INITIAL SEGMENTS OF MANY-ONE DEGREES

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Introduction. Our aim in this paper is to give a characterization of the order types of the countable initial segments of many-one degrees (m-degrees). The basic definitions and background information can be found in [**2**] from where we draw most of our notation and terminology. We expand the usual notion of m-reducibility by adopting the convention that $R \leqq_m \emptyset$ and $R \leqq_m N$ for every recursive set R. This has the effect of giving all recursive sets the same m-degree; that m-degree will be denoted by $\mathbf{0}$. We shall denote by $\leqq$ the partial ordering of m-degrees induced by $\leqq_m$, and shall denote by $\mathbf{a} \cup \mathbf{b}$ the least upper bound of the m-degrees $\mathbf{a}$, $\mathbf{b}$. We call $\mathbf{a} \cup \mathbf{b}$ the *union* of $\mathbf{a}$ and $\mathbf{b}$.

Consider a map θ from one partially ordered set into another. We say that θ *preserves unions* if for any a, b in dom θ having least upper bound $a \cup b$, $\theta(a \cup b)$ is defined and equal to $\theta(a) \cup \theta(b)$; similarly for intersections. The least member and greatest member of a partially ordered set will, if they exist, be denoted by 0 and 1, respectively. We say that θ *preserves* 0 if $\theta(0) = 0$; similarly for 1.

Let $\langle D_i \rangle$ be a sequence of finite distributive lattices each with $0 \neq 1$, and for each i let χ_i: $D_i \rightarrow D_{i+1}$ be a map preserving unions, 0, and 1, but not necessarily intersections. Then the direct limit L^* of the sequence

$$D_0 \xrightarrow{\chi_0} D_1 \xrightarrow{\chi_1} \ldots \xrightarrow{\chi_i} D_{i+1} \xrightarrow{\chi_{i+1}} \ldots$$

is an upper semilattice with 0 and 1.

Our principal result is: λ *is the order type of an initial segment of* m*-degrees with greatest member* (which is $>\mathbf{0}$) *if and only if* λ *is the order type of some upper semilattice* L^* *formed in the manner described above.* This provides a characterization of the order types of countable initial segments, which we shall not state any more explicitly, because any countable set of m-degrees has an upper bound.

The plan of the paper is as follows: In §1 we prove a closure property of the upper semilattice of m-degrees, in §2 we show that any order type of a countable initial segment is possible provided only that it is consistent with the closure property, and in §3 we mention some corollaries and related results.

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We conclude the introduction by mentioning some concepts that will be used below. Let D be a finite distributive lattice with $0 \neq 1$. Then D generates a unique Boolean algebra A with the same 0 and 1. The minimal non-zero elements of A are called *atoms*; since A is determined by D, we may speak of these atoms as being the *atoms of* D. Denote by $\mathscr{N}$ the Boolean algebra formed by the subsets of N. By an *isomorphism* $\pi: D \rightarrow \mathscr{N}$ we mean a one-to-one map π of D into $\mathscr{N}$ preserving unions, intersections, and such that $\pi(1) = N$. If π is any such isomorphism, we extend π to the atoms of D by letting $\pi(a) = \pi(d_1) - \pi(d_2)$, where $d_1 - d_2$ is the unique expression of the atom a as a difference of elements of D.

A class $\mathscr{C}$ of finite sets of natural numbers is called *canonically enumerable* [**2**, p. 76] if there exists an enumeration $\langle C_i \rangle$ of $\mathscr{C}$ such that the binary relation $x \in C_y$ is recursively enumerable (r.e.) and such that the cardinality of C_i is a recursive function of i. Any such enumeration $\langle C_i \rangle$ is called a *canonical enumeration* subject to the convention that canonical enumerations of infinite classes are to be without repetitions.

We say that the sets A and B *differ finitely* if the symmetric difference $(A - B) \cup (B - A)$ is finite. Let L_{m} denote the upper semilattice of m-degrees.

1. A closure property of m-degrees. Let U be a set of natural numbers with m-degree $\mathbf{u}$. For any non-empty r.e. set W, let w be a recursive function whose range is W. Define $\psi(W)$ to be the m-degree of $\{x | w(x) \in U\}$. If $W = \emptyset$, define $\psi(W) = \mathbf{0}$. It is easy to verify that $\psi(W)$ is independent of the choice of w, that ψ is onto the m-degrees $\leqq \mathbf{u}$, and that $\psi(W_1 \cup W_2) = \psi(W_1) \cup \psi(W_2)$ for any r.e. sets W_1, W_2. Thus ψ is an upper semilattice homomorphism from the r.e. sets onto the m-degrees $\leqq \mathbf{u}$.

We now prove the first half of the characterization which is our main concern in the paper.

