Canad. J. Math. Vol. **62** (2), 2010 pp. 305–319
doi:10.4153/CJM-2010-018-5

Approximation and Similarity Classification of Stably Finitely Strongly Irreducible Decomposable Operators

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Abstract. Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the collection of bounded linear operators on $\mathcal{H}$. In this paper, we show that for any operator $A \in \mathcal{L}(\mathcal{H})$, there exists a stably finitely (SI) decomposable operator A_ϵ, such that $\|A - A_\epsilon\| < \epsilon$ and $\mathcal{A}'(A_\epsilon)/\operatorname{rad}\mathcal{A}'(A_\epsilon)$ is commutative, where $\operatorname{rad}\mathcal{A}'(A_\epsilon)$ is the Jacobson radical of $\mathcal{A}'(A_\epsilon)$. Moreover, we give a similarity classification of the stably finitely decomposable operators that generalizes the result on similarity classification of Cowen–Douglas operators given by C. L. Jiang.

1 Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the collection of bound linear operators on $\mathcal{H}$. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be strongly irreducible if $\mathcal{A}'(T)$ (the commutant algebra of T) has no non-trivial idempotent. In what follows, $T \in$ (SI) means T is a strongly irreducible operator.

In matrix algebra, transforming a matrix into a Jordan standard form is really situated at the centre in the theory of linear transformation. It is also a prototype in the spectral theory of bounded linear operators on infinite dimensional space. The famous Jordan standard form theorem states that every $n \times n$ matrix can be written uniquely as the finite direct sum of Jordan blocks up to similarity. Zejian Jiang conjectured that the finite direct sums of strongly irreducible operators should be dense in $\mathcal{L}(\mathcal{H})$ [12]. This conjecture has been proved by Jiang [7] and Herrero and Jiang [6]. So (SI) operators are a suitable analogue of Jordan blocks in $\mathcal{L}(\mathcal{H})$.

The similarity classification of operators is a basic problem in operator theory. When $\mathcal{H}$ is a finite dimensional Hilbert space, we know from the Jordan standard form theorem that the eigenvalues and the generalized eigenspaces of an operator form a complete set of similarity invariants. When $\mathcal{H}$ is an infinite-dimensional Hilbert space, in a real sense the problem has no general solution, but one can restrict attention to special classes of operators. For two star-cyclic normal operators A and B, Conway showed that A and B are similar if and only if the scalar-valued spectral measures induced by A and B are equivalent [3]. Shields characterized similarity for injective weighted shift operators [14]. Herrero and Jiang proved that the

Received by the editors July 11, 2007; revised March 1, 2008.
Published electronically December 4, 2009.
This work is supported by the National Natural Science Foundation of China(Grant No.10571041, No. 10701031) and Hebei Province Natural Science Foundation(Grant No.A2005000006).
AMS subject classification: **47A05**, 47A55, 46H20.
Keywords: K_0-group, strongly irreducible decomposition, Cowen–Douglas operators, commutant algebra, similarity classification.

operator class $\mathcal{F} = \{T : T$ can be written as the direct sum of finite (SI) operators$\}$ is dense in $\mathcal{L}(\mathcal{H})$ under the norm topology. Therefore, it is an interesting problem to find the complete similarity invariants of $\mathcal{F}$.

Cowen and Douglas introduced a class of operators related to complex geometry, now referred to as Cowen–Douglas operators [5]. The Cowen–Douglas operators play an important role in studying the structure of non-self-adjoint operators.

Definition 1.1 Let Ω be a bounded open set in C, and n a positive integer. The set $B_n(\Omega)$ of Cowen–Douglas operators of index n is the set of operators B in $\mathcal{L}(H)$ satisfying

(i) $\Omega \subset \sigma(B) = \{z \in C \mid B - z \text{ is not invertible}\}$;
(ii) $\operatorname{ran}(B - z) = \mathcal{H}, z \in \Omega$;
(iii) $\bigvee_{z\in\Omega} \ker(B - z) = \mathcal{H}$;
(iv) $\dim\ker(B - z) = n, z \in \Omega$.

In this paper, we will need the case of $n = \infty$.

Jiang and Wang [10] proved that every Cowen–Douglas operator can be written as the direct sum of finitely many strongly irreducible Cowen–Douglas operators.

It was shown that two operators S and T in $B_n(\Omega)$ are unitarily equivalent if and only if the corresponding Hermitian bounds E_S and E_T are equivalent [5]. As a consequence of this, it was shown that the curvature function of E_T is a complete set of unitary invariants for operators T in $B_1(\Omega)$. However the curvature function of E_T is not a complete set of similarity invariants of Cowen–Douglas operators. Using techniques of complex geometry and K-theory, Jiang proved that the scaled ordered K_0-group of the commutant algebra is a similarity invariant of a strongly irreducible Cowen–Douglas operator [8].

Recently, Jiang, Guo, and Ji generalized the above result by removing the restriction of strong irreducibility of operators, and proved that the ordered K_0-group of the commutant algebra is a complete similarity invariant of Cowen–Douglas operators [9].

In this paper, we focus on studying the operators with stably finite strongly irreducible decomposable operators.

Definition 1.2 Let $T \in \mathcal{L}(\mathcal{H})$, $\mathcal{P} = \{P_i\}_{i=1}^n$ and $\mathcal{Q} = \{Q_i\}_{i=1}^m$ be two units of finite (SI) decompositions of T. Then $\mathcal{P}$ and $\mathcal{Q}$ are said to be equivalent if following conditions are satisfied:

(i) $m = n$;
(ii) There is an invertible operator $X \in \mathcal{A}'(T)$ and a permutation $\Pi \in S_n$ such that $XQ_{\Pi(i)}X^{-1} = P_i$ for $1 \leq i \leq n$.

We say that T has unique finitely (SI) decomposition up to similarity if all units of finite (SI) decompositions of T are equivalent. We say that T is a stably finite strongly irreducible decomposable operator if $T^{(n)}$ has unique finitely (SI) decomposition up to similarity for all $n = 1, 2, 3, \dots$.

By Theorem CFJ [2], we know that the K_0-group of the commutant algebra of a stably finite strongly irreducible decomposable operator is isomorphic to the direct sum of several integer groups. We let $\mathcal{F}_1$ denote the class of all stably finite strongly irreducible decomposable operators. By [9], $\mathcal{F}_1$ contains all Cowen-Douglas operators. In [2], the following theorem was proved.

