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CHARACTERIZATIONS OF FUNCTIONALLY COMPACT SPACES

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Abstract

This paper gives characterizations of functionally compact spaces in terms of filterbases and nets. Also, a topological property that is weaker than countable compactness but stronger than first countable minimality is investigated.

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1. Introduction

Dickman and Zame (1969) introduced the notion of functionally compact spaces. Covering properties of functionally compact spaces have been studied by Gross and Viglino (1970). Our primary interest is to characterize functional compactness in terms of nets and filterbases. These characterizations are obtained mainly through the concept of Θ-convergence. Also we investigate a topological property that is weaker than countable compactness but stronger than first countable minimality.

Throughout, $\mathrm{cl}(A)$ will denote the closure of a subset A in a topological space X and $G(f)$ will denote the graph of a function $f\colon X \to Y$.

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2. Preliminary definitions and theorems

DEFINITION 2.1. Let X be a topological space and let $\emptyset \neq A \subset X$. A filterbase $\mathcal{F} = \{F_\alpha: F_\alpha \subset A, \alpha \in \Delta\}$ on A *θ-converges* to $p \in X$ ($\mathcal{F} \to_\theta p$) if for each open $V \subset X$ containing p, there exists an $F_\alpha \in \mathcal{F}$ such that $F_\alpha \subset \mathrm{cl}(V)$. $\mathcal{F}$ *θ-accumulates* to $p \in X$ ($\mathcal{F} \propto_\theta p$) if for each open $V \subset X$ containing p and for each $F_\alpha \in \mathcal{F}$, $F_\alpha \cap \mathrm{cl}(V) \neq \emptyset$ (Velicko (1969)).

In an obvious way, the concept of Θ-convergence is defined for nets. It is pointed out by Velicko (1969) that the concepts Θ-convergence and Θ-accumulation in a topological space X for filterbases and nets are 'equivalent' in the sense that:

(1) Each net ϕ determines a filterbase $\mathcal{F}(\phi)$ such that $\phi \to_\theta p \in X$ ($\phi \propto_\theta p \in X$) if and only if $\mathcal{F}(\phi) \to_\theta p$ ($\mathcal{F}(\phi) \propto_\theta p$).

(2) Each filterbase $\mathcal{F}$ determines a net $\phi: \mathcal{D} \to X$ such that $\mathcal{F} \to_\theta p \in X$ ($\mathcal{F} \propto_\theta p$) if and only if $\phi \to_\theta p$ ($\phi \propto_\theta p$).

DEFINITION 2.2. A Hausdorff space X is *functionally compact* if for every open filterbase $\mathcal{F}$ in X such that the intersection A of elements of $\mathcal{F}$ equals the intersection of the closures of elements of $\mathcal{F}$, then $\mathcal{F}$ is a neighborhood filterbase of A (Dickman and Zame (1969)).

DEFINITION 2.3. A closed subset A of a topological space X is *Θ-closed* if for every $x \notin A$ there exist two disjoint open sets U and V containing x and A, respectively. An open set $U \subset X$ is *Θ-open* if and only if its complement is Θ-closed (Velicko (1969)).

Θ-closed sets are used in Theorem III (Gross and Viglino (1969)) to characterize functionally compact spaces. Following Definition 2.4, we show that the notions of Θ-closed sets and Θ-convergence can be used to characterize functional compactness in terms of filterbases and nets.

DEFINITION 2.4. Let A be a closed subset of a topological space X. If $U \subset X$ is a Θ-open set containing A, then we say that (A, U) is a *Θ-ordered pair* in X.

3. Filterbase characterizations of functionally compact spaces

THEOREM 3.1. *In a Hausdorff space X the following are equivalent*:

(a) *X is functionally compact.*

(*b*) *For each Θ-ordered pair (A, U) in X and each filterbase $\mathcal{F}$ on A, there exists a point $p \in U$ such that $\mathcal{F} \propto_\theta p$.*

(c) *For each Θ-ordered pair (A, U) in X and each maximal filterbase $\mathcal{M}$ on A, there exists a point $p \in U$ such that $\mathcal{M} \to_\theta p$.*

