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FINITELY PROJECTIVE MODULES OVER A DEDEKIND DOMAIN

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Abstract

As dual to the notion of "finitely injective modules" introduced and studied by Ramamurth and Rangaswamy (1973), we define a right R-module M to be *finitely projective* if it is projective with respect to short exact sequences of right R-modules of the form $0 \to A \to B \to C \to 0$ with C finitely generated. We have completely characterized finitely projective modules over a Dedekind domain. If R is a Dedekind domain, then an R-module M is finitely projective if and only if its reduced part is torsionless and coseparable.

For a Dedekind domain R, finite projectivity, unlike projectivity is not hereditary. But it is proved to be pure hereditary, that is, every pure submodule of a finitely projective R-module is finitely projective.

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Ramamurthi and Rangaswamy (1973) have introduced and studied finitely injective modules. They defined a right R-module M to be *finitely injective* if M is injective with respect to short exact sequences of right R-modules of form $0 \to A \to B \to C \to 0$ with A finitely generated.

In this paper, as dual to the notion of "finite injectivity", we have introduced the notion of "finite projectivity" for R-modules. A right R-module M is said to be *finitely projective* if it is projective with respect to short exact sequences of right R-modules of the form $0 \to A \to B \to C \to 0$ with C finitely generated. We have completely characterized finitely projective modules over a Dedekind domain. If R is a Dedekind domain, then an R-module M is finitely projective if and only if its reduced part is torsionless and coseparable.

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For a Dedekind domain R, finite projectivity, unlike projectivity, is not hereditary. But, it is proved to be pure hereditary, that is, every pure submodule of a finitely projective R-module is finitely projective.

DEFINITION 1. A right R-module M is said to be *torsionless* if for each $0 \neq m \in M$, there exists $\varphi \in \mathrm{Hom}_R(M, R)$ such that $\varphi(m) \neq 0$.

DEFINITION 2. A right R-module M is said to be *finitely projective* if it is projective with respect to short exact sequences of the form $0 \to A \to B \to C \to 0$ with C finitely generated.

REMARK 3. Clearly, every projective right R-module is finitely projective, but not conversely. For example, the additive group Q of rational numbers is trivially finitely projective as a Z-module, since $\mathrm{Hom}_Z(Q, A) = 0$ for every finitely generated abelian group A. But Q is not projective Z-module.

Using standard arguments one proves the following proposition.

PROPOSITION 4. (i) *A direct sum of finitely projective right R-modules is finitely projective.*

(ii) *A direct summand of a finitely projective right R-module is finitely projective.*

(iii) *Every finitely generated, finitely projective right R-module is projective.*

REMARK 5. Azumaya, Mbuntum and Varadarajan (1975) have defined for a right R-module M, a right R-module A to be *M-projective* if for every epimorphism $\varphi: M \to B$ and every homomorphism $f: A \to B$ there is a homomorphism $g: B \to M$ such that $\varphi g = f$. Clearly, every finitely projective R-module is R-projective. By Azumaya, Mbuntum and Varadarajan (1975), p. 13, for an R-module A, the class $C^P(A)$, of all R-module M such that A is M-projective, is closed under the formation of finite direct sums and the homomorphic images. From this it follows that if A is R-projective, then $C^P(A)$ contains all finitely generated R-modules and hence, A is finitely projective.

Now, we study finitely projective modules over a Dedekind domain R. In the rest of this paper, *R will stand for a Dedekind domain.* The basic results, notations and terminology for modules over a Dedekind domain used in the rest of the paper are from Kaplansky (1952).

PROPOSITION 6. *An R-module M is finitely projective if and only if its reduced part is finitely projective.*

PROOF. Since R is a Dedekind domain, by Kaplansky (1952), p. 335, M can be expressed as $M = D \oplus E$, where D and E are respectively the divisible and the reduced parts of M. Next, we observe that any divisible R-module A is trivially finitely projective, since $\mathrm{Hom}_R(A, B) = 0$ for every finitely generated R-module B. So, by (i) and (ii) of Proposition 4, M is finitely projective if and only if its reduced part E is finitely projective.

In view of Proposition 6, the study of finitely projective modules over a Dedekind domain is reduced to the study of reduced, finitely projective modules. Before proving the main theorem, we introduce the notion of coseparability for R-modules which generalizes the notion of coseparability introduced by Griffith (1968) in the case of abelian groups.

DEFINITION 7. An R-module M is said to be *coseparable* if for every submodule N of M with M/N finitely generated, there exists a direct summand K of M contained in N such that M/K is finitely generated.