LEMMA 1. *Let S be a finite set of* m-*degrees closed under finite unions. There exists a finite distributive lattice D and mappings $\varphi: S \rightarrow D$ and $\psi: D \rightarrow L_{\mathrm{m}}$ both preserving unions such that $\psi\varphi$ is the identity on S.*

Proof. Let U be a representative of the greatest member $\mathbf{u}$ of S. Let ψ be as above and let $W_0, W_1, \ldots, W_m$ be r.e. sets such that

$$S = \{\psi(W_0), \ldots, \psi(W_m)\}.$$

For each $i \leqq m$ let w_i be a recursive function with range W_i. For each triple of numbers (i, j, k), $i, j, k \leqq m$, we define an r.e. relation Q_{ijk} as follows. If $\psi(W_i) \leqq \psi(W_j) \cup \psi(W_k)$, then $\psi(W_i) \leqq \psi(W_j \cup W_k)$ and thus

$$\{x | w_i(x) \in U\} \leqq_{\mathrm{m}} \{x | w_{jk}(x) \in U\},$$

where w_{jk} is a recursive function whose range is $W_j \cup W_k$. It follows that a recursive function r can be chosen such that for each x, $w_i(x)$ and $w_{jk}r(x)$

are either both in U or in its complement. Let Q_{ijk} consist of all the pairs $(w_i(x), w_{jk}r(x))$. If $\psi(W_i) \nleqq \psi(W_j) \cup \psi(W_k)$, let Q_{ijk} be the empty relation. Let Q be the least equivalence relation including all the relations Q_{ijk}. It is clear that Q is r.e. and that for all x, y, we have

$$Q(x, y) \rightarrow (x \in U \Leftrightarrow y \in U),$$

since each Q_{ijk} has both these properties. For any r.e. set W define $W' = \{x | \exists y[y \in W \,\&\, Q(x, y)]\}$; then W' is r.e. and $\psi(W') = \psi(W)$. To prove the latter, let w and w' be recursive functions whose ranges are W and W', respectively. Since Q is r.e., there exists a recursive function f such that $Q(w'(x), wf(x))$. Thus $\{x| w'(x) \in U\}$ is m-reducible to $\{x| w(x) \in U\}$, whence $\psi(W') \leqq \psi(W)$. We also have $\psi(W) \leqq \psi(W')$ since $W \subseteq W'$.

Let D be the distributive lattice generated by $W_0', \ldots, W_m'$ under union and intersection. Suppose that $\psi(W_i) \leqq \psi(W_j) \cup \psi(W_k)$; then $W_i' \subseteq W_j' \cup W_k'$ since Q contains Q_{ijk}; in particular, taking $j = k$ we see that $\psi(W_i) \leqq \psi(W_j)$ implies $W_i' \subseteq W_j'$. Let φ be defined by $\varphi\psi(W_i) = W_i'$; then from the preceding remarks it is clear that φ preserves unions. Finally, $\psi\varphi$ is the identity on S since $\psi(W') = \psi(W)$ for every r.e. set W and in particular for $W_0, \ldots, W_m$.

Let L be a non-trivial upper semilattice; we say that "L has the closure property" if for every finite subset S of L closed under unions there exist a finite distributive lattice D and mappings $\varphi: S \rightarrow D$ and $\psi: D \rightarrow L$, both preserving unions such that $\psi\varphi$ is the identity on S.

LEMMA 2. *Let L be a countable upper semilattice with* 0 *and* 1, $0 \neq 1$. *There exists a sequence $\langle D_i \rangle$ of finite distributive lattices each with* $0 \neq 1$, *and maps $\chi_i: D_i \rightarrow D_{i+1}$ preserving unions,* 0, *and* 1, *such that L is isomorphic to the direct limit of the sequence*

$$D_0 \xrightarrow{\chi_0} D_1 \xrightarrow{\chi_1} \ldots \xrightarrow{\chi_i} D_{i+1} \xrightarrow{\chi_{i+1}} \ldots,$$

if and only if L has the closure property.

Proof. For the "only if" part suppose that the sequences $\langle D_i \rangle$ and $\langle \chi_i \rangle$ are given such that L is isomorphic to the direct limit. From the definition of direct limit, for each i there is a map $\xi_i: D_i \rightarrow L$ preserving unions, 0, and 1, and such that $\xi_i = \xi_{i+1}\chi_i$. For all i, j with $i < j$ let χ_{ij} denote the composition $\chi_{j-1}\chi_{j-2} \ldots \chi_i$ which maps D_i into D_j. Let S be a finite subset of L, closed under unions. Choose i such that $S \subseteq \xi_i(D_i)$, choose $j > i$ such that ξ_j is one-to-one on $\chi_{ij}(D_i)$. Define D to be D_j, $\varphi: S \rightarrow D$ by

$$\varphi(s) = \text{the unique member of } \chi_{ij}(D_i) \cap \xi_j^{-1}(\{s\}),$$

and ψ to be ξ_j. It is easily verified that φ and ψ both preserve unions and that $\psi\varphi$ is the identity on S. Hence L has the closure property.