Theorem CFJ *Let $T \in \mathcal{L}(\mathcal{H})$, $\mathcal{H}^{(n)}$ denote the direct sum of n copies of Hilbert space $\mathcal{H}$, and $T^{(n)}$ is the operator $\bigoplus_1^n T$ on $\mathcal{H}^{(n)}$. Then following are equivalent:*

(i) *$T \in \mathcal{F}_1$ is similar to $(\sim)$ $\bigoplus_{i=1}^k A_i^{(n_i)}$ respect to the decomposition $\mathcal{H} = \bigoplus_{i=1}^k \mathcal{H}_i^{(n_i)}$, where $k, n_i < \infty$, $A_1, \dots, A_k$ are all strongly irreducible operators, and $A_i \not\sim A_j$ for $1 \le i \ne j \le k$.*

(ii) *$K_0(\mathcal{A}'(T)) \cong Z^{(k)}$ and $V(\mathcal{A}'(T)) \cong N^{(k)}$. Then h denotes the isomorphism from $V(\mathcal{A}'(T))$ to $N^{(k)}$; h sends $[I]$ to $(n_1, n_2, \dots, n_k)$,* i.e., *$h([I]) = n_1e_1 + n_2e_2 + \cdots + n_ke_k$, where $N = (0, 1, 2, 3, \dots)$, $k, n_1, \dots, n_k$ are natural numbers, $\{e_i\}_{i=1}^k$ are generators of $N^{(k)}$.*

By techniques of the theory of operator approximation and K-theory, we prove that $\mathcal{F}_1$ is dense in $\mathcal{L}(\mathcal{H})$ in the norm topology. Moreover, we get the similarity classification of stably finite (SI) decomposable operators following the similarity classification of Cowen–Douglas operators by Jiang. We prove the following.

Theorem 3.9 *Let $A \in \mathcal{L}(\mathcal{H})$ and $\epsilon > 0$. Then there exists a stably finite strongly irreducible decomposable operator A_ϵ such that*

(i) *$\|A - A_\epsilon\| < \epsilon$;*
(ii) *$\mathcal{A}'(A_\epsilon)/\operatorname{rad}\mathcal{A}'(A_\epsilon)$ is commutative;*
(iii) *$V(\mathcal{A}'(A_\varepsilon)) \cong N^{k_\epsilon}$, $K_0(\mathcal{A}'(A_\varepsilon)) \cong Z^{k_\epsilon}$.*

Corollary 3.12 *Let $A, B \in \mathcal{L}(\mathcal{H})$, such that A, B both have unique stably finite* (SI) *decomposition up to similarity. Assume $A = A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots \oplus A_k^{(n_k)}$, where $0 \ne n_i \in N$, $A_i \in$* (SI)*, $i = 1, 2, \dots, k$, and $A_i \not\sim A_j$, when $A_i \not\sim A_j$. Then $A \sim B$ if and only if*

(i) $\big(K_0(\mathcal{A}'(A \oplus B)), \bigvee(\mathcal{A}'(A \oplus B)), I\big) \cong (Z^{(k)}, N^{(k)}, 1)$.

(ii) *The isomorphism h from $\bigvee(\mathcal{A}'(A \oplus B))$ to $N^{(k)}$ sends $[I]$ to $(2n_1, 2n_2, \dots, 2n_k)$,* i.e., *$h([I]) = 2n_1e_1 + 2n_2e_2 + \cdots + 2n_ke_k$, where I is the unit of $\mathcal{A}'(A \oplus B)$ and $\{e_i\}_{i=1}^k$ are the generators of $N^{(k)}$.*

This paper is organized as follows. In Section 2, we introduce some basic tools and concepts. In Section 3, we prove the main result of this paper and give a similarity classification of the stably finitely strongly irreducible decomposable operators.

2 Preliminary Results

To express our results more carefully we need to introduce the following definitions, notations and theorems.

For a unital Banach algebra $\mathcal{A}$, $\operatorname{rad}\mathcal{A}$ denotes the Jacobson radical of $\mathcal{A}$.

Lemma 2.1 ([1]) *Let $\mathcal{A}$ is a unital Banach algebra. Then the following are equivalent:*

(i) *$\mathcal{A}/\operatorname{rad}(\mathcal{A})$ is commutative,*

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(ii) $\sigma(xy - yx) = \{0\}$ *for every* $x, y \in \mathcal{A}$.

Lemma 2.2 (**[1]**) *Let* $A, B \in \mathcal{L}(\mathcal{H})$. *Then the following are equivalent for* τ_{AB}:

(i) τ_{AB} *is surjective;*
(ii) $\sigma_r(A) \cap \sigma_l(B) = \varnothing$;
(iii) $\operatorname{ran} \tau_{AB}$ *contains the set of finite rank operators.*

Lemma 2.3 (**[10]**) *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two Hilbert spaces,* $A \in \mathcal{L}(\mathcal{H})_1$, $B \in \mathcal{L}(\mathcal{H})_2$, *and assume that* $\sigma_l(A) \cap \sigma_r(B) = \varnothing$. *Then* τ_{AB} *is injective.*

Lemma 2.4 (**[10]**) *Let* $A, B \in \mathcal{L}(\mathcal{H})$. *Assume that*

$$\mathcal{H} = \bigvee \{\ker(\lambda - B)^k : \lambda \in \Gamma, k \geq 1\}$$

for a certain subset Γ *of the point spectrum* $\sigma_p(B)$ *of* B, *and* $\sigma_p(A) \cap \Gamma = \varnothing$. *Then* τ_{AB} *is injective.*

Lemma 2.5 (**[11]**) *Let* $\mathcal{A}$ *be a unital Banach algebra and let* P *be an idempotent in* $\mathcal{A}$ *and* $R \in \operatorname{rad} \mathcal{A}$. *If* $P + R \in \mathcal{A}$ *is still an idempotent in* $\mathcal{A}$, *then there exists an invertible element* $X \in \mathcal{A}$ *such that* $X(P + R)X^{-1} = P$.

Lemma 2.6 (**[10]**) *Let* Ω *be a connected and bounded open subset of* $\mathbf{C}$, n *a natural number, and* $T \in \mathcal{L}(\mathcal{H})$ *satisfy*

(a) $\Omega \subset \sigma(T)$;
(b) $\operatorname{ran}(\lambda - T) = \mathcal{H}$ *and* $\operatorname{nul}(\lambda - T) = n$ *for all* $\lambda \in \Omega$.