THEOREM 8. *An R-module M is reduced, finitely projective if and only if M is torsionless and coseparable.*

PROOF. *Necessity.* Suppose M is reduced and finitely projective. If M were not torsion-free, by Kaplansky (1952), p. 336, M will have a direct summand isomorphic to R/P^n for some non-zero prime ideal P of R and positive integer n. Then R/P^n will be finitely projective so that it is projective by (iii) of Proposition 4, since it is finitely generated. This is a contradiction since P^n is not a direct summand of R. Hence M is torsion-free.

Next, we prove that M is torsionless. Since M is torsion-free and reduced, $\bigcap_{0 \neq r \in R} Mr = 0$. Let $0 \neq m \in M$. Then there exists $0 \neq r \in R$ such that $m \notin Mr$. Let N be a submodule of M maximal with respect to the property that $Mr \subseteq N$ and $m \notin N$. Then M/N is subdirectly irreducible and hence indecomposable. Since $(M/N)r = 0$, M/N is of bounded order and hence is reduced. Since M/N is reduced, indecomposable and is of bounded order, by Kaplansky (1952), p. 336, it follows that M/N is isomorphic to R/P^n, for some non-zero prime ideal P of R and positive integer n. Now, by finite projectivity of M, the diagram

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow{\scriptstyle \eta} & & \\ & & M/N & & \\ & & \downarrow{\scriptstyle \psi} & & \\ R & \xrightarrow{\ \varphi\ } & R/P^n & \longrightarrow & 0 \end{array}$$

where η, φ are the natural maps and ψ is an isomorphism, yields a morphism $\alpha: M \to R$ satisfying $\varphi\alpha = \psi\eta$. Since $\eta(m) \neq 0$ and ψ is monic, $\alpha(m) \neq 0$. Thus M is torsionless.

Finally, we prove that M is coseparable. Let N be a submodule of M such that M/N is finitely generated. Let F be a finitely generated free R-module with an epimorphism $\varphi: F \to M/N$. Let $\eta: M \to M/N$ be the natural map. Then, by the finite projectivity of M, there is a map $\psi: M \to F$ such that $\varphi\psi = \eta$. Since $\operatorname{Im}\psi$ is projective, $\operatorname{Ker}\psi$ is a direct summand of M. Also, $\operatorname{Ker}\psi \subseteq N$ and $M/\operatorname{Ker}\psi \cong \operatorname{Im}\psi$ is finitely generated. So, M is coseparable.

Sufficiency. Let M be torsionless and coseparable. Since M is torsionless and R is reduced, it follows that M is reduced.

It remains to prove that M is finitely projective. Let $A \xrightarrow{\eta} B \longrightarrow 0$ be an exact sequence of R-modules with B finitely generated. Let $\alpha: M \to B$ be any R-map. Then $M/\operatorname{Ker}\alpha \cong \operatorname{Im}\alpha$ is finitely generated. Now, by coseparability of M, $M = K \oplus L$, for some submodule K of M contained in $\operatorname{Ker}\alpha$, where $M/K \cong L$ is finitely generated. Since L is torsion-free and finitely generated, it is projective. Let $\alpha' = \alpha/L: L \to B$. Then, by projectivity of L, there is $\beta': L \to A$ such that $\eta\beta' = \alpha'$. Now define $\beta: M \to A$ by setting $\beta = \beta'$ on L and $\beta = 0$ on K. Then $\eta\beta = \alpha$, noting that $\alpha = 0$ on K. So, M is finitely projective. This completes the proof of the theorem.

REMARK 9. From the proof of the sufficiency it follows that every torsion-free, coseparable module over a Dedekind domain is finitely projective.

REMARK 10. A direct product of (finitely) projective modules need not be finitely projective.

EXAMPLE. Let $M = \pi_\infty Z$. Griffith (1968), p. 655, has proved that M is not coseparable as a Z-module. So it is not finitely projective, by Theorem 8. But Z is projective as a Z-module.

Azumaya, Mbuntum and Varadarajan (1975), p. 10, have proved that every reduced Z-projective abelian group G is torsion-free and has the property that every pure subgroup of G of finite rank is a free direct summand of G. The above example shows that the converse of this statement is not true, that is, there exists a reduced abelian group having the said property that which is not Z-projective.

REMARK 11. We know that over a Dedekind domain every submodule of a projective module is projective. But, if R is a Dedekind domain, every submodule of a finitely projective R-module need not be finitely projective.