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PROOF. (a) implies (b). Suppose there exists a Θ-ordered pair (A, U) in X and a filterbase, $\mathscr{F}$, on A that does not θ-accumulate in the Θ-open set U. Let $K = X-U$ and define $H = X-A$. Then H is an open set containing the Θ-closed set K. Now for each $x \in U$, there exists an open set $V(x)$ containing x (where $\mathrm{cl}(V(x)) \subset U$) such that $\mathrm{cl}(V(x)) \cap F_x = \varnothing$ for some $F_x \in \mathscr{F}$. Thus the collection $\mathscr{C} = \{V(x): x \in U\}$ forms an open cover of $X-K$. Since $\mathscr{F}$ does not θ-accumulate to any $x \in U$, it follows that for each finite subcollection $\{V(x_i): i = 1, 2, \ldots, n\}$ of $\mathscr{C}$,

$$\bigcup_{i=1}^{n} \mathrm{cl}(V(x_i)) \cup H \neq X.$$

Consequently, X is not functionally compact according to Theorem III by Gross and Viglino (1970).

(b) implies (a). Suppose that the topological space X is not functionally compact. Then there exists an open set U containing a Θ-closed set A and an open cover $\mathscr{C} = \{U_\alpha: \alpha \in \Delta\}$ of $X-A$ such that for each finite subcollection $\{U_{\alpha_i}: i = 1, 2, \ldots, n\}$ of $\mathscr{C}$, $\bigcup_{i=1}^{n} \mathrm{cl}(U_{\alpha_i}) \cup U \neq X$. Thus the sets $(X-\mathrm{cl}(U_\alpha)) \cap (X-U)$ (where $\alpha \in \Delta$) together with all finite intersections of the form $\bigcap_{i=1}^{n}(X-\mathrm{cl}(U_{\alpha_i})) \cap (X-U)$ form a filterbase, $\mathscr{F}$, on the closed set $X-U$. Now if $x \in X-A$ and if $U_\alpha \supset \{x\}$ (where $U_\alpha \in \mathscr{C}$), then $\mathrm{cl}(U_\alpha) \cap (X-\mathrm{cl}(U_\alpha)) \cap (X-U) = \varnothing$. Consequently, $\mathscr{F}$ does not θ-accumulate in the Θ-open set $X-A$ which contains the closed set $X-U$.

(b) implies (c). Let $\mathscr{M}$ be a maximal filterbase on the closed set A. Then $\mathscr{M}$ θ-accumulates to some $p \in U$ and hence θ-converges to p according to Theorem 2 of Velićko (1969).

(c) implies (b). Let $\mathscr{F}$ be a filterbase of the closed set $A \subset X$. Then there exists a maximal filterbase, $\mathscr{M}$, on A stronger than $\mathscr{F}$ that Θ-converges to some point $p \in U$. Applying Lemma 4 and Theorem 2 by Velicko (1969), we see that $\mathscr{F}$ Θ-accumulates to p.

Since filterbases and nets are 'equivalent' in the sense of θ-convergence, we can now characterize functional compactness in terms of nets.

THEOREM 3.2. *In a Hausdorff space X the following are equivalent:*

(a) *X is functionally compact.*

(b) *For each Θ-ordered pair (A, U) in X and each net Φ on A, there exists a point $p \in U$ such that $\Phi \propto_\theta p$.*

(c) *For each Θ-ordered pair (A, U) in X and each universal net Φ on A, there exists a point $p \in U$ such that $\Phi \to_\theta p$.*

DEFINITION 3.3. A function f: $(X, \sigma) \to (Y, \tau)$ is *weakly-continuous* at $x \in X$ if for each open set V containing $f(x)$, there exists an open set U containing x such that $fU) \subset \mathrm{cl}(V)$. If $f(X) \subset A \subset Y$, then f is *weakly-continuous with respect to A* if f: $(X, \sigma) \to (A, \tau_A)$ is weakly-continuous (Levine (1961)).

It is noted in Theorem 6 by Herrington and Long (1975b) that a function $f: X \to Y$ is weakly-continuous at $x \in X$ if and only if for every net $\{x_\alpha\}$ in X such that $\{X_\alpha\} \to x$, $\{f(x_\alpha)\} \to_\theta f(x)$.