Then the following are equivalent

(i) $\bigvee \{\ker(\lambda - T) : \lambda \in \Omega\} = \mathcal{H}$;
(ii) $\bigvee \{\ker(\lambda_0 - T)^n : n \geq 1\} = \mathcal{H}$, $\forall \lambda_0 \in \Omega$;
(iii) $\bigvee \{\ker(\lambda_n - T) : n \geq 1\} = \mathcal{H}$ *for all sequences* $\{\lambda_n\}_{n=1}^{\infty} \subset \Omega$ *such that* $\lim_{n\to\infty} \lambda_n = \lambda_0 \in \Omega$;
(iv) $\bigvee \{\ker(\lambda_n - T) : n \geq 1\, k \geq 1\} = \mathcal{H}$ *for all sequences* $\{\lambda_n\}_{n=1}^{\infty} \subset \Omega$ *such that* $\lim_{n\to\infty} \lambda_n = \lambda_0 \in \Omega$;

Lemma 2.7 (**[8]**) *Let* $A \in B_n(\Omega) \cap (\mathrm{SI})$, *then* $\mathcal{A}'(A)/\operatorname{rad} \mathcal{A}'(A)$ *is commutative.*

Lemma 2.8 (**[9]**) *Let* $T \in B_n(\Omega)$. *Then we know that* $T \sim A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots \oplus A_k^{(n_k)}$, *where* A_i *is a strongly irreducible Cowen–Douglas operator and* A_i *is not similarly equivalent to* A_j *for* $i \neq j$. *Then*

$$\bigvee(\mathcal{A}')(T) \cong N^{(k)}, \quad K_0((\mathcal{A}')(T)) \cong Z^{(k)}.$$

Lemma 2.9 (**[13]**) *If* $\lambda \in \partial\sigma(A)$ *and* λ *is not an isolated point of* $\sigma(A)$, *then* $\lambda \in \sigma_{\mathrm{lre}}(A)$.

Definition 2.10 Let T be a semi-Fredholm operator. Then the minimal index of T is defined by $\operatorname{min\,ind} T = \min\{\dim\ker T, \operatorname{nul} \dim\ker T^*\}$.

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Lemma 2.11 (**[4]**) *Let λ be an isolated point of $\sigma(A)$. Then the following are equivalent:*

(i) $\lambda \notin \sigma_{\mathrm{lre}}(A)$;
(ii) *Riesz idempotent $E(\lambda, A)$ has finite rank;*
(iii) *$A - \lambda$ is a Fredholm operator and* $\mathrm{ind}(A - \lambda) = 0$.

We let $\sigma_0(A)$ denote the set of all isolated points satisfying the above conditions.

Theorem 2.12 (**[10]**) *Let $T \in \mathcal{L}(\mathcal{H})$, $\varepsilon > 0$ and let Φ be an analytic Cauchy domain satisfying $\sigma_{\mathrm{lre}}(T) \subset \Phi \subset [\sigma_{\mathrm{lre}}(T)]_\varepsilon$. Then there exists a $T_\varepsilon \in \mathcal{L}(\mathcal{H})$ such that*

(i) $\sigma_{\mathrm{lre}}(T_\varepsilon) = \bar{\Phi}$;
(ii) $\rho^r_{s-F}(T_\varepsilon) = \rho_{s-F}(T_\epsilon) \backslash \sigma_0(T_\varepsilon)$, $\mathrm{ind}(\lambda - T_\varepsilon) = \mathrm{ind}(\lambda - T)$ *and* $\min \mathrm{ind}(\lambda - T_\varepsilon)^k \leq \min \mathrm{ind}(\lambda - T)^k$ *for all $\lambda \in \rho_{s-F}(T_\varepsilon)$ and $k = 1, 2, \ldots$;*
(iii) *$\sigma(T_\varepsilon)$ consists of finitely many connected components; the number of connected components is less than or equal to the number of connected components of $\sigma(T)$;*
(iv) $\|T - T_\varepsilon\| < \varepsilon$.

Theorem 2.13 (**[10]**) *Let $A, T \in \mathcal{L}(\mathcal{H})$ satisfy*

(i) $\sigma_0(T) \subset \sigma_0(A)$, $\dim \mathcal{H}(\lambda, A) = \dim \mathcal{H}(\lambda, T)$ *for all* $\lambda \in \sigma_0(T)$;
(ii) *each component of $\sigma_{\mathrm{lre}}(T)$ meets $\sigma_e(A)$;*
(iii) *for all $\lambda \in \rho_{s-F}(T)$ and $k \geq 1$,* $\rho_{s-F}(T) \subset \rho_{s-F}(A)$, $\mathrm{ind}(\lambda - A) = \mathrm{ind}(\lambda - T)$, *and* $\min \mathrm{ind}(\lambda - A)^k \leq \min \mathrm{ind}(\lambda - T)^k$;
(iv) *$\sigma_e(A)$ has no isolated points.*

Then $T \in \mathcal{S}(A)^-$, where $\mathcal{S}(A) = \{XAX^{-1} : X$ is invertible$\}$, is the similarity orbit of A.

3 Approximation and Similarity Classification of Stably Finitely Strongly Irreducible Decomposable Operators

Lemma 3.1 *Given $T \in \mathcal{L}(\mathcal{H})$ with the representation*

$$T = \begin{pmatrix} T_1 & * & \cdots & * \\ \mathbf{0} & T_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & T_k \end{pmatrix}$$

satisfying

(i) $\mathcal{A}'(T_i)/\operatorname{rad}\mathcal{A}'(T_i)$ *is commutative,*
(ii) $\ker \tau_{T_i T_j} = \{0\}$, $(1 \leq j < i \leq k)$,

then $\mathcal{A}'(T)/\operatorname{rad}\mathcal{A}'(T)$ is commutative.

Proof Let $A, B \in \mathcal{A}'(T)$. Note that $AT = TA$ and $BT = TB$, by (ii),

$$A = \begin{pmatrix} A_{11} & * & \cdots & * \\ \mathbf{0} & A_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & A_{kk} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & * & \cdots & * \\ \mathbf{0} & B_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & B_{kk} \end{pmatrix},$$

and $A_{ii}, B_{ii} \in \mathcal{A}'(T_i)$ for all $i : 1 \leq i \leq k$. Then

$$AB - BA = \begin{pmatrix} A_{11}B_{11} - B_{11}A_{11} & * & \cdots & * \\ \mathbf{0} & A_{22}B_{22} - B_{22}A_{22} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & A_{kk}B_{kk} - B_{kk}A_{kk} \end{pmatrix}.$$

Note that $\mathcal{A}'(T_i)/\operatorname{rad}\mathcal{A}'(T_i)$ is commutative. By Lemma 2.1, $\sigma(A_{ii}B_{ii} - B_{ii}A_{ii}) = \{0\}$. Then

$$\sigma(AB - BA) = \bigcup_{i=1}^{k} \sigma(A_{ii}B_{ii} - B_{ii}A_{ii}) = \{0\}.$$

By Lemma 2.1, $\mathcal{A}'(T)/\operatorname{rad}\mathcal{A}'(T)$ is commutative. ■

Lemma 3.2 *Let $T \in \mathcal{L}(H)$, $T = \bigoplus_{i=1}^{k} T_i$, where for each natural number n, $T_i^{(n)} \in H_i^{(n)}$ has unique finite* (SI) *decomposition, and*

$$\mathcal{A}'(T) = \begin{pmatrix} \mathcal{A}'(T_1) & * & \cdots & * \\ \mathbf{0} & \mathcal{A}'(T_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathcal{A}'(T_k) \end{pmatrix},$$

i.e., $\ker \tau_{T_i T_j} = \{0\}$, $(1 \leq j < i \leq k)$. *Then for each natural number n, $T^{(n)}$ has unique finite* (SI) *decomposition and $\bigvee \mathcal{A}'(T) \cong \bigoplus_{i=1}^{k} \bigvee \mathcal{A}'(T_i)$.*

Proof By Theorem CFJ, we only need to prove that $T^{(n)}$ has unique finite (SI) decomposition for each natural number n.

