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ON THE DIOPHANTINE EQUATION $x^2 - py^2 = \pm 4q$ AND THE CLASS NUMBER OF REAL SUBFIELDS OF A CYCLOTOMIC FIELD*)

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Introduction

Let $H(m)$ denote the class number of the field $K = \boldsymbol{Q}(\zeta_m + \zeta_m^{-1})$, where $\boldsymbol{Q}$ is the rational number field and ζ_m is a primitive m-th root of unity for a positive rational integer m.

It has been proved by Ankeny, Chowla and Hasse in [2] that if $p = (2nq)^2 + 1$ is a prime, with prime q and integer $n > 1$, then $H(p) > 1$. Later, S.-D. Lang proved in [5] that if $p = ((2n + 1)q)^2 + 4$ is a prime, with odd prime q and integer $n \geq 1$, then $H(p) > 1$.

Both results are based on the fact that the diophantine equation $x^2 - py^2 = \pm 4m$ has no solution (x, y) in integers unless $m \geq nq$ (resp. $m \geq (2n + 1)q$).

In this paper, we shall first consider the diophantine equation $x^2 - py^2 = \pm 4q$ for distinct odd primes p, q, and give a necessary and sufficient condition for its solvability (§ 1). Next, we shall show that for distinct odd primes p, q satisfying $p = ((2n + 1)q)^2 \pm 2$ with integer $n \geq 0$ the diophantine equation $x^2 - py^2 = \pm q$ has no solution (x, y) in integers except for the case $p = 7$ $(n = 0, q = 3)$ (§2).

Moreover, in Section 3, for a prime p of such type, we shall give a sufficient condition for the class number $h(p)$ of the real quadratic field $\boldsymbol{Q}(\sqrt{p})$ to be greater than 1, and by applying this result to maximal real subfield of a cyclotomic field we shall also give a sufficient condition for $H(4p) > 1$.

Finally, we shall list up all primes $p < 100{,}000$ satisfying $p = ((2n + 1)q)^2 - 2$ with prime $q \equiv 1$ or $3 \pmod 4$, $(n \geq 0)$, and $p = ((2n + 1)q)^2 + 2$ with prime $q \equiv 1$ or $7 \pmod 4$, $(n \geq 0)$, for which both $h(p)$ and $H(4p)$ are

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greater than 1.

§ 1. Solvability of the equation $x^2 - py^2 = \pm 4q$

We consider, in this section, the diophantine equation $x^2 - py^2 = \pm 4q$ for distinct odd primes p, q. However, the following fact is noteworthy: When the equation $x^2 - py^2 = \pm q$ has a solution (u, v) in integers, the double of the solution $(2u, 2v)$ is also a solution of the equation $x^2 - py^2 = \pm 4q$. Conversely, in the case $p \not\equiv 1 \pmod 4$ all the solutions of $x^2 - py^2 = \pm 4q$ can be obtained from the solutions $x^2 - py^2 = \pm q$ in such a way, while in the case $p \equiv 1 \pmod 4$ not all the solutions can necessarily be found from the solutions of $x^2 - py^2 = \pm q$.

The following fact, which gives a relation between the solvability of the equation $x^2 - py^2 = \pm 4q$ and the class number of the real quadratic field $\boldsymbol{Q}(\sqrt{p})$, is already known[1], but is fundamental in our investigation. Therefore, we state it as a theorem and, for the sake of completeness, add a simple proof:

THEOREM 1. *Let p and q be two distinct odd primes. Then, the diophantine equation $x^2 - py^2 = \pm 4q$ has at least one solution (x, y) in integers if and only if the prime q splits completely in the real quadratic field $\boldsymbol{Q}(\sqrt{p})$ into the product of a principal prime ideal $\mathfrak{q}$ with degree one and its conjugate $\mathfrak{q}'$: $q = \mathfrak{q} \cdot \mathfrak{q}'$, ($\mathfrak{q} \neq \mathfrak{q}'$, $N\mathfrak{q} = N\mathfrak{q}' = q$, $\mathfrak{q} = (\omega)$, $\mathfrak{q}' = (\omega')$ with ω, ω' in $\boldsymbol{Q}(\sqrt{p})$).*

