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ON α-LIKE RADICALS

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Abstract

A radical ρ is called prime-like if for every prime ring A, the polynomial ring $A[x]$ is ρ-semisimple. Let α be a radical satisfying the polynomial equation $\alpha(A[x]) = (\alpha(A))[x]$ for every ring A. A radical γ is called α-like if for every α-semisimple ring A, the polynomial ring $A[x]$ is γ-semisimple. In this paper, we study properties of α-like radicals. We show that α-likeness is a generalization of prime-likeness and extend some results concerning prime-like radicals. This allows us easily to find distinct special radicals which coincide on simple rings and on polynomial rings, which answers a question put by Ferrero.

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1. Introduction

In this paper all rings are associative and all classes of rings are closed under isomorphisms and contain the one-element ring 0. The fundamental definitions and properties of radicals can be found in [1, 7]. A class μ of rings is called hereditary if μ is closed under ideals. If μ is a hereditary class of rings, $\mathcal{U}(\mu)$ denotes the upper radical generated by μ, that is, the class of all rings which have no nonzero homomorphic images in μ. For any class μ of rings an ideal I of a ring A is called a μ-ideal if the factor ring A/I is in μ. As usual, for a radical γ, the γ radical of a ring A is denoted by $\gamma(A)$ and the class of all γ-semisimple rings is denoted by $\mathcal{S}(\gamma)$. π denotes the class of all prime rings and $\beta = \mathcal{U}(\pi)$ denotes the prime radical. The notation $I \lhd A$ means that I is a two-sided ideal of a ring A. An ideal I of a ring A is called essential in A if $I \cap J \neq 0$ for every nonzero two-sided ideal J of A. A ring A is called an essential extension of a ring I if I is an essential ideal of A. A class μ of rings is called essentially closed if $\mu = \mu_k$, where

$$\mu_k = \{A : A \text{ is an essential extension of some } I \in \mu\}$$

is the essential cover of μ. A hereditary and essentially closed class of prime rings is called a special class and the upper radical generated by a special class is called a special radical. A hereditary radical containing the prime radical β is called a supernilpotent radical. Given a ring A, the polynomial ring over A in a commuting indeterminate x is denoted by $A[x]$. We say that a radical γ has the Amitsur property if $\gamma(A[x]) = (\gamma(A[x]) \cap A)[x]$ for every ring A. A radical γ is called polynomially extensible if $A[x] \in \gamma$ for every ring $A \in \gamma$. It is well known [7, Proposition 4.9.21] that γ is polynomially extensible if and only if $\gamma = \gamma_x$, where $\gamma_x = \{A : A[x] \in \gamma\}$. A semiprime ring R is called a $*$-ring [2–4, 9] if $R/I \in \beta$ for every nonzero ideal I of R. The nonnil Jacobson radical ring

$$W = \{2x/(2y+1) : x,\ y \in \mathbb{Z} \text{ and } (2x,\ 2y+1) = 1\}$$

is an example of a commutative $*$-ring without minimal ideals, as observed in [2, 3, 9]. The class of all $*$-rings is denoted by $*$. The importance of the class $*_k$ is underlined by the two facts that follow.

THEOREM 1.1 [3, 9]. *If R is a nonzero $*$-ring, then the smallest special (respectively, supernilpotent) radical $\widehat{l_R}$ (respectively, $\bar{l}_R$) containing R is an atom in the lattice of all special (respectively, supernilpotent) radicals.*

THEOREM 1.2 [4, Proposition 2]. *If $R \in *_k$ and μ is a special class of rings, then $R \in \mathcal{S}(\mathcal{U}(\mu))$ if and only if $R \in \mu$. Thus, in particular, a ring $R \in *_k$ is Jacobson semisimple if and only if R is primitive.*

A radical α is said to satisfy the polynomial equation if $\alpha(A[x]) = (\alpha(A))[x]$ for every ring A. It was proved in [8] that α satisfies the polynomial equation if and only if it is polynomially extensible and has the Amitsur property. In this paper α always denotes a radical that satisfies the polynomial equation.

A radical γ is called prime-like [11] if $A[x] \in \mathcal{S}\gamma$ for any prime ring A. The importance of prime-like radicals stems from the fact that, as was shown in [11], they allow us to easily construct pairs of distinct special radicals that coincide on simple rings and on polynomial rings, which answers a question posed by Ferrero [12]. Also, the long-standing open question of Gardner [6, Problem 1], which asks whether $\beta = \mathcal{U}(*_k)$, is equivalent to the question whether the radical $\mathcal{U}(*_k)$ is prime-like.

It was shown in [11] that if γ is a prime-like radical, then $A[x] \in \mathcal{S}\gamma$ for every semiprime ring A. Inspired by this fact, we introduce the following definition.