DEFINITION 3.4. A function $f: (X, \sigma) \to (Y, \tau)$ has a *strongly-closed graph with respect to* $A \subset Y$ if for each $(x, a) \notin G(f)$ (where $a \in A$), there exist open sets $U \in \sigma$ and $V \in \tau$ containing x and a, respectively, such that $U \times \mathrm{cl}(V) \cap G(f) = \varnothing$.

The example in Remark 1 by Herrington and Long (1975a) shows that a function may have a graph that is strongly-closed with respect to A but not strongly-closed with respect to Y. Also, the example illustrates that a function may be weakly-continuous with respect to Y but not weakly-continuous with respect to A.

Following Lemma 3.5, we show how to use weakly-continuous functions and functions with a strongly-closed graph to characterize functional compactness. To do this we rely on a particular class of spaces, described by Kasahara (1973) as class S. (S is a class of spaces containing the class of Hausdorff completely normal *and fully normal spaces.)*

LEMMA 3.5. *If $f: X \to Y$ has a strongly-closed graph and if $\{x_\alpha\} \to p \in X$ and $\{f(x_\alpha)\} \not\propto_\theta q \in Y$, then $f(p) = q$.*

PROOF. Suppose $\{x_\alpha\} \to p \in X$, $\{f(x_\alpha)\} \propto_\theta q \in Y$, and $f(p) \neq q$. Then the strongly-closed graph hypothesis gives the existence of open sets U and V containing p and q, respectively, such that $U \times \mathrm{cl}(V) \cap G(f) = \varnothing$. Now since $\{x_\alpha\} \to p$, there exists an α_0 such that for all $\alpha > \alpha_0$, $x_\alpha \in U$. Thus for all $\alpha > \alpha_0$, $(x_\alpha, f(x_\alpha)) \notin U \times \mathrm{cl}(V)$. This implies that $\{f(x_\alpha)\} \propto_\theta q$ which is a contradiction. We conclude that $f(p) = q$.

THEOREM 3.6. *A Hausdorff space Y is functionally compact if and only if for every topological space X belonging to class S and for every Θ-ordered pair (A, U) in Y, each mapping, $f: X \to A$, of X into A with a strongly-closed graph with respect to U is weakly-continuous with respect to Y.*

PROOF. Let (A, U) be a Θ-ordered pair in Y and assume that $f: X \to A$ has a strongly-closed graph with respect to U. If $\{x_\alpha\}$ is a net in X converging to $p \in X$, then the net $\{f(x_\alpha)\}$ θ-accumulates to $f(p)$ according to Theorem 3.2 and Lemma 3.5. Now assume that $\{f(x_\alpha)\}$ does not θ-converge to $f(p)$. Then $\{f(x_\alpha)\}$ has a subnet θ-accumulating to some point $y \neq f(p)$. Using Lemma 3.5 we see that $f(p) = y$ which is a contradiction. Thus, $\{f(x_\alpha)\} \to_\theta f(p)$ showing that f is weakly-continuous.

Conversely, assume that Y is not functionally compact. Then by Theorem 3.2, there exists a Θ-ordered pair (A, U) in Y and a net, $\Phi: \mathscr{D} \to A$, in A which has no

Θ-accumulation point in U. Let $\infty \notin \mathscr{D}$ and define $X = \mathscr{D} \cup \{\infty\}$. Then the power set of $\mathscr{D}, \mathscr{P}(\mathscr{D})$, together with $\{T_d \cup \{\infty\}: d \in \mathscr{D}\}$, is a base for a Hausdorff fully normal topology, σ, on X. It follows that the space (X, σ) belongs to class S (see Theorem 1 of Kasahara (1973)). Let $p \in A$ and define a map $f: X \to A$ by $f | \mathscr{D} = \Phi$ and $f(\infty) = p$. It follows that $G(f)$ is strongly-closed and f is not weakly-continuous at the point $x = \infty$. This completes the proof.

4. Countably functionally compact spaces

In this section we investigate a topological property which is weaker than countable compactness but stronger than first countable minimality.

DEFINITION 4.1. A first countable Hausdorff space X is *countably functionally compact* if whenever $\mathscr{F} = \{U_n: n \in N\}$ is a countable open filterbase on X such that $\bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} \operatorname{cl}(U_n) = A$, then $\mathscr{F}$ forms a base for the neighborhoods of A.