For the "if" part suppose that L has the closure property. Let $u_0, u_1, \ldots$ be an enumeration of L. For each $i \geqq 0$ we define a subset S_i of L, a finite distributive lattice D_i and maps φ_i, ψ_i as follows. Let $S_0 = \{0, 1\}$. Suppose that S_i has been defined and is closed under unions, then let D_i be a finite distributive lattice and φ_i: $S_i \to D_i$, ψ_i: $D_i \to L$ be maps preserving unions such that $\psi_i\varphi_i$ is the identity on S_i. Finally, define S_{i+1} to be the closure under unions of $\psi_i(D_i) \cup \{u_i\}$. Let $\chi_i = \varphi_{i+1}\psi_i$; then it is easily seen that the upper semilattice L is the direct limit of the sequence

$$D_0 \xrightarrow{\chi_0} D_1 \xrightarrow{\chi_1} \ldots \xrightarrow{\chi_i} D_{i+1} \xrightarrow{\chi_{i+1}} \ldots .$$

2. The characterization of initial segments. In view of Lemma 2, the characterization of the order types of initial segments of the m-degrees with greatest member, which was stated in the introduction, is immediate from the following result.

THEOREM. *Let L be a countable upper semilattice with* 0. *There exists an initial segment of* m-*degrees isomorphic to L if and only if L has the closure property.*

The "only if" part of the theorem is immediate from Lemma 1. The "if" part will be proved in the remainder of this section. Thus suppose that L is a countable upper semilattice with 0 which has the closure property. Since a greatest member may be adjoined to any upper semilattice, we may suppose that L has a 1, $1 \neq 0$.

We shall now state three propositions and prove the theorem from them, but first some definitions. A *recursive partition* $\mathscr{R}$ of the natural numbers is a canonically enumerable class of disjoint non-empty finite sets such that $\bigcup\mathscr{R} = N$. If $\mathscr{R}$ is a recursive partition and A is a set, then the *closure of A under $\mathscr{R}$*, denoted by $\mathscr{R}(A)$, is the set

$$\bigcup\{R \mid R \in \mathscr{R} \;\&\; R \cap A \neq \emptyset\}.$$

A recursive partition $\mathscr{R}$ is a *refinement* of the recursive partition $\mathscr{R}^*$ if every member of $\mathscr{R}$ is a subset of some member of $\mathscr{R}^*$. The sets A and B are said to be *equivalent with respect to $\mathscr{R}$* if $\mathscr{R}(A) = \mathscr{R}(B)$. When we speak of *quintuples* below we mean those of the form $(U, V, D, \pi, \mathscr{R})$, where $\mathscr{R}$ is a recursive partition, U and V are disjoint non-empty sets in $\mathscr{R}$ and are subsets of $\pi(0)$, D is a finite distributive lattice with 0 and 1, π: $D \to \mathscr{N}$ is an isomorphism, $\pi(a)$ is infinite, recursive, and closed under $\mathscr{R}$ for each atom a of D, and $\pi(0)$ is infinite. Observe that $\pi(0)$ is necessarily recursive and closed under $\mathscr{R}$.

PROPOSITION 1. *Let $(U, V, D, \pi, \mathscr{R})$ be a quintuple, D^* a finite distributive lattice with $0 \neq 1$, and χ: $D \to D^*$ a map preserving unions,* 0, *and* 1. *Then there exists a quintuple $(U, V, D^*, \pi^*, \mathscr{R}^*)$ such that $\mathscr{R}$ is a refinement of $\mathscr{R}^*$ and such that $\pi^*\chi(d) = \mathscr{R}^*\pi(d)$ for all d in D.*

PROPOSITION 2. *Let $(U, V, D, \pi, \mathscr{R})$ be a quintuple, d_1, d_2 members of D such that $d_1 \nleq d_2$, f a unary partial recursive* (*p.r.*) *function, and W_1, W_2 sets* *equivalent to* $\pi(d_1)$, $\pi(d_2)$, *respectively, with respect to* $\mathscr{R}$. *Then there is a quintuple* $(U^*, V^*, D, \pi^*, \mathscr{R}^*)$ *such that* $\mathscr{R}$ *is a refinement of* $\mathscr{R}^*$, $\pi^*(d) = \mathscr{R}^*\pi(d)$ *for all* d *in* D, $U \subseteq U^*$, $V \subseteq V^*$, $(U^* - U) \cap \pi(0) = \emptyset$, *and such that for some* n *in* W_1 *one of the following three possibilities holds:*

(i) $f(n)$ *is undefined or in* $N - W_2$,

(ii) n *is in* U^* *and* $f(n) \in V^*$,

(iii) n *is in* V^* *and* $f(n) \in U^*$.