Proof. If there exists one solution (u, v) in integers of $x^2 - py^2 = \pm 4q$, then $u^2 - pv^2 = \pm 4q$ implies $u^2 \equiv pv^2 \pmod q$. Hence $1 = (pv^2/q) = (p/q)$ holds, and so by the law of decomposition in quadratic fields q splits completely in $\boldsymbol{Q}(\sqrt{p})$. On the other hand, it follows from $\pm q = (u + v\sqrt{p})/2 \cdot (u - v\sqrt{p})/2$ that both

$$\mathfrak{q} = \left(\frac{u + v\sqrt{p}}{2}\right) \quad \text{and} \quad \mathfrak{q}' = \left(\frac{u - v\sqrt{p}}{2}\right)$$

are principal ideals in $\boldsymbol{Q}(\sqrt{p})$ and $N\mathfrak{q} = \mathfrak{q} \cdot \mathfrak{q}' = q$ holds. Therefore $\mathfrak{q}$ and $\mathfrak{q}'$ are principal prime ideals in $\boldsymbol{Q}(\sqrt{p})$ with degree one.

Conversely, if q splits completely in $\boldsymbol{Q}(\sqrt{p})$ into the product of two principal prime ideals $\mathfrak{q}, \mathfrak{q}'$ with degree one, then there exist two rational

1) Cf. e. q. [2], [3] etc.

integers u, v such that both $\omega = (u + v\sqrt{p})/2$ and $\omega' = (u - v\sqrt{p})/2$ are integers in $\boldsymbol{Q}(\sqrt{p})$ and $\mathfrak{q} = (\omega)$, $\mathfrak{q}' = (\omega')$. Hence

$$q = \mathfrak{q}\cdot\mathfrak{q}' = N\mathfrak{q} = |N(\omega)| = \left|\frac{u^2 - pv^2}{4}\right|$$

implies $u^2 - pv^2 = \pm 4q$. Therefore $x^2 - py^2 = \pm 4q$ has the solution (u, v) in integers, which completes the proof of Theorem 1.

For example, let p and q be two odd primes satisfying $p = 4q^2 + 1$ or $p = q^2 + 4$. Then, the equation $x^2 - py^2 = \pm 4q$ has a solution $(2q \pm 1, 1)$ or $(q \pm 2, 1)$ in integers respectively. On the other hand, the prime q splits completely in $Q(\sqrt{p})$ such as

$$q = \mathfrak{q}\cdot\mathfrak{q}'; \quad \mathfrak{q} = \left(\frac{2q \pm 1 + \sqrt{p}}{2}\right), \quad \mathfrak{q}' = \left(\frac{2q \pm 1 - \sqrt{p}}{2}\right)$$

or

$$\mathfrak{q} = \left(\frac{q \pm 2 + \sqrt{p}}{2}\right), \quad \mathfrak{q}' = \left(\frac{q \pm 2 - \sqrt{p}}{2}\right)$$

respectively.

From Theorem 1 we deduce easily:

COROLLARY. *Let p and q be two odd primes satisfying $p = (nq)^2 + r^2$ for natural numbers n, r. Then, the class number $h(p)$ of the real quadratic field $\boldsymbol{Q}(\sqrt{p})$ is not equal to one i.e. $h(p) > 1$ if $x^2 - py^2 = \pm 4q$ has no solution (x, y) in integers.*

Proof. Since the condition $p = (nq)^2 + r^2$ implies immediately $(p/q) = 1$, prime q splits completely in $\boldsymbol{Q}(\sqrt{p})$. Hence, if we suppose $h(p) = 1$, then it follows from Theorem 1 that $x^2 - py^2 = \pm 4q$ has at least one solution (x, y) in integers. This is a contradiction. Therefore $h(p) = 1$ is impossible, which proves the assertion of Corollary.

§ 2. Solvability of the equation $x^2 - py^2 = \pm q$ for $p = ((2n + 1)q)^2 \pm 2$

After Ankeny-Chowla-Hasse and S.-D. Lang, H. Takeuchi proved in [6] that if both $12m + 7$ and $p = (3(8m + 5))^2 - 2$ are primes or both $12m + 11$ and $p = (3(8m + 7))^2 - 2$ are primes with an integer $m \geq 0$, then the equation $x^2 - py^2 = \pm 3$ has no solution (x, y) in integers.

