \mathbb{Z})$. Setting $r_2 = j \circ r_1$, we find that $F = S^{[2]}$ is ambiguous.

We digress to give another beautiful example of ambiguous varieties.

PROPOSITION 6.2.2. *Assume that* $3|d$ *and that the Fano variety* F *of a generic cubic fourfold in* $\mathcal{C}_d$ *is isomorphic to* $S_1^{[2]}$ *for some K3 surface* S_1. *Then* F *is ambiguous.*

This follows immediately from Proposition 6.1.1 and the results of Section 5.2, which imply that $\mathcal{C}_d^{\mathrm{lab}}$ imbeds into a $\mathbb{Z}/2\mathbb{Z}$-quotient of $\mathcal{N}_d$ if $3|d$.

6.3. CONSTRUCTION OF THE EXAMPLES

Let $X_0 \in \tilde{\mathcal{C}}_6$, F_0 its Fano variety of lines, and S the sextic K3 surface associated to X_0 (see Section 4.2). Let $\phi\colon \mathcal{X} \to \Delta$ be a family in $\tilde{\mathcal{C}}$ with central fiber X_0 and X_t smooth for $t \neq 0$. Let $\mathcal{F} \to \Delta$ be the corresponding family of Fano varieties and $\mathcal{X}' \to \Delta'$ a semistable reduction of $\mathcal{X} \to \Delta$. For simplicity, we assume that the central fiber of the semistable family is of the form $X_0' = \overline{X_0} \cup Q$ where $\overline{X_0} = \mathrm{Bl}_S(\mathbb{P}^4)$ is the desingularization of X_0, Q is a smooth quadric fourfold, and $Q_0 = \overline{X_0} \cap Q$ is the smooth quadric in $\mathbb{P}^4$ containing S. This is the case if ϕ is a sufficiently generic smoothing of X_0.

LEMMA 6.3.1. *F_0 is singular along the lines through the double point, which are parametrized by S. These singularities are ordinary codimension-two double points and the blow-up* $\sigma\colon \mathrm{Bl}_S F_0 \to F_0$ *desingularizes* F_0. *If* S_0 *does not contain a line then* $\mathrm{Bl}_S F_0 \cong S^{[2]}$.

The first part follows from Section 4.2 and [1] 1.10. For the second part, we realize σ by blowing up the Grassmannian $\mathbb{G}(1, 5)$ along the locus $L(p)$ of lines containing p. The fiber square

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

$$\begin{array}{ccc} S & \longrightarrow & F_0 \\ \downarrow & & \downarrow \\ L(p) & \longrightarrow & \mathbb{G}(1,5) \end{array}$$

gives a natural closed imbedding of normal cones $C_S F_0 \hookrightarrow C_{L(p)}\mathbb{G}(1,5)|S$. The projectivization $\mathbb{P}(C_{L(p)}\mathbb{G}(1,5))$ corresponds to $\mathbb{P}(\mathbb{C}^6/\mathcal{S})$, where $\mathcal{S}$ is the restriction of the universal subbundle. Note $L(p) \cong \mathbb{P}^4$ and $C_{L(p)}\mathbb{G}(1,5)_\ell$ corresponds to the lines λ such that $\ell \in \lambda \subset \mathbb{P}^4$. For $\ell \in \mathrm{Sing}(F_0)$ the fiber of $\mathbb{P}(C_S F_0)_\ell$ corresponds to those lines λ such that $\ell \in \lambda \subset Q_0$. These are parametrized by a smooth conic curve, hence F_0 has codimension-two ordinary double points along S and $\mathrm{Bl}_S F_0$ is smooth.

This description implies that we can regard $\mathrm{Bl}_S F_0$ as a parameter space for certain curves on $\overline{X_0}$. These curves are of the following types:

(1) lines on X_0 disjoint from p;
(2) unions of proper transforms of lines through p and lines contained in $Q_0 \subset \overline{X_0}$.

