CORRESPONDENCE.

ON THE VALUE OF CONTINGENT REVERSIONS SUBJECT TO CERTAIN LIMITATIONS.

To the Editors of the Assurance Magazine.

GENTLEMEN,—The two following Problems appear to me to be of sufficient importance in the doctrine of Life Contingencies, and of sufficient moment in a practical point of view, to merit a page in your Magazine.

The second of these Problems was, for the first time, solved by Mr. David Jones in his work on Reversionary Payments, page 184; but by a different method from that adopted in Problem II., and I think in a less practical and commodious form than the one that I have arrived at.

Problem I., however, which is manifestly the fundamental theorem, has not been to my knowledge solved but by myself. I first gave the formulæ for the solution of this, and two other similar and dependant problems, in my little work on Notation, published in 1840, but unaccompanied by any reasoning or analysis. But the analysis, although brief, is, in my opinion, so full of interest and instruction to the student, and the resulting expression so simple and so elegant, that you may perhaps consider it not altogether unworthy of the attention of your readers.

I am, Gentlemen,
Yours very faithfully,
PETER HARDY.

London Assurance,
7, *Royal Exchange, August* 2, 1851.

PROBLEM I.—To determine the present value of a reversion of £1, payable on the death of A, provided he survives another life, B, by at least n years.

Solution.—Supposing the sum assured to be paid as usual, at the termination of the year in which the death occurs, this reversion cannot possibly become payable until the end of the $n+1$st year at the very least. In the $n+1$st year, its becoming payable will depend on the probability that B died in the *first year*, and that A died in the $n+1$st year, in particular equal to

$$\frac{b-b_1}{b}\times\frac{a_n-a_{n+1}}{a_n}\times\frac{a_n}{a}=\frac{b-b_1}{b}\times\frac{a_n-a_{n+1}}{a};$$

but as B may possibly die *at the end* of the first year, and as A may die at *the beginning* of the $n+1$st year, it is an even chance whether full n years will have elapsed or not, consequently *one-half* of the foregoing expression discounted for $n+1$ years will be the value of the expectation for the $n+1$st year, equal to

$$\frac{b-b_1}{b}\times\frac{a_n-a_{n+1}}{a}\times\frac{1}{2}\times\frac{1}{r^{n+1}}$$

$$=\frac{1}{2}\left(\frac{a_n b}{abr^{n+1}}-\frac{a_n b_1}{abr^{n+}}-\frac{a_{n+1} b}{abr^{n+1}}+\frac{a_{n+1} b_1}{abr^{n+1}}\right).$$

In the $n+2$nd year, the value will depend, firstly, on the probability that A dies at the *end* of the $n+2$nd year, and that B dies at the *beginning* of the second year, equal by the foregoing reasoning to

$$\frac{1}{2}\left(\frac{a_{n+1} b_1}{abr^{n+2}}-\frac{a_{n+1} b_2}{abr^{n+2}}-\frac{a_{n+2} b_1}{abr^{n+2}}+\frac{a_{n+2} b_2}{abr^{n+2}}\right);$$

and secondly, on the probability that A dies in *any part* of the $n+2$nd year, B having died in the *first* year, which also discounted for $n+2$ years is equal to

$$\frac{b-b_1}{b}\times\frac{a_{n+1}-a_{n+2}}{a}\times\frac{1}{r^{n+2}}=\frac{a_{n+1} b}{abr^{n+2}}-\frac{a_{n+1} b_1}{abr^{n+2}}-\frac{a_{n+2} b}{abr^{n+2}}+\frac{a_{n+2} b_1}{abr^{n+2}}.$$

The sum of these two expressions, as below, viz.—

$$\frac{a_{n+1} b}{abr^{n+2}}-\frac{a_{n+2} b}{abr^{n+2}}+\frac{1}{2}\left(-\frac{a_{n+1} b_1}{abr^{n+2}}-\frac{a_{n+1} b_2}{abr^{n+2}}+\frac{a_{n+2} b_1}{abr^{n+2}}+\frac{a_{n+2} b_2}{abr^{n+2}}\right),$$

will be the value of the expectation in the $n+2$nd year.