Without loss of generality, we will show Lemma 3.2 only for case of $T^{(2)}$,

$$T^{(2)} = \begin{pmatrix} {T_1}^{(2)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & {T_2}^{(2)} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & {T_k}^{(2)} \end{pmatrix}$$

and

$$\mathcal{A}'(T^{(2)}) = \begin{pmatrix} \mathcal{A}'(T_1^{(2)}) & * & \cdots & * \\ \mathbf{0} & \mathcal{A}'(T_2^{(2)}) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathcal{A}'(T_k^{(2)}) \end{pmatrix}.$$

Let $\{P_i\}_{i=1}^{m}$ and $\{Q_j\}_{j=1}^{n}$ be two units of finite (SI) decompositions of $T^{(2)}$. Then

$$P_i = \begin{pmatrix} P_{i1} & * & \cdots & * \\ \mathbf{0} & P_{i2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & P_{ik} \end{pmatrix}.$$

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Let

$$\widetilde{P}_i = \begin{pmatrix} P_{i1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & P_{i2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & P_{ik} \end{pmatrix}.$$

It is easily shown that $\widetilde{P}_i \in \mathcal{A}'(T^{(2)})$. Thus $P_i - \widetilde{P}_i \in \operatorname{rad} \mathcal{A}'(T^{(2)})$. By Lemma 2.5, there exists an invertible operator $X_i \in \mathcal{A}'(T^{(2)})$, such that $X_i P_i X_i^{-1} = \widetilde{P}_i$. Since P_i is a minimal idempotent, $\widetilde{P}_i$ is also a minimal idempotent. It is easy to see that there exists a natural number k_i satisfying $1 \le k_i \le k$ and $P_{it} = \mathbf{0}$, when $t \ne k_i$. Since $\sum_{i=1}^{k} P_i = I$, we have $\sum_{i=1}^{k} \widetilde{P}_i = I$. Hence,

$$\sum_{k_i = t} \widetilde{P}_i = \begin{pmatrix} \mathbf{0} & & & & \mathbf{0} \\ & \ddots & & & \\ & & I_{\mathcal{A}'(T_t^{(2)})} & & \\ & & & \ddots & \\ \mathbf{0} & & & & \mathbf{0} \end{pmatrix} H_t,$$

i.e., $\{\widetilde{P}_i|_{H_t} : k_i = t\}$ is a unit finite (SI) decomposition of $T_t^{(2)}$. Let

$$X = X_1|_{\operatorname{ran} P_1} \dot{+} X_2|_{\operatorname{ran} P_2} \dot{+} \cdots \dot{+} X_m|_{\operatorname{ran} P_m}.$$

Then $X \in \mathcal{A}'(T^{(2)})$, and $\{P_i\}_{i=1}^m$ is equivalent to $\{\widetilde{P}_i\}_{i=1}^m$ with respect to X.

Similarly, we can define $\{\widetilde{Q_j}\}_{j=1}^k$ and $\widetilde{k}_j$. A similar argument shows that $\{\widetilde{Q_j}|_{H_t} : \widetilde{k}_j = t\}$ is also a unit finite (SI) decomposition of $T_t^{(2)}$. Note that $T_t^{(2)}$ has unique finitely (SI) decomposition up to similarity. Then there exists a $Y_t \in \mathcal{A}'(T_t^{(2)})$, such that $\{\widetilde{P}_i|_{H_t} : k_i = t\}$ is equivalent to $\{\widetilde{Q_j}|_{H_t} : \widetilde{k}_j = t\}$ with respect to Y_t. Let $Y = \bigoplus_{t=1}^k Y_t$. Then it is easy to see that $Y \in \mathcal{A}'(T^{(2)})$, and that $\{\widetilde{P}_i\}_{i=1}^m$ is equivalent to $\{\widetilde{Q_j}\}_{j=1}^n$ with respect to Y.

Note that $\{P_i\}_{i=1}^m$ is equivalent to $\{\widetilde{P}_i\}_{i=1}^m$, $\{Q_i\}_{j=1}^n$ is equivalent to $\{\widetilde{Q_j}\}_{j=1}^n$, and $\{\widetilde{P}_i\}_{i=1}^m$ is equivalent to $\{\widetilde{Q_j}\}_{j=1}^n$. Hence $\{P_i\}_{i=1}^m$ is equivalent to $\{Q_j\}_{j=1}^n$, *i.e.*, $T^{(2)}$ has unique finite (SI) decomposition up to similarity. ■

Lemma 3.3 **([10])** *Let Ω be a connected analytic Cauchy domain and let n be a natural number. Then there exists $B \in B_n(\Omega) \cap$ (SI) such that $\sigma(B) = \bar{\Omega}$ and $\Omega = \rho_F(B) \cap \sigma(B)$.*

Lemma 3.4 *Let Ω be a connected analytic Cauchy domain. Then there exists $B = B_1 \oplus B_2$ satisfying*

(i) $\sigma(B) = \sigma(B_1) = \sigma(B_2) = \bar{\Omega}$;

(ii) $B_1 \in B_1(\Omega)$, $B_2^* \in B_1(\Omega^*)$;

(iii) $\rho_F(B) \cap \sigma(B) = \Omega$, $\operatorname{ind}(\lambda - B) = 0$, $\min\operatorname{ind}(\lambda - B) = 1$ *for all* $\lambda \in \Omega$;

(iv) $\ker \tau_{B_2 B_1} = \{0\}$ *and* $\mathcal{A}'(B)/\operatorname{rad}\mathcal{A}'(B)$ *is commutative;*

(v) $\bigvee \mathcal{A}'(B) \cong N^{(2)}$.