These in turn may be identified with:

(1) two-secants λ to $S \subset \mathbb{P}^4$;
(2) three-secants λ with a distinguished point $s \in \lambda \cap S$.

We emphasize that each line meeting S in more than two points is contained in Q_0 but not in S, and thus is a three-secant to S. We claim elements of $S^{[2]}$ naturally correspond to curves of this type. For each ideal sheaf I of colength two there is a unique line λ containing the corresponding subscheme. Either λ is a two-secant, or λ is a three-secant and s is the support of $I/I_{\lambda\cap S}$. □

LEMMA 6.3.2. *Retain the notation and assumptions introduced above. The family of Fano varieties* $\mathcal{F} \times_\Delta \Delta'$ *has ordinary codimension-three double points along the surface* S. *The variety* $\mathcal{F}' = \mathrm{Bl}_S(\mathcal{F} \times_\Delta \Delta')$ *is smooth, and the exceptional divisor* $E \subset F_0'$ *is a smooth quadric surface bundle over* S. *The component of* F_0' *dominating* F_0 *is isomorphic to* $S^{[2]}$.

The proof is essentially the same as the first lemma. Our next result is:

PROPOSITION 6.3.3. *Retain the notation and assumptions introduced above. Then there is a smooth family* $\overline{\mathcal{F}} \to \Delta'$, *birational to* $\mathcal{F} \times \Delta'$, *such that* $\overline{F}_u = F_u$ *and* $\overline{F}_0 = S^{[2]}$.

We start with the family $\mathcal{F}'$ described in the previous lemma. The fibers of $E \to S$ are all smooth quadric surfaces, so the variety parametrizing rulings of E is an étale double cover of S. Since S has no nontrivial étale coverings we may choose a ruling of E. Blowing down E in the direction of this ruling, we obtain a smooth family $\overline{\mathcal{F}}$. This map induces an isomorphism from the proper transform of F_0 in

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

F_0' to the central fiber of $\overline{\mathcal{F}}$. The proper transform to F_0 in F_0' is isomorphic to $S^{[2]}$, so $\overline{\mathcal{F}}$ satisfies the conditions of the proposition. □

We now prove Theorem 6.1.4. Let S be an algebraic K3 surface with Picard group

$$P = \begin{array}{c|cc} & f_6 & f_4 \\ \hline f_6 & 6 & n+5 \\ f_4 & n+5 & 4 \end{array}$$

and $n \geqslant 2$. By Lemma 4.3.2, such a surface exists and we may assume that $|f_6|$ imbeds it as a smooth sextic surface. The divisor f_4 is effective because it has positive degree with respect to f_6. We claim that f_4 is very ample. If f_4 were not ample, then there would exist a (-2)-curve E with $f_4 E \leqslant 0$. This follows from the structure of the Kähler cone of S ([16] Section 1, Section 10). Note that $f_4 E \neq 0$ because P does not contain a rank-two sublattice of discriminant -8. Recall that the Picard–Lefschetz reflection associated to E is given by the equation $r_E(x) = x + (E, x)E$. Applying this to the class f_4, we find that $r_E(f_4)^2 = 4$ and $(f_6, r_E(f_4)) < (f_6, f_4)$. Hence f_6 and $r(f_4)$ span a sublattice with discriminant smaller than that of P, which is impossible. Finally, applying Lemma 4.3.3 we see that the linear system $|f_4|$ imbeds S as a smooth quartic surface.