PROPOSITION 3. *Let* $(U, V, D, \pi, \mathscr{R})$ *be a quintuple and* W *an infinite r.e. set. There is a quintuple* $(U, V, D, \pi^*, \mathscr{R}^*)$ *such that* $\mathscr{R}$ *is a refinement of* $\mathscr{R}^*$, $\pi^*(d) = \mathscr{R}^*\pi(d)$ *for all* d *in* D, *and for some* d *in* D, $\pi^*(d)$ *and* $\mathscr{R}^*(W)$ *differ finitely.*

To prove the theorem from the propositions we start with the sequences $\langle D_i \rangle$ and $\langle \chi_i \rangle$ defined above. Since $S_0 = \{0, 1\}$, note that each D_i has 0 and 1 and that each χ_i preserves unions, 0, and 1. From the propositions, it is clear that there exists a sequence $\langle Q_i \rangle = \langle (U_i, V_i, E_i, \pi_i, \mathscr{R}_i) \rangle$ of quintuples with the following properties:

(q1) $U_{i+1} \supseteq U_i$, $V_{i+1} \supseteq V_i$, and $(U_{i+1} - U_i) \cap \pi_i(0) = \emptyset$ for all i;

(q2) $\mathscr{R}_i$ is a refinement of $\mathscr{R}_{i+1}$ for all i;

(q3) for all i there exists j such that E_i is D_j and such that E_{i+1} is either D_j or D_{j+1}; define θ_i to be the identity if $E_i = E_{i+1} = D_j$ and to be χ_j if E_i and E_{i+1} are D_j, D_{j+1}, respectively; then $\pi_{i+1}\theta_i(e) = \mathscr{R}_{i+1}\pi_i(e)$ for all i and all e in E_i;

(q4) for all j there exists i such that $E_i = D_j$;

(q5) for all i and e_1, e_2 in E_i and each unary p.r. function f, either there exists $k > i$ such that $\theta_{ik}(e_1) \leqq \theta_{ik}(e_2)$, where $\theta_{ik} = \theta_{k-1}\,\theta_{k-2}\ldots\theta_i$ $(i < k)$, or there exists $k > i$ and n in $\pi_i(e_1)$ such that one of the following three possibilities holds:

(i) $f(n)$ is undefined or in $N - \pi_i(e_2)$,

(ii) n is in U_k and $f(n)$ is in V_k,

(iii) n is in V_k and $f(n)$ is in U_k;

(q6) for every infinite r.e. set W there exists i and e in E_i such that $\pi_i(e)$ and $\mathscr{R}_i(W)$ differ finitely.

To see that the sequence $\langle Q_i \rangle$ does indeed exist, notice that we may take $U_0 = \{0\}$, $V_0 = \{1\}$, $E_0 = D_0$, $\mathscr{R}_0 = \{\{x\} \mid x \in N\}$, and π_0 to be any isomorphism of D_0 into $\mathscr{N}$ satisfying the stated conditions. For the rest, (q1–q3) will automatically be satisfied provided that for each i, Q_{i+1} is obtained from Q_i by one of the propositions and that χ is the appropriate χ_j when Proposition 1 is used. We have (q4) provided that Proposition 1 is used an infinite number of times. We can satisfy (q5) by including an infinite number of applications of Proposition 2 in the formation of $\langle Q_i \rangle$ since by induction on k for each $k > i$ we have $\pi_k\theta_{ik}(e_1) = \mathscr{R}_k\pi_i(e_1)$ and $\pi_k\theta_{ik}(e_2) = \mathscr{R}_k\pi_i(e_2)$. Finally, we can ensure (q6) by including an infinite number of applications of Proposition 3 in the formation of $\langle Q_i \rangle$.

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Since $\langle E_i \rangle$ is merely $\langle D_i \rangle$ with repetitions, it is clear that L is isomorphic to the direct limit L^* of

$$E_0 \xrightarrow{\theta_0} E_1 \xrightarrow{\theta_1} \ldots \xrightarrow{\theta_i} E_{i+1} \xrightarrow{\theta_{i+1}} \ldots .$$

We shall complete the proof of the theorem by showing that L^* is the order type of the m-degrees $\leqq \mathbf{u}$, where $\mathbf{u}$ is the m-degree of $U = \bigcup\{U_i | i \geqq 0\}$.

Observe that U is closed under each $\mathscr{R}_i$ since for each i, U_i is closed under $\mathscr{R}$. Also, let $V = \bigcup\{V_i | i \geqq 0\}$; then $U \cap V = \emptyset$ since $U_i \cap V_i = \emptyset$ for all i. Let ψ be the upper semilattice homomorphism from the r.e. sets onto the m-degrees $\leqq \mathbf{u}$ defined in § 1. We shall need the following lemmas.

LEMMA 3. *Let B be an r.e. set such that $B \cap U$ and $B - U$ are both non-empty and let A be any r.e. set; then $\psi(A) \leqq \psi(B)$ if and only if there is a unary p.r. function f mapping A into B such that for all n, we have $n \in U \Leftrightarrow f(n) \in U$.*

Proof. Suppose that A and B are as in the statement of the lemma. If A is finite, then $\psi(A) = \mathbf{0}$ and the lemma is trivially true; thus in what follows we suppose that A is infinite. Let a be a one-to-one recursive function enumerating A, let b be a recursive function whose range is B. Let c be a p.r. function such that $bc(x) = x$ for x in B.