DEFINITION 1.3. Let α be a radical that satisfies the polynomial equation. We say that a radical γ is α-like if $A[x] \in \mathcal{S}\gamma$ for any $A \in \mathcal{S}\alpha$.

It is well known [7, p. 275] that $\beta(A[x]) = (\beta(A))[x]$ for every ring A. Thus we have the following lemma.

LEMMA 1.4. *γ is a prime-like radical if and only if γ is β-like.*

In this paper we study properties of α-like radicals containing α. In particular, we give necessary and sufficient conditions for a radical $\gamma \supseteqq \alpha$ to be α-like. These generalize some results of [11] and allow us easily to construct pairs of distinct special

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radicals that meet Ferrero conditions [12]. We also show that $\beta = \mathcal{U}(*_k)$ if and only if $\mathcal{U}(*_k)$ is β-like. This gives a reason for studying α-like radicals.

2. Main results

We will start by describing some properties of α-like radicals.

LEMMA 2.1. *α is α-like.*

PROOF. Since α satisfies the polynomial equation, for any $A \in \mathcal{S}(\alpha)$ we have $\alpha(A[x]) = (\alpha(A))[x] = 0[x] = 0$. Thus α is α-like. □

LEMMA 2.2. *A polynomially extensible radical $\gamma \supseteq \alpha$ is α-like if and only if $\gamma = \alpha$.*

PROOF. Let $\gamma \supseteq \alpha$ be a polynomially extensible radical.

If $\gamma = \alpha$, then γ is α-like by Lemma 2.1.

Conversely, let γ be α-like and suppose that $\gamma \supsetneqq \alpha$. Then there exisits $0 \neq A \in \gamma \cap \mathcal{S}(\alpha)$. But then, since γ is α-like and is polynomially extensible, it follows that $0 \neq A[x] \in \mathcal{S}(\gamma) \cap \gamma$, a contradiction. Thus $\gamma = \alpha$. □

COROLLARY 2.3 [11, Corollary 4]. *A polynomially extensible radical $\gamma \supseteq \beta$ is prime-like if and only if $\gamma = \beta$.*

It was shown in [5] that the special radical $\mathcal{U}(*_k) \supseteq \beta$ is polynomially extensible. Thus Corollary 2.3 implies the following.

COROLLARY 2.4. *$\mathcal{U}(*_k) = \beta$ if and only if $\mathcal{U}(*_k)$ is β-like.*

LEMMA 2.5. *If $\alpha \supseteq \beta$ and γ is β-like, then γ is α-like.*

PROOF. Let $A \in \mathcal{S}\alpha$. Then $A \in \mathcal{S}\beta$ since $\alpha \supseteq \beta$ implies $\mathcal{S}\alpha \subseteq \mathcal{S}\beta$. But then $A[x] \in \mathcal{S}\gamma$ because γ is β-like, which shows that γ is α-like. □

LEMMA 2.6. *If γ and ρ are radicals with $\gamma \subseteq \rho$ and ρ is α-like, then γ is also α-like.*

PROOF. Let $A \in \mathcal{S}\alpha$. Then, as ρ is α-like, it follows that $A[x] \in \mathcal{S}\rho$. But $\mathcal{S}\rho \subseteq \mathcal{S}\gamma$ since $\gamma \subseteq \rho$. So $A[x] \in \mathcal{S}\gamma$ which shows that γ is α-like. □

COROLLARY 2.7. *Neither the locally nilpotent radical $\mathcal{L}$, nor the nil radical $\mathcal{N}$, nor the Jacobson radical $\mathcal{J}$, nor the Brown–McCoy radical $\mathcal{G}$ is β-like.*

PROOF. Since $\mathcal{L}$ is polynomially extensible [13, Example 2.1(ii)] and $\beta \subsetneqq \mathcal{L}$, $\mathcal{L}$ is not β-like by Corollary 2.3. Since $\mathcal{L} \subset \mathcal{N} \subset \mathcal{J} \subset \mathcal{G}$, the result follows from Lemma 2.6. □

REMARK 2.8. Note that for some radicals α, in particular for β, there exist radicals $\gamma \supseteq \alpha$ that are not α-like. Consider, for example, $\mathcal{L} \supset \beta$. Since β satisfies the polynomial equation, it follows from Lemma 2.1 that β is β-like but $\mathcal{L}$ is not by Corollary 2.7.

The general question is interesting: do there exist radicals $\gamma \supseteqq \alpha$ that are not α-like for any α?

Our next result gives various characterizations of α-like radicals that contain α and forms a generalization of [11, Corollary 13, Theorem 14].