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In the $n+3$rd year, the value of the expectation may, by similar reasoning, be shown to be—

$$\frac{a_{n+2}b}{abr^{n+3}} - \frac{a_{n+3}b}{abr^{n+3}} + \frac{1}{2}\left(-\frac{a_{n+2}b_2}{abr^{n+3}} - \frac{a_{n+2}b_3}{abr^{n+3}} + \frac{a_{n+3}b_2}{abr^{n+3}} + \frac{a_{n+3}b_3}{abr^{n+3}}\right);$$

and so on for the $n+4$th, $n+5$th, &c. years, to the termination of the risk; and these several expressions for each year may be arranged as follows:—

$$\text{1st year} = \frac{a_n b}{2abr^{n+1}} - \frac{a_{n+1}b}{2abr^{n+1}} - \frac{1}{2}\left(\qquad - \frac{a_{n+1}b_1}{abr^{n+1}} + \frac{a_n b_1}{abr^{n+1}} \qquad \right)$$

$$\text{2nd year} = \frac{a_{n+1}b}{abr^{n+2}} - \frac{a_{n+2}b}{abr^{n+2}} - \frac{1}{2}\left(\frac{a_{n+1}b_1}{abr^{n+2}} - \frac{a_{n+2}b_2}{abr^{n+2}} + \frac{a_{n+1}b_2}{abr^{n+2}} - \frac{a_{n+2}b_1}{abr^{n+2}}\right)$$

$$\text{3rd year} = \frac{a_{n+2}b}{abr^{n+3}} - \frac{a_{n+3}b}{abr^{n+3}} - \frac{1}{2}\left(\frac{a_{n+2}b_2}{abr^{n+3}} - \frac{a_{n+3}b_3}{abr^{n+3}} + \frac{a_{n+2}b_3}{abr^{n+3}} - \frac{a_{n+3}b_2}{abr^{n+3}}\right)$$

And if the perpendicular columns be then summed, the first and second columns having every term multiplied and divided by the number living at the age of B, will be found equal to

$$\frac{a_n}{ar_n}\left(\mathrm{A}_n\mathrm{I}\mathring{1} + \frac{a_n - a_{n+1}}{2a_n r}\right);$$

the third column will be found equal to

$$\frac{\mathrm{I}\overline{\mathrm{A}_n\mathrm{B}}^{1}+1}{2r}\cdot\frac{a_n}{ar^n} - \frac{a_n}{2a_n r};$$

the fourth to

$$\frac{\mathrm{I}\overline{\mathrm{A}_n\mathrm{B}}^{1}}{2} + \frac{a_n}{ar^n};$$

the fifth to

$$\left(\frac{\mathrm{I}\overline{\mathrm{A}_n\mathrm{B}_1}^{1}+1}{2r}\cdot\frac{b_1}{b}\right)\frac{a_n}{ar^n};$$

the sixth to

$$\left(\frac{\mathrm{I}\overline{\mathrm{A}_{n+1}\mathrm{B}}^{1}+1}{2r}\cdot\frac{a_{n+1}}{a_n}\right)\frac{a_n}{ar^n} + \frac{a_{n+1}}{2a_n r};$$

and if the sum of these separate sums be again taken we shall have

$$\frac{a_n}{ar^n}\left[A_n\mathrm{I}\overset{\circ}{1} + \frac{a_n - a_{n+1}}{2a_n r} - \frac{1}{2}\left(\frac{\mathrm{I}\overline{A_nB}^{1}+1}{r} - \mathrm{I}\overline{A_nB}^{1}\right.\right.$$

$$\left.\left. + \frac{\mathrm{I}\overline{A_nB_1}^{1}+1}{r}\cdot\frac{b_1}{b} - \frac{\mathrm{I}\overline{A_{n+1}B}^{1}+1}{r}\cdot\frac{a_{n+1}}{a_n}\right) - \frac{a_n - a_{n-1}}{2a_n r}\right]$$

$$= \frac{a_n}{ar^n}\left[A_n\mathrm{I}\overset{\circ}{1} - \frac{1}{2}\left(\frac{\mathrm{I}\overline{A_nB}^{1}+1}{r} - \mathrm{I}\overline{A_nB}^{1}\right.\right.$$

$$\left.\left. + \frac{\mathrm{I}\overline{A_nB_1}^{1}+1}{r}\cdot\frac{b_1}{b} - \frac{\mathrm{I}\overline{A_{n+1}B}^{1}+1}{r}\cdot\frac{a_{n+1}}{a_n}\right)\right]$$