If $\psi(A) \leqq \psi(B)$, then $\{x | a(x) \in U\}$ is m-reducible to $\{x | b(x) \in U\}$; whence there is a recursive function g such that for all n, $a(n) \in U \Leftrightarrow bg(n) \in U$. Clearly we can define f by $f(x) = bga^{-1}(x)$ to satisfy the conclusion of the lemma.

Now suppose that instead of $\psi(A) \leqq \psi(B)$ we are given the existence of a p.r. function f mapping A into B such that $x \in U \Leftrightarrow f(x) \in U$. Defining g by $g(x) = cfa(x)$, it is clear that for all n, $a(n) \in U \Leftrightarrow fa(n) \in U \Leftrightarrow bg(n) \in U$. From this it follows immediately that $\psi(A) \leqq \psi(B)$.

LEMMA 4. *Let $\mathscr{R}$ be a recursive partition under which U is closed. If A and B are r.e. sets equivalent with respect to $\mathscr{R}$, then $\psi(A) = \psi(B)$.*

Proof. Let A and B be r.e. sets equivalent with respect to $\mathscr{R}$. For any m in A there exists n in B such that m and n are in the same member of $\mathscr{R}$. Given m in A, such n can be found effectively since $\mathscr{R}$ is canonically enumerable and B is r.e. Hence there is a p.r. function f mapping A into B such that m and $f(m)$ are always in the same member of $\mathscr{R}$. Since U is closed under $\mathscr{R}$, $x \in U \Leftrightarrow f(x) \in U$. By Lemma 3, $\psi(A) \leqq \psi(B)$. Similarly, $\psi(B) \leqq \psi(A)$, which completes the proof.

We now return to the proof of the theorem. Let p be any member of L^* and let e in E_i be a representative of p; define $\kappa(p) = \psi\pi_i(e)$. For any $k > i$, $\pi_i(e)$ and $\pi_k\theta_{ik}(e)$ are equivalent with respect to $\mathscr{R}_k$, whence by Lemma 4 we have $\psi\pi_i(e) = \psi\pi_k\theta_{ik}(e)$, and so $\kappa(p)$ depends only on p. Let p_1 and p_2 be members of L^* such that $p_1 \leqq p_2$; then there exist i and e_1, e_2 in E_i representing p_1, p_2, respectively, such that $e_1 \leqq e_2$. By definition, $\kappa(p_1) = \psi\pi_i(e_1)$, $\kappa(p_2) = \psi\pi_i(e_2)$. However, $\pi_i(e_1) \subseteq \pi_i(e_2)$ since π_i is an isomorphism, whence $\kappa(p_1) \leqq \kappa(p_2)$. Suppose now that p_1, p_2, e_1, e_2, and i are as defined above but that now $p_1 \nleqq p_2$. For *reductio ad absurdum* suppose that $\kappa(p_1) \leqq \kappa(p_2)$; then $\psi\pi_i(e_1) \leqq \psi\pi_i(e_2)$. Now $\pi_i(e_2) \supseteq \pi_i(0) = \mathscr{R}_i\pi_0(0) \supseteq U_0 \cup V_0$. Hence $\pi_i(e_2) \cap U$ and $\pi_i(e_2) - U$ are both non-empty. By Lemma 3 there exists a unary p.r. function f mapping $\pi_i(e_1)$ into $\pi_i(e_2)$ such that for all n, we have $n \in U \Leftrightarrow f(n) \in U$. But this contradicts (q5), whence $\kappa(p_1) \nleqq \kappa(p_2)$.

We have shown above that κ is an embedding of L^* in the m-degrees $\leqq \mathbf{u}$. It only remains to prove that κ is onto. Let W be an infinite r.e. set; then by (q6) there exist i and e in E_i such that $\pi_i(e)$ and $\mathscr{R}_i(W)$ differ finitely. Since ψ preserves unions and $\psi(F) = \mathbf{0}$ for any finite F, we have $\psi\pi_i(e) = \psi\mathscr{R}_i(W)$. Since W and $\mathscr{R}_i(W)$ are equivalent with respect to $\mathscr{R}_i$, $\psi(W) = \psi\mathscr{R}_i(W)$. Hence $\psi(W) = \kappa(p)$, where p is the member of L^* represented by e. Finally, let W be a finite set; then as noted above, $\psi(W) = \mathbf{0}$. Clearly $\kappa(0) = \psi\pi_0(0)$. Notice by setting $e = 0$ in (q3) that $\pi_{i+1}(0) \supseteq \pi_i(0)$ for all i. From (q1), $\pi_0(0) \cap U \subseteq U_0$, which is finite, whence

$$\psi(\pi_0(0)) = \psi(\pi_0(0) - U) = \mathbf{0}$$

by definition of ψ. Thus when W is finite, $\psi(W) = \kappa(0)$. Since ψ is onto the m-degrees $\leqq \mathbf{u}$, κ is onto the m-degrees $\leqq \mathbf{u}$, and the proof of the theorem is complete.