Our hypothesis on P implies that the image of S under $|f_6|$ lies on a smooth quadric hypersurface and does not contain a line, and that the image of S under $|f_4|$ also does not contain a line. In particular, S corresponds to a singular cubic fourfold $X_0 \in \tilde{\mathcal{C}}_6$. Furthermore $S^{[2]}$ is ambiguous, with an involution $j: S^{[2]} \to S^{[2]}$ so that $\delta_2 := j^*\delta = 2f_4 - 3\delta$. Using Proposition 6.3.3 and the arguments of Section 4.3, X_0 has a smoothing $\phi: \mathcal{X} \to \Delta$ such that (after base change) the corresponding family of smooth symplectic varieties $\overline{\mathcal{F}} \to \Delta'$ is a deformation of $S^{[2]}$ for which δ_2 remains algebraic. By Theorem 6.1.2 the Fano variety $\overline{F_u}$ of X_u' is isomorphic to $S_u^{[2]}$.

If we choose ϕ generally, we may assume that the X_u' are typical and that $\mathrm{Pic}(S_u)$ is generated by the polarization f'. Let $\Pi = \mathrm{Pic}(\overline{F_u})$, a lattice (with respect to the canonical form) of discriminant $-2\deg(S_u)$. On the other hand, Π is the saturation of $\mathbb{Z}g + \mathbb{Z}\delta_2$. Specializing to $S^{[2]}$ we obtain $\Pi = \mathbb{Z}(2f_6 - 3\delta) + \mathbb{Z}(f_6 - f_4)$ with discriminant $-4(n^2 + n + 1)$. In particular, the S_u have degree $d(n) = 2(n^2 + n + 1)$ and the X_u are special of discriminant $d(n)$. □

We have shown that the pure limiting Hodge structures parametrized by $\mathcal{D}_6$ actually arise from smooth symplectic varieties. This may be interpretted as a weak surjectivity result for the corresponding Torelli map. It also explains the computation of the limiting mixed Hodge structure H^4_{lim} in Section 4.2.

There are a number of ways Theorem 6.1.4 might be generalized. We need not assume that the polarizations f_6 and f_4 actually generate the Picard lattice of S. Another approach is to replace $\tilde{\mathcal{C}}_6$ by some other divisor $\mathcal{C}_d$ parametrizing special cubic fourfolds whose Fano varieties are of the form $S^{[2]}$. To make precise

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press

statements one requires explicit descriptions of two complicated closed sets: the complement $\mathcal{D}_d - \mathcal{C}_d$ and the locus in $\mathcal{C}_d$ where the isomorphism between the Fano varieties and the blown-up symmetric squares breaks down. Finally, Mukai's philosophy suggests that whenever we have an associated K3 surface S, the Fano variety F might be interpretted as a suitable moduli space of simple sheaves on S. It would be interesting to find such interpretations when F cannot be a blown-up symmetric square.

Acknowledgements

I would like to acknowledge the help I received in the course of this project. I benefitted from conversations with David Eisenbud and Elham Izadi, and from suggestions by Walter Baily and Robert Friedman. Barry Mazur provided important insights and comments, and Johan de Jong made many helpful observations and pointed out some errors in an early version of this paper. Finally, Joe Harris introduced me to this beautiful subject; his inspiration and advice have been invaluable.