Proof By Lemma 3.3, there exists $B_1 \in B_1(\Omega)$ and $B_2^* \in B_1(\Omega)$ such that $\sigma(B_1) = \bar{\Omega}$, $\rho_F(B_1) \cap \sigma(B_1) = \Omega$ and $\sigma(B_2) = \bar{\Omega}$, $\rho_F(B_2) \cap \sigma(B_2) = \Omega$. Note that

$$\bigvee\{\ker(\lambda - B) : \lambda \in \Omega\} = H$$

and $\sigma_p(B_2) = \varnothing$. By Lemma 2.5, $\ker \tau B_2 B_1 = \{0\}$. Since $\mathcal{A}'(B_i)/\operatorname{rad}\mathcal{A}'(B_i)$ is commutative, $i = 1, 2$, Lemma 3.1 implies $\mathcal{A}'(B)/\operatorname{rad}\mathcal{A}'(B)$ is commutative. By Lemma 3.2, $\bigvee \mathcal{A}'(B) \cong N^{(2)}$. ∎

Lemma 3.5 **([10])** *Given $B \in B_1(\Omega)$ and $\epsilon > 0$, there exists a sequence $\{B_i\}_{i=1}^{\infty} \subset B_1(\Omega)$ such that*

(i) *$B_i = B + K_i$, where $K_i \in \mathcal{K}(H)$, $\sup_i\{\|K_i\|\} < \epsilon$ and $\lim_{i\to\infty} \|K_i\| = 0$;*
(ii) *$\ker \tau B_i B_j = \{0\}$ $(i \neq j)$.*

Lemma 3.6 *Given $B \in B_1(\Omega)$, let $B_{i=1}^{\infty}$ be given as in Lemma 3.5 and*

$$T = \begin{pmatrix} B_1 & C_2 & C_3 & \cdots \\ \mathbf{0} & B_2 & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & B_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $C_j \in \mathcal{K}(H)$, $C_j \notin \operatorname{ran} \tau_{B_1 B_j}$. Then

(i) *$T \in B_\infty(\Omega) \cap (\mathrm{SI})$;*
(ii) *$\sigma(T) = \sigma(B)$;*
(iii) *$\mathcal{A}'(T)/\operatorname{rad}\mathcal{A}'(T)$ is commutative;*
(iv) *$\bigvee \mathcal{A}'(T) \cong N$.*

Proof The proofs of (i) and (ii) are omitted; the reader is referred to [10]. Lemma 3.1 and Lemma 3.5 imply (iii). For (iv), by Theorem CFJ, we only need to prove for each natural number n, that $T^{(n)}$ has unique finite (SI) decomposition up to similarity.

We consider $T^{(n)}$ with the representation

$$T^{(n)} = \begin{pmatrix} B_1^{(n)} & C_2^{(n)} & C_3^{(n)} & \cdots \\ \mathbf{0} & B_2^{(n)} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & B_3^{(n)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

Let $P \in \mathcal{A}'(T)$ be the corresponding representation of $T^{(n)}$. Then we have

$$P = \begin{pmatrix} P_1 & P_{12} & P_{13} & \cdots \\ P_{21} & P_2 & P_{23} & \cdots \\ P_{31} & P_{32} & P_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

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Note that $\ker \tau B_i B_j = \{0\}$ $(i \neq j)$. Then computation shows that $P_{ij} = \mathbf{0}, i > 2$ and $P_i \in \mathcal{A}'(T_i^{(n)})$. Hence

$$P = \begin{pmatrix} P_1 & P_{12} & P_{13} & \dots \\ \mathbf{0} & P_2 & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & P_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Given an idempotent $P \in \mathcal{A}'(T)$, P_i is also a idempotent in $\mathcal{A}'(B_i^{(n)})$. Note that $B_i^{(n)}$ has unique finite (SI) decomposition up to similarity. We can assume $P_i = P_1^i + P_2^i + \cdots + P_{n_i}^i$, where $\{P_r^i\}_{r=1}^n$ is a unit finite (SI) decomposition of $B_i^{(n)}$. Hence there exists $\{X_r^i\}_{r=1}^n$ such that $X_r^i P_r^i (X_r^i)^{-1} = E_r^i$, where $X_r^i \in \mathcal{L}(\operatorname{ran} P_r^i, \mathcal{H}_i)$ and $\{E_r^i\}_{r=1}^n$ is the standard unit finite (SI) decomposition of $B_i^{(n)}$. Let $Z = \bigoplus_{i=1}^{\infty}(\bigoplus_{r=1}^{n} X_r^i)$, $T_z^{(n)} = Z^{-1}T^{(n)}Z$, $P_z = Z^{-1}PZ$. Then $T_z^{(n)}$ and P_z have the representations

$$T_z^{(n)} = \begin{pmatrix} \bigoplus_{r=1}^{n} B_{1r} & \bigoplus_{r=1}^{n} C_{2r} & \bigoplus_{r=1}^{n} C_{3r} & \cdots \\ \mathbf{0} & \bigoplus_{r=1}^{n} B_{2r} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & \bigoplus_{r=1}^{n} B_{3r} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$P_z = \begin{pmatrix} P_{z_1} & P_{z_{12}} & P_{z_{13}} & \cdots \\ \mathbf{0} & P_{z_2} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & P_{z_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathcal{A}'(T_z^{(n)}),$$

where

$$P_{z_i} = \Big(\bigoplus_{r=1}^{n} X_r^i\Big)^{-1} P_i \Big(\bigoplus_{r=1}^{n} X_r^i\Big)$$

$$= \left(\begin{array}{cccc|cccc} I_1 & \mathbf{0} & \cdots & \mathbf{0} & & & & \\ \mathbf{0} & I_2 & \cdots & \mathbf{0} & & \mathbf{0}_{\mathbf{n_i}\times(\mathbf{n}-\mathbf{n_i})} & & \\ \vdots & \vdots & \ddots & \vdots & & & & \\ \mathbf{0} & \mathbf{0} & \cdots & I_{n_i} & & & & \\ & & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & \mathbf{0}_{(\mathbf{n}-\mathbf{n_i})\times\mathbf{n_i}} & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ & & & & \vdots & \vdots & \ddots & \vdots \\ & & & & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{array}\right) \begin{array}{l} \operatorname{ran} P_1^{z_i} \\ \operatorname{ran} P_2^{z_i} \\ \vdots \\ \operatorname{ran} P_{n_i}^{i} \\ \operatorname{ran} P_{n_i+1}^{i} \\ \operatorname{ran} P_{n_i+2}^{i} \\ \vdots \\ \operatorname{ran} P_n^{i} \end{array}.$$