For example, the Prüfer group $Z(p^\infty)$ (where p is a prime) being divisible is finitely projective as a Z-module. But the subgroup $Z(p)$, a cyclic group of order p, of $Z(p^\infty)$ is not finitely projective as a Z-module, since it is not projective as a Z-module.

But it turns out that every pure submodule of a finitely projective module over a Dedekind domain is finitely projective.

Before proving this, we prove the following proposition.

PROPOSITION 12. *Let R be a Dedekind domain. Then every pure submodule of a torsion-free, coseparable R-module is coseparable.*

For the proof we need the following lemma.

LEMMA 13. *Let M be a torsion-free module over a Dedekind domain R. Then M is coseparable if and only if for every submodule N of M such that M/N is finitely generated, torsion, there exists a direct summand K of M contained in N such that M/K is finitely generated.*

PROOF. We need only prove the sufficiency. Let N be a submodule of M such that M/N is finitely generated. Let $M/N = A/N \oplus B/N$, where A/N and B/N are respectively torsion and torsion-free submodules of M/N (here A, B are submodules of M with $A \cap B = N$). Then M/B is finitely generated, torsion. So, by hypothesis, $M = P \oplus Q$, where $P \subseteq B$ and $M/P \cong Q$ is finitely generated. Since M/A is finitely generated, torsion-free, it is projective so that A is a direct summand of M. Let $M = A \oplus C$ for some submodule C of M. Then C is finitely generated. Since $A/A \cap P \cong A+P/P \subseteq M/P \cong Q$, $A/A \cap P$ is finitely generated, torsion-free and, hence, $A \cap P$ is a direct summand of A. Let $A = (A \cap P) \oplus T$ for some submodule T of A so that T is finitely generated. Then

$$M = A \oplus C = ((A \cap P) \oplus T) \oplus C = (A \cap P) \oplus T \oplus C,$$

where $A \cap P \subseteq A \cap B = N$ and $M/A \cap P \cong T \oplus C$ is finitely generated. So M is coseparable.

PROOF OF PROPOSITION 12. Let M be a torsion-free, coseparable R-module and let N be a pure submodule of M. Let K be a submodule of N such that N/K is finitely generated, torsion. Since N is pure in M, N/K is pure in M/K. Since N/K is of bounded order, by Kaplansky (1952), p. 333, M/K is a direct summand of M/K. Let $M/K = N/K \oplus T/K$ for some submodule T of M containing K. Now M/T is finitely generated. So, by the coseparability of M, M has a direct summand A contained in T so that $M = A \oplus B$ for some submodule B of M and $M/A \cong B$

is finitely generated. Now

$$N/N\cap A \cong N+A/A \subseteq M/A \cong B$$

and so, $N/N\cap A$ is finitely generated, torsion-free, so that $N=(N\cap A)\oplus P$ for some submodule P of N. Clearly, $N\cap A\subseteq N\cap T=K$. So, by Lemma 13, N is coseparable.

We now prove the result stated before Proposition 12.

PROPOSITION 14. *Let R be a Dedekind domain. Then every pure submodule of a finitely projective R-module is finitely projective.*

PROOF. Let M be a finitely projective R-module and let N be a pure submodule of M. Now $M = D\oplus E$, where D, E are respectively the divisible and the reduced parts of M. Since E is reduced, finitely projective, by Theorem 8, it is torsion-free and coseparable, so that $t(M) = t(D)$ (where $t(X)$ denotes the torsion part of a module X). Since $t(N) = t(D)\cap N$ is pure in D, it is a (divisible) summand of M. Factoring out $t(N)$ we may assume, without loss of generality, that $t(N) = 0$, that is, $N\cap t(D) = 0$. By the divisibility of $t(D)$, we can choose D' so that $D = t(D)\oplus D'$ and $N\subseteq D'\oplus E$. Then, by the modular law, we get

$$N\cap D = N\cap(D'\oplus E)\cap D = N\cap[D'\oplus(E\cap D)] = N\cap D',$$

as $E\cap D = 0$. Since N is pure and D' is divisible in the torsion-free module $D'\oplus E$, $N\cap D'$ is pure in D' and hence is divisible. Thus $N\cap D$ is divisible and hence $N=(N\cap D)\oplus N'$. Choosing E so that $N'\subseteq E$ we note that N' is pure in the reduced, torsion-free, coseparable module E, and so, by Proposition 12 and Remark 9, N' (and hence N) is finitely projective.

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