References

1. Altman, A. and Kleiman, S.: Foundations of the theory of Fano schemes, *Compositio Math.* **34** (1977), 3–47.
2. Baily, W. L. and Borel, A.: Compactifications of arithmetic quotients of bounded symmetric domains, *Ann. of Math.* **84** (1966), 442–528.
3. Beauville, A.: Variétés kähleriennes dont la première classe de Chern est nulle, *J. Differential Geom.* **18** (1983), 755–782.
4. Beauville, A.: Surjectivité de l'application des périodes, In: *Géométrie des Surfaces K3: Modules et Périodes-Séminaire Palaiseau, Astérisque* **126**, (1985), 122–128.
5. Beauville, A. and Donagi, R.: La variété des droites d'une hypersurface cubique de dimension 4, *C.R. Acad. Sci. Paris, Sér. I* **301** (1985), 703–706.
6. Borel, A.: Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem, *J. Differential Geom.* **6** (1972), 543–560.
7. Clemens, C.H.: Degenerations of Kähler manifolds, *Duke Math J.* **44** (1977), 215–290.
8. Debarre, O.: Un contre-exemple au théorème de Torelli pour les variétés symplectiques irréductibles, *C.R. Acad. Sci. Paris, Sér. I* **299** (1984), 681–684.
9. Dolgachev, I.: Integral quadratic forms, in *Séminaire Bourbaki Exposé 611, Astérisque* **105-106** (1983), 251–275.
10. Fano, G.: Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate rationali de 4° ordine, *Comment. Math. Helv.* **15** (1943), 71–80.
11. Gauss, C.F.: *Disquisitiones Arithmeticae*, Yale University Press, New Haven, 1966.
12. Griffiths, P. and Schmid, W.: Recent developments in Hodge theory, In: *International Colloquium on Discrete Subgroups of Lie Groups and Applications to Moduli*, Tata Institute of Fundamental Research and Oxford University Press, Bombay, 1973, pp. 31–127.
13. Hassett, B.: Special cubic hypersurfaces of dimension four, Harvard University Thesis, Cambridge, Mass., 1996.
14. Hassett, B.: Some rational cubic fourfolds, to appear in *J. Algebraic Geom*. (1997).
15. Izadi, E.: A Prym construction for the cohomology of a cubic hypersurface, Preprint in alg-geom/9701012. (1997).

16. Loojenga, E. and Peters, C.: Torelli theorem for Kähler K3 surfaces, *Compositio Math.* **42** (1981), 145–186.
17. Morin, U.: Sulla razionalità dell'ipersuperficie cubic generale dello spazio lineare S^5, *Rend. Sem. Mat. Univ. Padova* **11** (1940), 108–112.
18. Mumford, D., Fogarty, J., and Kirwan, F.: *Geometric Invariant Theory*, Springer, Berlin. (1994).
19. Mukai, S.: Symplectic structure of the moduli space of sheaves on an abelian or K3 surface, *Invent. Math.* **77** (1984), 101–116.
20. Mukai, S.: On the moduli space of bundles on K3 surfaces, I, *Vector Bundles on Algebraic Varieties, Proc. Bombay Conference, 1984,* Tata Inst. Fund. Res. Stud. 11, Oxford University Press, 1987, pp. 341–413.
21. Mukai, S.: Moduli of vector bundles on K3 surfaces, and symplectic manifolds, *Sugaku Expositions* **1** (1988), 139–174.
22. Nikulin, V.V.: Integral symmetric bilinear forms and some of their applications, *Math. USSR Izvest*. **14** (1) (1980), 103-167.
23. Saint-Donat, B.: Projective models of K3 surfaces, *Amer. J. Math.* **96** (1974), 602–639.
24. Satake, I.: *Algebraic Structures of Symmetric Domains*, Princeton University Press, Princeton, 1980.
25. Serre, J.P.: *A Course in Arithmetic*, Springer, New York, 1973.
26. Siu, Y.T.: A simple proof of the surjectivity of the period map of K3 surfaces, *Manuscripta Math.* **35** (1981), 311–321.
27. Tregub, S.L.: Three constructions of rationality of a cubic fourfold, *Moscow Univ. Math. Bull.* **39** (3) (1984), 8–16.
28. Tregub, S.L.: Two remarks on four dimensional cubics, *Russian Math. Surveys* **48** (2) (1993), 206–208.
29. Voisin, C.: Théorème de Torelli pour les cubiques de $\mathbb{P}^5$, *Invent. Math.* **86** (1986), 577–601.
30. Zarhin, Y.: Algebraic cycles over cubic fourfolds, *Boll. Un. Mat. Ital. B (7)*, **4b** (1990), 833–847.
31. Zucker, S.: The Hodge conjecture for cubic fourfolds, *Compositio Math.* **34** (1977), 199–209.

https://doi.org/10.1023/A:1001706324425 Published online by Cambridge University Press