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$$P_{z_{1i}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & R^i_{1,n_i+1} & R^i_{1,n_i+2} & \cdots & R^i_{1n} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & R^i_{2,n_i+1} & R^i_{2,n_i+2} & \cdots & R^i_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & R^i_{n_1,n_i+1} & R^i_{n_1,n_i+2} & \cdots & R^i_{n_1,n} \\ R^i_{n_1+1,1} & R^i_{n_1+1,2} & \cdots & R^i_{n_1+1,n_i} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ R^i_{n_1+2,1} & R^i_{n_1+2,2} & \cdots & R^i_{n_1+2,n_i} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R^i_{n,1} & R^i_{n,2} & \cdots & R^i_{n,n_i} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \begin{matrix} \operatorname{ran} P^1_1 \\ \operatorname{ran} P^1_2 \\ \vdots \\ \operatorname{ran} P^1_{n_1} \\ \operatorname{ran} P^1_{n_1+1} \\ \operatorname{ran} P^1_{n_1+2} \\ \vdots \\ \operatorname{ran} P^1_n \end{matrix}.$$

Claim 1: $C_{ir} \notin \operatorname{ran} \tau_{B_{1r},B_{ir}}$. If $C_{ir} \in \operatorname{ran} \tau_{B_{1r},B_{ir}}$, there exists $G \in \mathcal{L}(\operatorname{ran} P^i_r, \operatorname{ran} P^1_r)$ such that $B_{1r}G - GB_{ir} = C_{ir}$. Then

$$X^1_r B_{1r}(X^1_r)^{-1} X^1_r G(X^i_r)^{-1} - X^1_r G(X^i_r)^{-1} X^i_r B_{ir}(X^i_r)^{-1} = X^1_r C_{ir}(X^i_r)^{-1},$$

i.e., $B_1G - GB_i = C_i$. Since $\ker \tau B_i B_j = \{0\}$, this is a contradiction. Thus $C_{ir} \notin \operatorname{ran} \tau_{B_{1r},B_{ir}}$.

Claim 2: $n_i = n_1$, for all natural numbers i. Assume $n_1 > n_i$. Since $P_z T^{(n)}_z = T^{(n)}_z P_z$, we have

$$\Big(\bigoplus_{r=1}^{n} B_{1r}\Big) P_{z_{1i}} + \Big(\bigoplus_{r=1}^{n} C_{ir}\Big) P_{z_i} = P_{z_1}\Big(\bigoplus_{r=1}^{n} C_{ir}\Big) + P_{z_{1i}}\Big(\bigoplus_{r=1}^{n} B_{ir}\Big).$$

Hence $B_{1,n_1} R^i_{n_1,n_1} = C_{i,n_1} + R^i_{n_1,n_1} B_{i,n_1}$. By Claim 1, this is a contradiction. Thus Claim 2 holds.

Claim 3: Let P be a minimal idempotent in $\mathcal{A}'(T^{(n)})$, then P_i is a minimal idempotent in $\mathcal{A}'(B^{(n)}_i)$. Since P is a minimal idempotent in $\mathcal{A}'(T^{(n)})$, P_z is a minimal idempotent in $\mathcal{A}'(T^{(n)}_z)$. If P_i is not a minimal idempotent in $\mathcal{A}'(B^{(n)}_i)$, we assume $P_i = P^i_1 + P^i_2$. Construct the following idempotent P_v in $\mathcal{A}'(T^{(n)}_z)$:

$$P_z = \begin{pmatrix} P_{z_1} & P_{z_{12}} & P_{z_{13}} & \cdots \\ \mathbf{0} & P_{z_2} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & P_{z_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathcal{A}'(T^{(n)}_z),$$

$$P_{z_i} = \begin{pmatrix} I_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & I_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \begin{matrix} \operatorname{ran} P^i_1 \\ \operatorname{ran} P^i_2 \\ \operatorname{ran} P^i_3 \\ \vdots \\ \operatorname{ran} P^i_n \end{matrix},$$

$$P_{z_{1i}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & R^i_{13} & R^i_{14} & \cdots & R^i_{1n} \\ \mathbf{0} & \mathbf{0} & R^i_{23} & R^i_{24} & \cdots & R^i_{2n} \\ R^i_{31} & R^i_{32} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ R^i_{41} & R^i_{42} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R^i_{n1} & R^i_{n2} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \begin{matrix} \operatorname{ran} P^{z_1}_1 \\ \operatorname{ran} P^{z_1}_2 \\ \operatorname{ran} P^{z_1}_3 \\ \operatorname{ran} P^{z_1}_4 \\ \vdots \\ \operatorname{ran} P^{z_1}_n \end{matrix},$$

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$$P_v = \begin{pmatrix} P_{v_1} & P_{v_{12}} & P_{v_{13}} & \cdots & P_{v_{1k}} & \cdots \\ \mathbf{0} & P_{v_2} & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & P_{v_3} & \cdots & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & P_{v_k} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \in \mathcal{A}'(T_z^{(n)}),$$

$$P_{v_i} = \begin{pmatrix} I_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \begin{matrix} \operatorname{ran} P_1^i \\ \operatorname{ran} P_2^i \\ \operatorname{ran} P_3^i \\ \vdots \\ \operatorname{ran} P_n^i \end{matrix},$$

$$P_{v_{1i}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & R_{13}^i & R_{14}^i & \cdots & R_{1n}^i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ R_{31}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ R_{41}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{n1}^i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix} \begin{matrix} \operatorname{ran} P_1^1 \\ \operatorname{ran} P_2^1 \\ \operatorname{ran} P_3^1 \\ \operatorname{ran} P_4^1 \\ \vdots \\ \operatorname{ran} P_n^{z_1} \end{matrix}.$$

Since $P_z T_z^{(n)} = T_z^{(n)} P_z$, it is proved that $P_v T_z^{(n)} = T_z^{(n)} P_v$. Computation shows that $P_z P_v = P_v P_z = P_v \neq 0$. Hence P_z is not a minimal idempotent in $\mathcal{A}'(T_z^{(n)})$. Note that $T_z^{(n)} = Z^{-1} T^{(n)} Z$, $P_z = Z^{-1} P Z$. Then P is not a minimal idempotent in $\mathcal{A}'(T^{(n)})$. This is a contradiction. Thus Claim 3 holds.

For

$$P_z = \begin{pmatrix} P_{z_1} & P_{z_{12}} & P_{z_{13}} & \cdots \\ \mathbf{0} & P_{z_2} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & P_{z_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathcal{A}'(T_z^{(n)}),$$

we define

$$\widetilde{P}_z = \begin{pmatrix} P_{z_1} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & P_{z_2} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & P_{z_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is easy to prove that $\widetilde{P}_z \in \mathcal{A}'(T_z^{(n)})$.