It only remains to prove the three propositions stated above.

Proof of Proposition 1. Let A and A^* be the Boolean algebras generated by D and D^*, respectively. We first observe that for every atom a^* of A^* there is a class $C(a^*)$ of atoms of A such that for all d in D we have

$$a^* \leqq \chi(d) \Leftrightarrow \exists a[a \in C(a^*) \mathbin{\&} a \leqq d].$$

Suppose for *reductio ad absurdum* that for the atom a^* of A^* no such class $C(A^*)$ exists. It follows that for some d in D, $a^* \leqq \chi(d)$ and every atom a of D, either $a \nleqq d$ or there exists e in D such that $a \leqq e$ and $a^* \nleqq \chi(e)$. Let

$$e_1 = \bigcup\{e \mid e \in D \mathbin{\&} a^* \nleqq \chi(e)\}.$$

Since d is a union of atoms ($d \leqq e_1$), whence $\chi(d) \leqq \chi(e_1)$. Since χ preserves unions,

$$\chi(e_1) = \bigcup\{\chi(e) \mid e \in D \mathbin{\&} a^* \nleqq \chi(e)\};$$

whence $a^* \nleqq \chi(e_1)$. It follows that $a^* \nleqq \chi(d)$. This contradiction shows that $C(a^*)$ exists.

For each atom a of A let $\mathscr{R}[a]$ be the subclass of $\mathscr{R}$ consisting of those members of $\mathscr{R}$ which are subsets of $\pi(a)$. Note that $\pi(a)$ is infinite from the definition of quintuple and closed under $\mathscr{R}$ since $\pi(d)$ is closed under $\mathscr{R}$ for each d in D. Clearly, $\mathscr{R}[a]$ is infinite and $\bigcup\mathscr{R}[a] = \pi(a)$. For each atom a of D which occurs in at least one of the classes $C(a^*)$, partition $\mathscr{R}[a]$ into n canonically enumerable infinite subclasses where n is the number of a^*s such that

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$a \in C(a^*)$, and assign a different member of the partition, $\mathscr{R}[a, a^*]$ say, to each a^* such that $a \in C(a^*)$. Let $\langle R_i[a, a^*]\rangle$ be a canonical enumeration of $\mathscr{R}[a, a^*]$. Let $\mathscr{R}[0]$ consist of those members of $\mathscr{R}$ which are subsets of $\pi(0) - (U \cup V)$; then $\bigcup\mathscr{R}[0] = \pi(0) - (U \cup V)$. Let $\langle R_i[0]\rangle$ be a canonical enumeration of $\mathscr{R}[0]$. Let $\langle T_i\rangle$ be a canonical enumeration of all the members of $\mathscr{R} - \{U, V\}$ which are neither in $\mathscr{R}[0]$ nor in any $\mathscr{R}[a, a^*]$. We recall that by convention, canonical enumerations of infinite classes are to be without repetitions.

Define $\mathscr{R}^*$ to be the recursive partition

$$\{U, V\} \cup \{R_i[0] \cup T_i | i \geqq 0\} \cup \{\bigcup\{R_i[a, a^*]| a \in C(a^*)\}|i \geqq 0 \;\&\; a^* \text{ is an atom of } D^*\}.$$

For each atom a^* of D^* define

$$\pi^*(a^*) = \bigcup(\bigcup\{\mathscr{R}[a, a^*]| a \in C(a^*)\}),$$

and define

$$\pi^*(0) = U \cup V \cup \bigcup\{R_i[0] \cup T_i| i \geqq 0\}.$$

Define

$$\pi^*(d^*) = \pi^*(0) \cup \bigcup\{\pi^*(a^*)| a^* \leqq d^*\}$$

for each d^* in D^*.

It is easy to check that $(U, V, D^*, \pi^*, \mathscr{R}^*)$ is a quintuple and that $\mathscr{R}$ is a refinement of $\mathscr{R}^*$. We have only to prove that $\pi^*\chi(d) = \mathscr{R}^*\pi(d)$. Consider n in $\pi^*\chi(d)$. If n is in $\pi^*(0)$, then certainly n is in $\mathscr{R}^*\pi(d) \supseteq \mathscr{R}^*\pi(0)$, since each member of $\mathscr{R}^*[0]$ includes a member of $\mathscr{R}[0]$. If n is not in $\pi^*(0)$, there exists an atom a^* of D^* such that n is in $\pi^*(a^*)$ and $a^* \leqq \chi(d)$. There exists a in $C(a^*)$ such that $a \leqq d$, and thus every member of $\mathscr{R}^*[a^*]$ contains a member of $\mathscr{R}[a]$ and hence of $\pi(d)$. Again n is in $\mathscr{R}^*\pi(d)$. Thus $\pi^*\chi(d) \subseteq \mathscr{R}^*\pi(d)$. We can reverse the argument to show that $\mathscr{R}^*\pi(d) \subseteq \pi^*\chi(d)$ also. This completes the proof of Proposition 1.