A space (X, τ) is called *first countable and minimal Hausdorff* if τ is first countable and Hausdorff and if no first countable topology on X which is strictly weaker than τ is Hausdorff (Stephenson (1969)). In our next theorem we give a characterization of first countable minimality.

THEOREM 4.2. *A first countable Hausdorff space X is first countable and minimal Hausdorff if and only if for every point $x \in X$ and for every open countable filterbase $\mathscr{F} = \{U_n: n \in N\}$ on X such that $\{x\} = \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} \operatorname{cl}(U_n)$, $\mathscr{F}$ is a base for the neighborhoods of $\{x\}$.*

PROOF. The proof parallels that of Theorem 2 by Dickman and Zame (1969) and is therefore omitted.

In Example 4.9 we show that a first countable minimal Hausdorff space X need not be countably functionally compact. However, using Definition 4.1 and Theorem 4.2 we have the following corollary.

COROLLARY 4.3. *A first countable Hausdorff space X is first countable and minimal Hausdorff if it is countably functionally compact.*

THEOREM 4.4. *A first countable Hausdorff space X is countably functionally compact if and only if for every closed set $A \subset X$ and for every countable open cover $\mathscr{C} = \{U_n: n \in N\}$ of A (where $\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} \operatorname{cl}(U_n)$), there exists a finite subcollection $\{U_{n_i}: i = 1, 2, \ldots, k\}$ of $\mathscr{C}$ such that $A \subset \bigcup_{i=1}^{k} \operatorname{cl}(U_{n_i})$.*

PROOF. The straightforward proof is omitted.

Theorem 3 (Dickman and Zame (1969)) shows that a space X is functionally compact if and only if every continuous mapping of X into a Hausdorff space Y is closed. In our next theorem we give a similar result for countably functionally compact spaces. However, we omit the proof since it closely parallels that of Theorem 3 by Dickman and Zame (1969).

THEOREM 4.5. *A first countable Hausdorff space X is countably functionally compact if and only if every continuous mapping of X into a first countable Hausdorff space Y is closed.*

LEMMA 4.6. *A first countable Hausdorff space Y is countably functionally compact if for each Θ-ordered pair (A, U) in Y and each sequence (y_n) in A, there exists a point $p \in U$ such that $(y_n) \propto_\theta p$.*

In view of Lemma 4.6, we can give an additional sufficient condition for a space to be countably functionally compact. We rely on the space $\bar{\mathcal{N}}$ defined by Kasahara (1973). (The topology on $\mathcal{N} = N \cup \{\infty\}$, where $N = \{1, 2, 3, \ldots\}$, is constructed like the topology σ defined in Theorem 3.6.)

THEOREM 4.7. *A first countable Hausdorff space Y is countably functionally compact if for every Θ-ordered pair (A, U) in Y, each mapping, $f: \bar{\mathcal{N}} \to A$, of $\bar{\mathcal{N}}$ into A with a strongly-closed graph with respect to U is weakly-continuous with respect to Y.*

PROOF. The result parallels that of Theorem 3.6 and the proof is therefore omitted.

REMARK 4.8. Let X be a non-compact first countable Hausdorff space. If X is countably compact, then X is countably functionally compact according to Theorem 4.4. However, X is not functionally compact. (If X was functionally compact, then X would be a functionally compact regular space and, hence, compact.)

We note that the space X constructed in Section 3 (Lim and Tan (1974)) is first countable and countably functionally compact. We conclude this section by giving an example of a non-countably functional compact space that is first countable and minimal Hausdorff. The construction of the space closely parallels that of Example 2 by Dickman and Zame (1969) and therefore many of the details are omitted.

EXAMPLE 4.9. Let $Z = [0,1]$ have the usual subspace topology of the reals, let $N = \{1,2,3,\ldots\}$, and define

$$Y = \left\{a_k \in Z: a_k = \frac{1}{k+2},\ k \in N\right\} \cup \left\{b_k \in Z: b_k = \frac{k+1}{k+2},\ k \in N\right\} \cup \{0,1\}.$$