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Claim 4: $P_z - \widetilde{P}_z \in \operatorname{rad} \mathcal{A}'(T_z^{(n)})$. Since $S \in \mathcal{A}'(T_z^{(n)})$,

$$S = \begin{pmatrix} S_1 & S_{12} & S_{13} & \cdots & S_{1k} & \cdots \\ \mathbf{0} & S_2 & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & S_3 & \cdots & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & S_k & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix}.$$

Hence $(P_z - \widetilde{P}_z)S$ and $S(P_z - \widetilde{P}_z)$ are both triangular operators whose diagonal entries are all $\mathbf{0}$ operators. Hence $\sigma((P_z - \widetilde{P}_z)S) = \sigma(S(P_z - \widetilde{P}_z)) = \{0\}$. Thus Claim 4 holds.

Let P_z be a minimal idempotent in $\mathcal{A}'(T_z^{(n)})$. By Lemma 2.5, there exists an invertible operator $X \in \mathcal{A}'(T_z^{(n)})$ such that $X^{-1} P_z X = \widetilde{P}_z$. Hence

$$T_z^{(n)} \widetilde{P}_z = \begin{pmatrix} B_{11} \oplus \mathbf{0^{(n-1)}} & C_{21} \oplus \mathbf{0^{(n-1)}} & C_{31} \oplus \mathbf{0^{(n-1)}} & \cdots & C_{i1} \oplus \mathbf{0^{(n-1)}} & \cdots \\ \mathbf{0} & B_{21} \oplus \mathbf{0^{(n-1)}} & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & B_{31} \oplus \mathbf{0^{(n-1)}} & \cdots & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & B_{i1} \oplus \mathbf{0^{(n-1)}} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix}$$

$$= Z^{-1} \begin{pmatrix} B_1 \oplus \mathbf{0^{(n-1)}} & C_2 \oplus \mathbf{0^{(n-1)}} & C_3 \oplus \mathbf{0^{(n-1)}} & \cdots & C_i \oplus \mathbf{0^{(n-1)}} & \cdots \\ \mathbf{0} & B_2 \oplus \mathbf{0^{(n-1)}} & \mathbf{0} & \cdots & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & B_3 \oplus \mathbf{0^{(n-1)}} & \cdots & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & B_i \oplus \mathbf{0^{(n-1)}} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix} Z.$$

Then $T^{(n)}|_{\operatorname{ran} P} \sim T^{(n)}|_{\operatorname{ran} P_z} \sim T^{(n)}|_{\operatorname{ran} \widetilde{P}_z} \sim T$, *i.e.*, $T^{(n)}$ has unique finite (SI) decomposition up to similarity. ∎

Corollary 3.7 *Let Ω be a connected analytic Cauchy domain. Then there exists a $B \in B_\infty(\Omega) \cap (\mathrm{SI})$ satisfying*

(i) $\sigma(B) = \bar{\Omega}$;
(ii) $\mathcal{A}'(B)/ \operatorname{rad} \mathcal{A}'(B)$ *is commutative;*
(iii) $\bigvee \mathcal{A}'(B) \cong N$.

Proof By Lemma 3.3, Lemma 3.5 and Lemma 3.6, Corollary 3.7 holds. ∎

Lemma 3.8 *Let $T \in L(\mathcal{H})$ satisfying that $\sigma_{\mathrm{lre}}(T)$ is the closure of an analytic Cauchy domain Φ, and that $\sigma(T)$ has finitely many connected components. Then there exists an $A \in L(\mathcal{H})$ satisfying the conditions of Theorem* 2.13 *such that*

(i) $\mathcal{A}'(A)/ \operatorname{rad} \mathcal{A}'(A)$ *is commutative;*

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(ii) *A has unique stably finite* (SI) *decomposition up to similarity.*

Proof By Lemma 2.11 we can assume

$$(\Omega_1, k_1), (\Omega_2, k_2), \dots, (\Omega_m, k_m), \{\lambda_1\}, \{\lambda_2\}, \dots, \{\lambda_n\}$$

to be the finitely many components of $\sigma(T)\backslash\bar{\Phi}$, where $k_i = \operatorname{ind}(\lambda - T), \lambda \in \Omega_i$. Note that $\lambda_j \in \sigma_0(T)$. Hence $\overline{\Omega_i}$ are pairwise disjoint, and Ω_i is a connected analytic Cauchy domain.

If $0 < k_i < \infty$, by Lemma 3.3 there exists an $A_i \in B_{k_i}(\Omega_i) \cap (\mathrm{SI})$ such that $\sigma(A_i) = \overline{\Omega_i}$. Hence $\operatorname{ind}(A_i - \lambda) = k_i$, and $\min\operatorname{ind}(A_i - \lambda) = 0, \lambda \in \Omega_i$.

If $-\infty < k_i < 0$, by Lemma 3.3 there exists an $A_i^* \in B_{-k_i}(\Omega_i^*) \cap (\mathrm{SI})$ such that $\sigma(A_i) = \overline{\Omega_i}$. Hence $\operatorname{ind}(A_i - \lambda) = k_i$ and $\min\operatorname{ind}(A_i - \lambda) = 0, \lambda \in \Omega_i$.

If $k_i = 0$, by Lemma 3.4 there exists an $A_i = A_{i1} \oplus A_{i2}, A_{i1} \in B_1(\Omega_i), A_{i2} \in B_1(\Omega_i^*)$, such that $\sigma(A_i) = \sigma(A_{i1}) = \sigma A_{i2} = \overline{\Omega_i}$, $\ker\tau_{A_{i2}A_{i1}} = \{0\}$. Hence $\operatorname{ind}(A_i - \lambda) = 0$ and $\min\operatorname{ind}(A_i - \lambda) = 1, \lambda \in \Omega_i$. Moreover, $\mathcal{A}'(A_i)/\operatorname{rad}\mathcal{A}'(A_i)$ is commutative and $\bigvee \mathcal{A}'(A_i) \cong N^{(2)}$.

If $k_i = +\infty$, by Corollary 3.7 there exists an $A_i \in B_\infty(\Omega_i) \cap (\mathrm{SI})$ such that $\sigma(A_i) = \overline{\Omega_i}$. Hence $\operatorname{ind}(A_i - \lambda) = +\infty$ and $\min\operatorname{ind}(A_i - \lambda) = 0$, $\lambda \in \Omega_i$. Moreover, $\mathcal{A}'(A_i)/\operatorname{rad}\mathcal{A}'(A_i)$ is commutative and $\bigvee \mathcal{A}'(A_i) \cong N$.

If $k_i = -\infty$, by Corollary 3.7 there exists an $A_i^* \in B_\infty(\Omega_i^*) \cap (\mathrm{SI})$ and $\sigma(A_i) = \overline{\Omega_i}$. Hence $\operatorname{ind}(A_i - \lambda) = -\infty$ and $\min\operatorname{ind}(A_i - \lambda) = 0$, $\lambda \in \Omega_i$. Moreover, $\mathcal{A}'(A_i)/\operatorname{rad}\mathcal{A}'(A_i)$ is commutative and $\bigvee \mathcal{A}'(A_i) \cong N$.