Here, we prove more generally:

THEOREM 2. *Let p and q be two odd primes satisfying $p = ((2n + 1)q)^2$*

± 2 *with an integer* $n \geq 0$, *Then, the diophantine equation* $x^2 - py^2 = \pm q$ *has at least one solution* (x, y) *in integers if and only if* $p = 7$ *and* $q = 3$ $(n = 0)$ *i.e. only the equation* $x^2 - 7y^2 = -3$ *has a solution in integers, for example* $(x, y) = (2, 1)$.

Proof. (1) Let p and q be two odd primes satisfying $p = ((2n+1)q)^2 - 2$ with an integer $n \geq 0$, and put $l = (2n+1)q$.

Assume first that $x^2 - py^2 = q$ has at least one solution in integers, and let (u, v) $(u > 0, v > 0)$ be the least such positive integral solution: $u^2 - pv^2 = q$.

In the case $q > 2v^2$, where $q = u^2 - pv^2 = u^2 - l^2v^2 + 2v^2$ implies easily $(u - lv)(u + lv) = q - 2v^2 > 0$, both $a = u - lv > 0$ and $b = u + lv > 0$ are positive rational integers, and $l = (b - a)/2v$, $q = ab + 2v^2$ holds. On the other hand, since $a \geq 1$, $b \geq 1$ and $(a-1)(b+1) = ab + a - b - 1$, we know $ab - 1 \geq b - a$. Therefore

$$0 \leq 2nq = l - q = \frac{b-a}{2v} - ab - 2v^2 = \frac{1}{2v}(b - a - 2vab - 4v^3)$$
$$\leq \frac{1}{2v}(ab - 1 - 2vab - 4v^3) = \frac{-1}{2v}((4v^3 + 1) + (2v - 1)ab) < 0\,.$$

It is clear that this is a contradiction.

In the case $q < 2v^2$, the norm form $1 = N\varepsilon = N((l^2 - 1) + l\sqrt{l^2 - 2})$ of the fundamental unit[2] $\varepsilon = (l^2 - 1) + l\sqrt{l^2 - 2}$ of $\boldsymbol{Q}(\sqrt{p})$ multiplied by the norm form $q = N(u - v\sqrt{l^2 - 2})$ of $u^2 - pv^2 = q$ yields

$$\begin{aligned} q &= N[\{(l^2 - 1)u - lv(l^2 - 2)\} + \{lu - (l^2 - 1)v\}\sqrt{l^2 - 2}] \\ &= \{(l^2 - 1)u - lv(l^2 - 2)\}^2 - (l^2 - 2)\{lu - (l^2 - 1)v\}^2\,. \end{aligned}$$

Because of the minimal choice of v, we have $|lu - (l^2 - 1)v| \geq v$. Here, if $lu - (l^2 - 1)v \geq v$ i.e. $u \geq lv$, we have

$$q = u^2 - (l^2 - 2)v^2 \geq l^2v^2 - (l^2 - 2)v^2 = 2v^2\,,$$

which contradicts $q < 2v^2$. If $(l^2 - 1)v - lu \geq v$ i.e. $(l^2 - 2)v \geq lu$, we have

$$l^2q = l^2u^2 - l^2(l^2 - 2)v^2 \leq (l^2 - 2)^2v^2 - l^2(l^2 - 2)v^2 = -2(l^2 - 2)v^2 < 0\,,$$

which is also a contradiction.

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2) Cf. [1], [3].

Therefore, it is impossible that for the prime $p = ((2n+1)q)^2 - 2$ the equation $x^2 - py^2 = q$ has a solution in integers.

Next, assume that $x^2 - py^2 = -q$ has at least one solution in integers, and let (u, v) $(u > 0, v > 0)$ be the least such positive integral solution: $u^2 - pv^2 = -q$.

In the case $q = 3$, $v = 1$, where $-3 = -q = u^2 - pv^2 = u^2 - l^2 + 2$ implies $(l-u)(l+u) = 5$, we have $l - u = 1$, $l + u = 5$, and so $l = 3$, $u = 2$, $p = 7$ is only one possible case as asserted in the Theorem.