Proof of Proposition 2. Assume the hypothesis of the proposition. Since $\pi(d_1) \nleqq \pi(d_2)$ and $\pi(a)$ is infinite for each atom a of D, it follows that $\pi(d_1) - \pi(d_2)$ is non-empty and hence that $W_1 - \pi(d_2)$ is non-empty. Let n be any member of $W_1 - \pi(d_2)$ and suppose that $f(n)$ is defined and in W_2 (for otherwise the result is trivial). Since $W_2 \subseteq \pi(d_2)$ and $\pi(d_2)$ is closed under $\mathscr{R}$, n and $f(n)$ are in different members of $\mathscr{R}$, say R_1 and R_2, respectively. Observe that $R_1 \cap \pi(0) = \emptyset$ since $\pi(0) \subseteq \pi(d_2)$. Hence $n \notin U \cup V$. There are now two cases according as $f(n) \in U$ or not. Suppose that $f(n) \notin U$ and define

$$\mathscr{R}^* = (\mathscr{R} - \{U, V, R_1, R_2\}) \cup \{U \cup R_1, V \cup R_2\},$$
$$\pi^*(d) = \pi(d) \cup R_1 \cup R_2 \quad \text{for all } d \text{ in } D.$$

Then it is easy to verify the conclusion of the proposition. The case $f(n) \in U$ may be treated similarly.

Proof of Proposition 3. Assume the hypothesis of the proposition. Let

$$\mathscr{A} = \{a \mid \pi(a) \cap W \text{ is infinite } \& \ [a \text{ is an atom of } D \text{ or } a = 0]\}.$$

Let e be the least member of D such that $a \leqq e$ for all a in $\mathscr{A}$. Let $\mathscr{B}$ consist of 0 together with all atoms $\leqq e$. Then there is map $\gamma: \mathscr{B} \rightarrow \mathscr{A}$ such that for all b in $\mathscr{B}$, we have

$$(1) \qquad (d)_{d \in D}[\gamma(b) \leqq d \rightarrow b \leqq d].$$

This is an elementary fact about finite Boolean algebras. We may assume that γ is the identity of $\mathscr{A}$. For each a in range γ, partition $\mathscr{R}[a]$ into $n + 1$ infinite canonically enumerable classes, where n is the cardinality of $\gamma^{-1}(a)$, such that at most one member of the partition has a member whose intersection with W is empty. Assign a different member of the partition, say $\mathscr{R}[a, b]$, to each b such that $\gamma(b) = a$, in such a way that the remaining member of the partition $\mathscr{R}^-[a]$ is that one (if there is one) which has members not intersecting W. Let $\langle R_i[a]\rangle$, $\langle R_i^-[a]\rangle$, and $\langle R_i[a, b]\rangle$ be canonical enumerations of the corresponding canonically enumerable classes. Define

$$\mathscr{R}^* = (\mathscr{R} - \bigcup \{\mathscr{R}[a] \mid a \in \mathscr{B}\}) \cup \{R_i[b] \cup R_i[\gamma(b), b] \mid i \geqq 0 \,\&\, b \in \mathscr{B} - \mathscr{A}\}$$
$$\cup \{R_i^-[a] \cup R_i[a, a] \mid i \geqq 0 \,\&\, a \in \mathscr{A}\}.$$

For all atoms a of D not in $\mathscr{B}$ define $\pi^*(a) = \pi(a)$; for b in $\mathscr{B} - (\mathscr{A} \cup \{0\})$ define

$$\pi^*(b) = \bigcup \{R_i[b] \cup R_i[\gamma(b), b] \mid i \geqq 0\}.$$

For $a \in \mathscr{A} - \{0\}$ define

$$\pi^*(a) = \bigcup \{R_i^-[a] \cup R_i[a, a] \mid i \geqq 0\}.$$

Define

$$\pi^*(0) = \begin{cases} U \cup V \cup \bigcup \{R_i[0] \cup R_i[\gamma(0), 0] \mid i \geqq 0\} & \text{if } 0 \in \mathscr{B} - \mathscr{A} \\ U \cup V \cup \bigcup \{R_i^-[0] \cup R_i[0, 0] \mid i \geqq 0\} & \text{if } 0 \in \mathscr{A}. \end{cases}$$

Finally, for all d in D define

$$\pi^*(d) = \pi^*(0) \cup \bigcup \{\pi^*(a) \mid a \text{ is an atom of } D \text{ and } a \leqq d\}.$$