For $\lambda_j \in \sigma_0(T)$, let B_j be the Jordan block on $E(\lambda, T)$ whose eigenvalue is λ. Hence B_j is an (SI) operator on a finite dimensional Hilbert space. Thus $\sigma(B_j) = \lambda_j$, $\operatorname{ind}(B_j - \lambda_j) = 0$, and $\min\operatorname{ind}(B_j - \lambda_j) = 1$.

Let $\Phi_1, \Phi_2, \dots, \Phi_l$ be all the components of Φ. By Lemma 3.3 there exists a $C_k \in B_1(\Phi_k)$ such that $\sigma(C_k) = \overline{\Phi_k}$.

Let $A = \left(\bigoplus_{k=1}^{l} C_k\right) \oplus \left(\bigoplus_{i=1}^{m} A_i\right) \oplus \left(\bigoplus_{j=1}^{n} B_j\right)$ and

$$\widetilde{A}_t = \begin{cases} C_t & 1 \le t \le l, \\ A_{t-l} & l+1 \le t \le m+l, \\ B_{t+m-l} & m+l+1 \le t \le m+l+n. \end{cases}$$

Hence $A = \bigoplus_{t=1}^{m+l+n} \widetilde{A}_t$. By Lemma 2.7 and Lemma 2.8, $\mathcal{A}'(\widetilde{A}_t)/\operatorname{rad}\mathcal{A}'(\widetilde{A}_t)$ is commutative, and $\bigvee(\mathcal{A}'(\widetilde{A}_t)) = N$ or $N^{(2)}$.

Claim 1: A and T satisfy the conditions of Theorem 2.13. Note that by the construction of A, the conditions (i) and (iii) are satisfied. Since $\partial\sigma_{\mathrm{lre}}(T) \supset \partial\sigma_e(T) \supset \partial\sigma(T)$, each component of $\sigma_{\mathrm{lre}}(T) = \overline{\Phi}$ meets $\sigma_e(A) \supset \bigcup_{i=1}^{m} \partial\Omega_i$, and $\sigma_e(A)$ has no isolated points.

Claim 2: $\ker\tau_{\widetilde{A_{t_1}}\widetilde{A_{t_2}}} = \{0\}$, $1 \le t_2 < t_1 \le m+l+n$. Note that $\{\overline{\Omega_i}\}_{i=1}^{m}$ are pairwise disjoint, $\{\overline{\Phi_k}\}_{k=1}^{l}$ are pairwise disjoint, and none of them meets $\sigma_0(A)$. By Lemma 2.3, we can get almost all the cases of Claim 2 except the case of $\ker\tau_{A_iC_k}$. Let

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$\Delta_k \subset \Phi_k$ be an open set such that $\Delta_k \cap \partial\Phi_k = \varnothing$. Hence $\Delta_k \cap \Omega_i = \varnothing$. By Lemma 2.6, $\mathcal{H} = \bigvee\{\ker(\lambda - B)^s : \lambda \in \Delta_k, s \geq 1\}$. By Lemma 2.5, $\ker \tau_{A_i C_k} = \{0\}$.

Claim 3: $\mathcal{A}'(A)/\operatorname{rad}\mathcal{A}'(A)$ is commutative and A has unique stably finite (SI) decomposition up to similarity. Note that $\mathcal{A}'(\widetilde{A}_t)/\operatorname{rad}\mathcal{A}'(\widetilde{A}_t)$ is commutative and either $\bigvee(\mathcal{A}'(\widetilde{A}_t)) = N$ or $N^{(2)}$. By Claim 2, Lemma 3.1, Lemma 3.2 and Theorem CFJ, Claim 3 holds. ∎

Theorem 3.9 *Let $A \in \mathcal{L}(\mathcal{H})$ and $\epsilon > 0$. Then there exists a stably finite strongly irreducible decomposable operator A_ϵ such that*

(i) $\|A - A_\epsilon\| < \epsilon$;
(ii) $\mathcal{A}'(A_\epsilon)/\operatorname{rad}\mathcal{A}'(A_\epsilon)$ *is commutative;*
(iii) $V(\mathcal{A}'(A_\varepsilon)) \cong N^{k_\epsilon}$, $K_0(\mathcal{A}'(A_\varepsilon)) \cong Z^{k_\epsilon}$.

Proof By Theorem 2.12, Theorem 2.13, Lemma 3.8, and Theorem CFJ, Theorem 3.9 holds. ∎

By Theorem 3.9 and Theorem CFJ, we get two corollaries.

Corollary 3.10 $\{T \in \mathcal{L}(\mathcal{H}) : K_0(\mathcal{A}'(A_\epsilon)) \cong Z^k, k \in N\}^- = \mathcal{L}(\mathcal{H})$.

Corollary 3.11 $\{T \in \mathcal{L}(\mathcal{H}) :$ *T is a stably finitely* (SI) *decomposable operator*$\}^- = \mathcal{L}(\mathcal{H})$.

Corollary 3.11 shows that the set of all stably finitely decomposable operators is dense in $\mathcal{L}(\mathcal{H})$, *i.e.,* the set of all the operators satisfying Theorem CFJ is dense in $\mathcal{L}(\mathcal{H})$.

Using techniques of complex geometry and K-theory, Jiang and others obtained the similarity classification of Cowen–Douglas operators. Following this result, we get the similarity classification of stably finite (SI) decomposable operators.

Corollary 3.12 *Let $A, B \in \mathcal{L}(\mathcal{H})$, such that A, B both have unique stably finite* (SI) *decomposition up to similarity. Assume $A = A_1^{(n_1)} \oplus A_2^{(n_2)} \oplus \cdots \oplus A_k^{(n_k)}$, where $0 \neq n_i \in N$, $A_i \in$ (SI), $i = 1, 2, \ldots, k$, and $A_i \not\sim A_j$, when $A_i \not\sim A_j$. Then $A \sim B$ if and only if*

(i) $\big(K_0(\mathcal{A}'(A \oplus B)), \bigvee(\mathcal{A}'(A \oplus B)), I\big) \cong (Z^{(k)}, N^{(k)}, 1)$;
(ii) *The isomorphism h from $\bigvee(\mathcal{A}'(A \oplus B))$ to $N^{(k)}$ sends $[I]$ to $(2n_1, 2n_2, \ldots, 2n_k)$,* i.e., *$h([I]) = 2n_1e_1 + 2n_2e_2 + \cdots + 2n_ke_k$, where I is the unit of $\mathcal{A}'(A \oplus B)$ and $\{e_i\}_{i=1}^k$ are the generators of $N^{(k)}$.*

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