In the case $q = 3, v \geq 2$ or $q > 3, v \geq q$, the norm form of the fundamental unit ε of $Q(\sqrt{p})$ multiplied by the norm form $-q = N(u - v\sqrt{l^2-2})$ of the equation $u^2 - pv^2 = -q$, together with the minimal choice of v, yields $|lu - (l^2-1)v| \geq v$. Here, if $lu - (l^2-1)v \geq v$, we have $-q = u^2 - (l^2-2)v^2 \geq l^2v^2 - (l^2-2)v^2 = 2v^2 > 0$, which is a contradiction. If $(l^2-1)v - lu \geq v$, we have

$$-l^2q = l^2u^2 - l^2(l^2-2)v^2 \leq (l^2-2)^2v^2 - l^2(l^2-2)v^2 = -2(l^2-2)v^2,$$

and hence $l^2q \geq 2(l^2-2)v^2$. Therefore, in the case of $q = 3$ and $v \geq 2$, $3l^2 \geq 2(l^2-2)v^2 \geq 8(l^2-2)$ implies $16 \geq 5l^2 \geq 45$, which is a contradiction. In the case of $v \geq q > 3$, $l^2v \geq l^2q \geq 2l^2v^2 - 4v^2$ implies $4v^2 \geq (2v^2 - v)l^2 \geq v(2v-1)q^2$, and hence $q^2 \leq 4v/(2v-1) = 2 + 2/(2v-1) < 2 + 2/5 < 3$ holds. This is also a contradiction.

In the case $q > 3, v < q$, where $-q = u^2 - pv^2 = u^2 - l^2v^2 + 2v^2$ implies $(lv - u)(lv + u) = q + 2v^2 > 0$, both $a = lv - u > 0$ and $b = lv + u > 0$ are positive rational integers, and $l = (a+b)/2v$, $q = ab - 2v^2$. On the other hand, since $a \geq 1, b \geq 1$ and $(a-1)(b-1) = ab - (a+b) + 1$, we know $ab + 1 \geq a + b$. Therefore

$$0 \leq 2nq = l - q = \frac{a+b}{2v} - ab + 2v^2 = \frac{1}{2v}(a + b - 2vab + 4v^3)$$
$$\leq \frac{1}{2v}(ab + 1 - 2vab + 4v^3) = \frac{1}{2v}((4v^3+1) - (2v-1)ab)$$

implies $4v^3 + 1 \geq (2v-1)ab$, and so $ab \leq (4v^3+1)/(2v-1)$. Hence

$$q = ab - 2v^2 \leq \frac{4v^3+1}{2v-1} - 2v^2 = \frac{2v^2+1}{2v-1} = v + \frac{v+1}{2v-1}.$$

Here, if $v = 1$ or 2, then $q \leq v + (v+1)/(2v-1) = 3$, which is a contradiction. If $v \geq 3$, then $0 < (v+1)/(2v-1) < 1$ implies $q \leq v + (v+1)/(2v-1) < v + 1$, which contradicts $q > v$.

Therefore, it is impossible except for the case of $p = 7$, $q = 3$ $(n = 0)$ that for $p = ((2n+1)q)^2 - 2$ the equation $x^2 - py^2 = -q$ has a solution in integers.

(2) Let p and q be two odd primes satisfying $p = ((2n+1)q)^2 + 2$ with an integer $n \geq 0$, and put $l = (2n+1)q$.

Assume first that $x^2 - py^2 = q$ has at least one solution in integers, and let (u, v) $(u > 0, v > 0)$ be the least such positive integral solution: $u^2 - pv^2 = q$.

In the case $q > v$, where $q = u^2 - l^2v^2 - 2v^2$ implies $(u - lv)(u + lv) = q + 2v^2 > 0$, both $a = u - lv > 0$ and $b = u + lv > 0$ are positive rational integers, and $l = (b - a)/2v$, $q = ab - 2v^2$ holds. Hence, we get

$$0 \leq 2nq = l - q = \frac{b-a}{2v} - (ab - 2v^2) = \frac{1}{2v}(b - a - 2vab + 4v^3)$$

$$\leq \frac{1}{2v}(ab - 1 - 2vab + 4v^3) = \frac{1}{2v}((4v^3 - 1) - (2v - 1)ab),$$

and so $ab \leq (4v^3 - 1)/(2v - 1)$. Therefore, we get

$$q = ab - 2v^2 \leq \frac{4v^3 - 1}{2v - 1} - 2v^2 = \frac{2v^2 - 1}{2v - 1} = v + \frac{v - 1}{2v - 1} < v + 1.$$

This, however, contradicts $q > v$.