THEOREM 2.9. *Let γ be a radical containing α. The following conditions are equivalent:*

(1) *γ is α-like;*
(2) *$\gamma_x = \alpha$ and γ has the Amitsur property;*
(3) *$\gamma(A[x]) = \alpha(A[x])$, for every ring A.*

PROOF. (1) $\Rightarrow$ (2). Let $\gamma \supseteqq \alpha$ be α-like. Then $\alpha_x \subseteqq \gamma_x$. But, since α satisfies the polynomial equation, it is polynomially extensible so $\alpha = \alpha_x$. So, it follows that $\alpha \subseteqq \gamma_x$. Suppose that there exists $A \in \gamma_x$ such that $A \notin \alpha$. Then $A[x] \in \gamma$ and $0 \neq A/\alpha(A) \in \mathcal{S}\alpha$. Now, since γ is α-like, it follows that $(A/\alpha(A))[x] \in \mathcal{S}\gamma$. On the other hand, since α satisfies the polynomial equation, we have $(A/\alpha(A))[x] \simeq A[x]/(\alpha(A)[x]) = A[x]/\alpha(A[x]) \in \gamma$ because $A[x] \in \gamma$ and γ is homomorphically closed. Thus $0 \neq (A/\alpha(A))[x] \in \mathcal{S}\gamma \cap \gamma$, a contradiction. Therefore $\gamma_x = \alpha$. In view of [13, Theorem 3.5], to show that γ has the Amitsur property it suffices to show that $A[x] \in \mathcal{S}\gamma$ for every $A \in \mathcal{S}\gamma_x$. Let $A \in \mathcal{S}\gamma_x$. Then, as seen above, $\alpha \subseteqq \gamma_x$ so $\mathcal{S}\gamma_x \subseteqq \mathcal{S}\alpha$. Therefore $A \in \mathcal{S}\alpha$. But, as γ is α-like, it then follows that $A[x] \in \mathcal{S}\gamma$, which shows that γ has the Amitsur property.

(2) $\Rightarrow$ (3). Let $\gamma_x = \alpha$ and let γ have the Amitsur property. Since α satisfies the polynomial equation, it suffices to show that $\gamma(A[x]) = (\alpha(A))[x]$. Now, since γ has the Amitsur property, it follows that $(\gamma(A[x]) \cap A)[x] = \gamma(A[x]) \in \gamma$ which implies that $\gamma(A[x]) \cap A \in \gamma_x$. This implies that $\gamma(A[x]) \cap A \subseteqq \gamma_x(A) = \alpha(A)$, because $\gamma_x = \alpha$. Then $\gamma(A[x]) = (\gamma(A[x]) \cap A)[x] \subseteqq \alpha(A)[x]$.

But, since $\alpha(A) = \gamma_x(A) \in \gamma_x$, it follows that $\alpha(A)[x] \in \gamma$. Thus, as $\alpha(A)[x] \lhd A[x]$, it follows that $\alpha(A)[x] \subseteqq \gamma(A[x])$. Thus $\gamma(A[x]) = (\alpha(A))[x]$.

(3) $\Rightarrow$ (1). Let $\gamma(A[x]) = \alpha(A[x])$, for every ring A. Let $B \in \mathcal{S}\alpha$. Then, since α satisfies the polynomial equation, we have $\gamma(B[x]) = \alpha(B[x]) = (\alpha(B))[x] = 0[x] = 0$. Therefore $B[x] \in \mathcal{S}\gamma$, which shows that γ is α-like. $\square$

Ferrero asked [12] whether two distinct special radicals can coincide on all simple rings as well as on polynomial rings. An affirmative answer was given in [10, 11, 14]. The following result shows that some α-like radicals also meet Ferrero's requirements.

COROLLARY 2.10. *Let α be a special radical satisfying the polynomial equation. For any special and α-like radical $\gamma \supsetneqq \alpha$ whose semisimple class contains all prime simple rings, α and γ satisfy Ferrero's requirements.*

PROOF. Since α is special, $\beta \subseteqq \alpha$. Since γ is α-like, it follows from Theorem 2.9 that $\gamma(A[x]) = \alpha(A[x])$, for every ring A. Let A be a simple ring. Then either $A^2 = 0$ or $A^2 = A \in \pi$. In the first case, $A \in \beta \subseteqq \alpha \subseteqq \gamma$ so $\alpha(A) = A = \gamma(A)$. In the second

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case, $\alpha(A) = 0 = \gamma(A)$ since all simple prime rings are in $\mathcal{S}\gamma$ and $\mathcal{S}\gamma \subsetneqq \mathcal{S}\alpha$ because $\alpha \subsetneqq \gamma$, which concludes the proof. □

COROLLARY 2.11 [11, Corollary 15]. *For any special and prime-like radical $\gamma \supsetneqq \beta$ whose semisimple class contains all prime simple rings (for example, $\widehat{l_W}$ is such a radical), the prime radical β and the radical γ satisfy Ferrero's requirements.*

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