The reader will easily verify that $(U, V, D, \pi^*, \mathscr{R}^*)$ is a quintuple, and that $\pi^*(e) = \pi(e)$. Further, by our construction, every member of $\mathscr{R}^*[e] - \{U, V\}$ intersects W. However, by definition of e, $W - \pi(e)$ is finite. Hence $\pi^*(e)$ and W differ finitely. It only remains to show that for all d in D, $\pi^*(d) = \mathscr{R}^*\pi(d)$. Consider n in $\pi^*(d)$; then n is in $\pi^*(a)$, where a is either 0 or an atom of D. If $a \notin \mathscr{B}$, then $n \in \pi(d)$ since $\pi^*(a) = \pi(a)$, whence $n \in \mathscr{R}^*\pi(d)$. If $a \in \mathscr{B} - (\mathscr{A} \cup \{0\})$, then there exists i such that

$$n \in R_i[a] \cup R_i[\gamma(a), a] \in \mathscr{R}^*.$$

Since $R_i[a] \subseteq \pi(a)$, again $n \in \mathscr{R}^*\pi(d)$. The cases $a \in \mathscr{A} - \{0\}$ and $a = 0$ may be treated similarly. Thus $\pi^*(d) \subseteq \mathscr{R}^*\pi(d)$. To prove the inclusion the other way suppose that $n \in \mathscr{R}^*\pi(d)$; let R be the member of $\mathscr{R}^*$ such that $n \in R$. If $R \in \mathscr{R} \cap \mathscr{R}^*$, then $n \in U \cup V$ or $n \in \pi(a)$ for some atom a of D not in $\mathscr{B}$. For any such a, $\pi^*(a) = \pi(a)$. Hence $n \in \pi^*(d)$ in this case. If there exist i and b in $\mathscr{B} - \mathscr{A}$ such that $n \in R_i[b] \cup R_i[\gamma(b), b]$, then $R_i[b] \cup R_i[\gamma(b), b]$ intersects $\pi(d)$. However, $R_i[\gamma(b), b] \in \mathscr{R}[\gamma(b)]$, and from (1) we deduce that $b \leqq d$. Hence $n \in \pi^*(b) \subseteq \pi^*(d)$. The only other possibility is that there exist i and a in $\mathscr{A}$ such that $n \in R_i^-[a] \cup R_i[a, a]$. Since $R_i^-[a] \cup R_i[a, a]$ intersects $\pi(d)$ and is a subset of $\pi(a)$, $a \leqq d$, and $n \in \pi^*(a) \subseteq \pi^*(d)$. This completes the proof.

3. Conclusion. One corollary to our theorem is that the order types of finite initial segments of the m-degrees are just the order types of finite initial segments of distributive lattices. To see this, let L be a non-trivial finite upper semilattice which has the closure property. Take S to be L in the definition of the closure property and let $\varphi: L \to D$ and $\psi: D \to L$ be the maps whose existence is required by that definition. We see at once that L has 0 since D has one. Therefore L is a lattice. Let a, b, and c be elements of L. To show that L is distributive, it suffices to prove that

$$(a \cup b) \cap c \leqq (a \cap c) \cup (b \cap c).$$

Since φ preserves unions and D is distributive, we have

$$\varphi((a \cup b) \cap c) \leqq \varphi(a \cup b) \cap \varphi(c) = (\varphi(a) \cup \varphi(b)) \cap \varphi(c)$$
$$= (\varphi(a) \cap \varphi(c)) \cup (\varphi(b) \cap \varphi(c)).$$

Applying ψ to both sides we have

$$(a \cup b) \cap c \leqq \psi(\varphi(a) \cap \varphi(c)) \cup \psi(\varphi(b) \cap \varphi(c)) \leqq (a \cap c) \cup (b \cap c).$$

Another corollary is that the elementary theory of the upper semilattice of m-degrees is not axiomatizable; for details see [**1**, § 3]. The methods of this paper are closely related to those of [**1**] where it was shown that any countable distributive lattice with least member can be embedded as an initial segment of the Turing degrees.

There is a characterization of the order types of initial segments of r.e. m-degrees which is very similar to that given here. The only difference is that the sequences $\langle E_i \rangle$ and $\langle \theta_i \rangle$ mentioned in the introduction must now be suitably effective. Of course, the construction of initial segments is now a good deal more complicated but the underlying algebra is still the same.

Finally, the method of § 2 can be modified so as to yield the following result about one-one degrees. Given any upper semilattice L with $0 \neq 1$ and consistent with the lemma, there exists a one-one degree $\mathbf{u}$ such that every one-one degree $\leqq \mathbf{u}$ is either the one-one degree of a cylinder, or of a finite or co-infinite set, and such that the one-one degrees of the cylinders $\leqq \mathbf{u}$ form an upper semilattice isomorphic to L. The main change that has to be made consists in working with partitions $\mathscr{R}$ of N into infinite recursive sets rather than finite sets. Otherwise the modification of our construction is quite straightforward.

References

1. A. H. Lachlan, *Distributive initial segments of the degrees of unsolvability*, Z. Math. Logik Grundlagen Math. *14* (1968), 457–472.

2. H. Rogers Jr., *Theory of recursive functions and effective computability* (McGraw-Hill, New York, 1967).

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