In the case $q \leq v$, the norm form $1 = N\varepsilon = N((l^2 + 1) + l\sqrt{l^2 + 2})$ of the fundamental unit[3] $\varepsilon = (l^2 + 1) + l\sqrt{l^2 + 2}$ of $\boldsymbol{Q}(\sqrt{p})$ multiplied by the norm form $q = N(u - v\sqrt{l^2 + 2})$ of the equation $u^2 - pv^2 = q$, yields

$$q = \{u(l^2 + 1) - lv(l^2 + 2)\}^2 - (l^2 + 2)\{lu - (l^2 + 1)v\}^2.$$

Because of the minimum choice of v, we have $|lu - (l^2 + 1)v| \geq v$. Here, if $lu - (l^2 + 1)v \geq v$, we have

$$l^2q = l^2u^2 - l^2(l^2 + 2)v^2 \geq (l^2 + 2)^2v^2 - l^2(l^2 + 2)v^2 = 2(l^2 + 2)v^2 \geq 2(l^2 + 2)q^2,$$

and hence $q \leq l^2/2(l^2 + 2) < 1/2$. This is a contradiction. If $(l^2 + 1)v - lu \geq v$, we have $q = u^2 - (l^2 + 2)v^2 \leq l^2v^2 - (l^2 + 2)v^2 = -2v^2 < 0$. This is also a contradiction.

Assume next that $x^2 - py^2 = -q$ has at least one solution in integers, and let (u, v) $(u > 0, v > 0)$ be the least such positive integral solution: $u^2 - pv^2 = -q$.

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3) Cf. [1], [3].

In the case $q > 2v^2$, where $-q = u^2 - l^2v^2 - 2v^2$ implies $(lv - u)(lv + u) = q - 2v^2 > 0$, both $a = lv - u > 0$ and $b = lv + u > 0$ are positive rational integers, and $l = (a + b)/2v$, $q = ab + 2v^2$ holds. Hence, we get

$$0 \leq l - q = \frac{a+b}{2v} - (ab + 2v^2) = \frac{1}{2v}(a + b - 2vab - 4v^3)$$
$$\leq \frac{1}{2v}(ab + 1 - 2vab - 4v^3) = \frac{-1}{2v}((2v - 1)ab + (4v^3 - 1)) < 0 .$$

This is a contradiction.

In the case $q < 2v^2$, the norm form of the fundamental unit ε of $\boldsymbol{Q}(\sqrt{p})$ multiplied by the norm form $-q = N(u - v\sqrt{l^2 + 2})$ of the equation $u^2 - pv^2 = -q$, together with the minimal choice of v, yields $|lu - (l^2 + 1)v| \geq v$. Here, if $lu - (l^2 + 1)v \geq v$, we have

$$-l^2q = l^2u^2 - l^2(l^2 + 2)v^2 \geq (l^2 + 2)^2v^2 - l^2(l^2 + 2)v^2 = 2(l^2 + 2)v^2 = 2pv^2 > 0 ,$$

which is a contradiction. If $(l^2 + 1)v - lu \geq v$, we have

$$-q = u^2 - (l^2 + 2)v^2 \leq l^2v^2 - (l^2 + 2)v^2 = -2v^2 ,$$

which contradicts $q < 2v^2$.

Therefore, it is impossible that for $p = ((2n + 1)q)^2 + 2$ the equation $x^2 - py^2 = \pm q$ has a solution in integers.

§ 3. The class number of real subfields of a cyclotomic field

In this section, we shall consider the class number $h(p)$ of the real quadratic subfield $\boldsymbol{Q}(\sqrt{p})$ and the class number $H(4p)$ of the maximal real subfield $\boldsymbol{Q}(\zeta_{4p} + \zeta_{4p}^{-1})$ of the cyclotomic field $\boldsymbol{Q}(\zeta_{4p})$:

$$\boldsymbol{Q} \subset \boldsymbol{Q}(\sqrt{p}) \subset \boldsymbol{Q}(\zeta_{4p} + \zeta_{4p}^{-1}) \subset \boldsymbol{Q}(\zeta_{4p}) .$$

From Theorems 1 and 2, we obtain first:

THEOREM 3. (1) *If $p = ((2n + 1)q)^2 - 2$ is a prime, where q is an odd prime satisfying $q \equiv 1$ or $3 \pmod 8$ and $n \geq 0$ is an integer, then the class number $h(p)$ of the real quadratic field $\boldsymbol{Q}(\sqrt{p})$ is not equal to one except for the case of $p = 7$ $(n = 0, q = 3)$.*

(2) *If $p = ((2n + 1)q)^2 + 2$ is a prime, where q is an odd prime satisfying $q \equiv 1$ or $7 \pmod 8)$ and $n \geq 0$ is an integer, then the class number $h(p)$ of the real quadratic field $\boldsymbol{Q}(\sqrt{p})$ is not equal to one i.e. $h(p) > 1$.*

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Proof. (1) It is evident that a prime $p=((2n+1)q)^2-2$ with an integer $n \geq 0$ and an odd prime q satisfies $(p/q)=(-2/q)$, and so by the law of decomposition in quadratic fields, the prime q splits in $\boldsymbol{Q}(\sqrt{p})$ completely if and only if $(-2/q)=1$ i.e. $q \equiv 1$ or 3 (mod 8). Hence, moreover if $h(p)=1$ is true, then by the Theorem 1 the equation $x^2-py^2=\pm q$ has at least one solution in integers x, y. This, however, contradicts the Theorem 2 except for the case of $p=7$ $(n=0, q=3)$. Therefore $h(p)=1$ is impossible except for the case of $p=7$ $(n=0, q=3)$.

(2) Since a prime $p=((2n+1)q)^2+2$ with an integer $n \geq 0$ and an odd prime q satisfies $(p/q)=(2/q)$, by the law of decomposition in quadratic fields implies that the prime q splits in $\boldsymbol{Q}(\sqrt{p})$ completely if and only if $(2/q)=1$ i.e. $q \equiv 1$ or 7 (mod 8). Hence, moreover if $h(p)=1$ is true, then by the Theorem 1 $x^2-py^2=\pm q$ has at least one solution in integers x, y. However, this contracts the Theorem 2. Therefore $h(p)=1$ is impossible, which proves the assertion of Theorem 3.

In order to prove Theorem 5, we need the following theorem[4]:

THEOREM 4. *For a positive integer m, let* ζ_m *be a primitive m-th root of unity and denote by* $H(m)$, $h(m)$ *the class number of the field* $K=\boldsymbol{Q}(\zeta_m+\zeta_m^{-1})$, $\boldsymbol{Q}(\sqrt{m})$ *respectively. If a prime p satisfies* $p \equiv 3 \pmod 4$, *then* $h(p) \mid H(4p)$ *holds.*

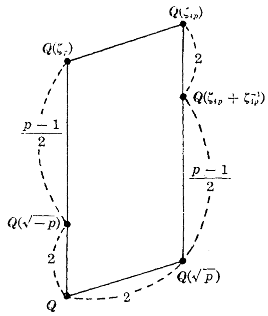
Proof. For a prime $p \equiv 3 \pmod 4$, we first know that the real quadratic field $\boldsymbol{Q}(\sqrt{p})$ and the imaginary quadratic field $\boldsymbol{Q}(\sqrt{-p})$ are imbedded respectively in the real cyclotomic field $K=\boldsymbol{Q}(\zeta_{4p}+\zeta_{4p}^{-1})$ and the imaginary cyclotomic field $\boldsymbol{Q}(\zeta_p)$ by means of the Gauss sum

$$\sqrt{d}=\sum_{a \bmod |d|}\left(\frac{d}{a}\right)\zeta_{|d|}^{a},$$

where d is the discriminant of a quadratic field $\boldsymbol{Q}(\sqrt{d})$ and (d/a) means the Kronecker symbol.