Now the first part of this expression is the value of a deferred reversion on the life of A,

$$= \frac{a_n}{ar_n}\left(A_n\mathrm{I}\overset{\circ}{1}\right);$$

and the second part is the value of a reversion contingent on B surviving a life n years older than A, multiplied into the value of £1 to be received if A lives n years,

$$= \left(\underset{\mathrm{I}}{\overline{A_nB}^{1}}\mathrm{I}\overset{\circ}{1}\right)\frac{a_n}{ar^n};$$

and the two together constitute the entire value of the reversion to be paid on the death of A, provided he lives n years at least after the death of B, equal to

$$\frac{a_n}{ar_n}\left(A_n\mathrm{I}\overset{\circ}{1} - \underset{\mathrm{I}}{\overline{A_nB}^{1}}\mathrm{I}\overset{\circ}{1}\right).$$

Q. E. D.

The rule enunciated in words at length consequently becomes as follows:—"From the value of an absolute reversion on a single life n years older than A, subtract the value of a reversion contingent on B surviving the said life n years older than A, and multiply the difference by the present value of £1 payable if A lives n years."

If instead of an algebraical analysis of the rule, as above given, we employ the more obvious, but less satisfactory, deduction by reasoning, it will be manifest (supposing even that we put the age of B at the extremity of life) that the limit of the value would be the value of a reversion on the life of A deferred for n years, obviously equal to the affirmative quantity

$$\frac{a_n}{ar^n}\left(A_n\mathrm{I}\overset{\circ}{1}\right);$$

but this value must be lessened by the probability of the life of A failing in the lifetime of B, or even after B, if within n years; this brings the life of B into actual competition of survivorship with a life n years older than A, provided that A lives the entire term of n years; hence the value of the deferred reversion as above is lessened by the value of the survivorship payable if a life n years older than A dies in the lifetime of B, multiplied into the value of £1 payable if A lives n years; hence the true value is equal to

$$\left(A_n\mathrm{I}\mathring{\imath} - \overset{1}{\underset{\mathrm{I}}{\overline{A_nB}}}\,\mathrm{I}\mathring{\imath}\right)\frac{a_n}{ar^n}.$$

Problem II.—To determine the present value of a reversion of £1 payable on the death of A, provided he dies before another life, B, or *within* n years after him.

Solution.—The value of this contingent reversion is manifestly very nearly equal to the value of an absolute reversion on the single life of A, as the assurance will not be paid only in the event of his surviving B by at least n years, as in the foregoing problem. Consequently

$$A\mathrm{I}\mathring{\imath} - \frac{a_n}{ar^n}\left(A_n\mathrm{I}\mathring{\imath} - \overset{1}{\underset{\mathrm{I}}{\overline{A_nB}}}\,\mathrm{I}\mathring{\imath}\right)$$

will be equal to the required value,

$$A\mathrm{I}\mathring{\imath} - \frac{a_n}{ar^n}\left(A_n\mathrm{I}\mathring{\imath}\right) + \frac{a_n}{ar^n}\left(\overset{1}{\underset{\mathrm{I}}{\overline{A_nB}}}\,\mathrm{I}\mathring{\imath}\right);$$

and since

$$A\mathrm{I}\mathring{\imath} - \frac{a_n}{ar^n}\left(A_n\mathrm{I}\mathring{\imath}\right) = \overset{1}{\underset{\mathrm{I}}{\overline{An}}}\,\mathrm{I}\mathring{\imath}$$

equals the value of a temporary reversion or assurance on the life of A, it is obvious that the value of the contingent reversion required by the problem will be

$$\overset{1}{\underset{\mathrm{I}}{\overline{An}}}\,\mathrm{I}\mathring{\imath} + \left(\overset{1}{\underset{\mathrm{I}}{\overline{A_nB}}}\,\mathrm{I}\mathring{\imath}\right)\frac{a_n}{ar^n}.$$

Q. E. D.

The rule in words at length will consequently be—"To the value of a temporary assurance on the life of A, add the value of a reversion contingent on B surviving a life n years older than A, multiplied into the present value of £1 payable if A lives n years."

ON A NEW EXPRESSION FOR THE VALUE OF THE ANNUAL PREMIUM FOR A LIFE ASSURANCE.

To the Editors of the Assurance Magazine.

Gentlemen,—In your last Number, p. 332, the very striking analogy was pointed out which subsists between a whole life assurance, and an assurance which is certainly payable at a specified age if it have not before become due. I may perhaps be pardoned for stating that this analogy had been previously shown by me, in proof of which I beg to refer you to "Tables and Formulæ for the Computation of Life Contingencies," p. 95, Corollary.

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