Now let

$$X = \bigcup_{n=1}^{\infty} V_n \cup Y,$$

where

$$V_n = \left\{\frac{1}{2+n} - \frac{1}{m}: m > (3)(n^2+5n+6), m \in N\right\}$$
$$\cup \left\{\frac{1}{2+n} + \frac{1}{m}: m > (3)(n^2+5n+6),\ m \in N\right\}$$
$$\cup \left\{\frac{1+n}{2+n} - \frac{1}{m}: m > (3)(n^2+5n+6),\ m \in N\right\}$$
$$\cup \left\{\frac{1+n}{2+n} + \frac{1}{m}: m > (3)(n^2+5n+6),\ m \in N\right\},$$

and define a function Φ from Y into the two-element subsets of Y by $\Phi(0) = \{0, a_1\}$, $\Phi(1) = \{1, b_1\}$, $\Phi(a_k) = \{a_{2k}, b_{2k}\}$ and $\Phi(b_k) = \{a_{2k+1}, b_{2k+1}\}$. It follows that the subset Y and the function Φ satisfy conditions (i) through (δ) given in Example 2 by Dickman and Zame (1969). Now define $Y^* = \{\Phi(y): y \in Y\}$ and topologize the set $X^* = (X - Y) \cup Y^*$ with open sets $U \subset X^*$ satisfying conditions (1) and (2) of Dickman and Zame (1969). By Proposition 1, X^* is first countable and minimal Hausdorff and by Proposition 2, X^* is not functionally compact (see Dickman and Zame (1969)). Moreover, X^* is not countably functionally compact. To see this, let

$$U_n = X^* \cap \left\{\left[0, \frac{1}{5+n}\right) \cup \left\{x \in Z: \frac{1}{5} - \frac{1}{m} < x < \frac{1}{5} + \frac{1}{m},\ m = (3)(30+n)\right\}\right.$$
$$\cup \left\{x \in Z: \frac{1}{3} - \frac{1}{m} < x < \frac{1}{3} + \frac{1}{m},\ m = (3)(12+n)\right\}$$
$$\cup \left\{x \in Z: \frac{2}{3} - \frac{1}{m} < x < \frac{2}{3} + \frac{1}{m},\ m = (3)(12+n)\right\}$$
$$\cup \left(\frac{4+n}{5+n}, 1\right] \cup \{\Phi(0), \Phi(1)\} \cup \{\Phi(a_k): k > (n+3)\}$$
$$\left.\cup \{\Phi(b_k): k > (n+3)\}\right\} \quad \text{if } n \text{ is odd,}$$

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$$U_n = X^* \cap \left\{ \left[0, \frac{1}{5+n}\right) \cup \left\{ x \in Z: \frac{1}{4} - \frac{1}{m} < x < \frac{1}{4} + \frac{1}{m}, m = (3)(n+20) \right\} \right.$$
$$\cup \left\{ x \in Z: \frac{1}{3} - \frac{1}{m} < x < \frac{1}{3} + \frac{1}{m}, m = (3)(n+12) \right\}$$
$$\cup \left\{ x \in Z: \frac{2}{3} - \frac{1}{m} < x < \frac{2}{3} + \frac{1}{m}, m = (3)(n+12) \right\}$$
$$\cup \left(\frac{4+n}{5+n}, 1 \right] \cup \{\Phi(0), \Phi(1)\} \cup \{\Phi(a_k): k > (n+3)\}$$
$$\left. \cup \{\Phi(b_k): k > (n+3)\} \right\}$$

if n is even,

and let $A = \{\Phi(0), \Phi(1)\}$. Then $\mathscr{F} = \{U_n: n \in N\}$ forms a countable open filterbase with

$$A = \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} \mathrm{cl}(U_n).$$

However, if

$$G = X^* \cap \left\{ \left[0, \frac{1}{6}\right) \cup \left\{ x \in Z: \frac{1}{3} - \frac{1}{m} < x < \frac{1}{3} + \frac{1}{m}, m = 36 \right\} \right.$$
$$\cup \left\{ x \in Z: \frac{2}{3} - \frac{1}{m} < x < \frac{2}{3} + \frac{1}{m}, m = 36 \right\} \cup \left(\frac{1}{5}, 1 \right]$$
$$\cup \{\Phi(0), \Phi(1)\} \cup \{\Phi(a_k): k > 4\} \cup \{\Phi(b_k): k > 4\}$$

then $A \subset G$ and $U_n \not\subset G$ for each $n \in N$. Therefore, X^* is not countably functionally compact.

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