Next, we shall show $\boldsymbol{Q}(\zeta_p) \cap \boldsymbol{Q}(\sqrt{p})=\boldsymbol{Q}$ and $\boldsymbol{Q}(\zeta_{4p})=\boldsymbol{Q}(\sqrt{p})\cdot\boldsymbol{Q}(\zeta_p)$. If we suppose $\boldsymbol{Q}(\zeta_p) \cap \boldsymbol{Q}(\sqrt{p}) \neq \boldsymbol{Q}$, namely $\boldsymbol{Q}(\sqrt{p}) \subset \boldsymbol{Q}(\zeta_p)$, then $\boldsymbol{Q}(\sqrt{p}) \subset \boldsymbol{Q}(\zeta_p+\zeta_p^{-1})$ follows. This, however, contradicts $p \equiv 3 \pmod 4$, which shows $\boldsymbol{Q}(\zeta_p) \cap \boldsymbol{Q}(\sqrt{p})=\boldsymbol{Q}$. Moreover, this assertion implies the following

4) This theorem was already stated by Yamaguchi in [4], with an incomplete proof, for any positive integer p satisfying $\varphi(p)>4$. But, the theorem is not true in such a general form.



relation between degrees:

$$[\boldsymbol{Q}(\sqrt{p})\cdot\boldsymbol{Q}(\zeta_p)\colon \boldsymbol{Q}] = [\boldsymbol{Q}(\sqrt{p})\colon \boldsymbol{Q}][\boldsymbol{Q}(\zeta_p)\colon \boldsymbol{Q}] = 2(p-1)\,.$$

On the other hand, since $[\boldsymbol{Q}(\zeta_{4p})\colon \boldsymbol{Q}] = 2(p-1)$ and $\boldsymbol{Q}(\zeta_{4p}) \supset \boldsymbol{Q}(\sqrt{p})\cdot\boldsymbol{Q}(\zeta_p)$, the assertion $\boldsymbol{Q}(\zeta_{4p}) = \boldsymbol{Q}(\sqrt{p})\cdot\boldsymbol{Q}(\zeta_p)$ is also true.

Furthermore, we can prove that no abelian unramified extension of $\boldsymbol{Q}(\sqrt{p})$ is contained in $\boldsymbol{Q}(\zeta_{4p} + \zeta_{4p}^{-1})$. For, if we suppose that there exists an abelian unramified extension field L of $\boldsymbol{Q}(\sqrt{p})$ contained in $\boldsymbol{Q}(\zeta_{4p} + \zeta_{4p}^{-1})$, then we have $n = [L\colon \boldsymbol{Q}(\sqrt{p})] > 2$ because $[\boldsymbol{Q}(\zeta_{4p} + \zeta_{4p}^{-1})\colon \boldsymbol{Q}(\sqrt{p})] = (p-1)/2$ is odd. Hence, the ramification index $e(p)$ of p in $\boldsymbol{Q}(\zeta_{4p})/\boldsymbol{Q}$, which is a divisor of $2(p-1)/n$, is less than $p-1$ i.e. $e(p) < p-1$. However, since p is completely ramified in $\boldsymbol{Q}(\zeta_p)/\boldsymbol{Q}$, $e(p)$ is not less than $p-1$ i.e. $e(p) \geq p-1$. This is a contradiction, which proves our assertion.

Finally, from this assertion, it follows immediately by Hasse-Chevalley's theorem[5] that the assertion of Theorem 4 $h(p)\,|\,H(4p)$ is true.

THEOREM 5. (1) *If* $p = ((2n+1)q)^2 - 2$ *is a prime, where* q *is an odd prime satisfying* $q \equiv 1$ *or* $3 \pmod 8$ *and* $n \geq 0$ *is an integer, then* the *class number* $H(4p)$ *of* $\boldsymbol{Q}(\zeta_{4p} + \zeta_{4p}^{-1})$ *is greater than one except for the case of* $p = 7$ $(n = 0,\ q = 3)$.

(2) *If* $p = ((2n+1)q)^2 + 2$ *is a prime, where* q *is an odd prime satisfying* $q \equiv 1$ *or* $7 \pmod 8$ *and* $n \geq 0$ *is an integer, then the class number* $H(4p)$ *of* $\boldsymbol{Q}(\zeta_{4p} + \zeta_{4p}^{-1})$ *is greater than one*: $H(4p) > 1$.

Proof. Since $p = ((2n+1)q)^2 \pm 2 \equiv 3 \pmod 4$, the assertion of the Theorem $H(4p) > 1$ follows immediately from Theorem 3 and 4.

Finally, we give the values of all primes p less than 10^5 satisfying

5) Cf. [2].

the conditions in Theorem 3 and the class number $h(p)$ of the corresponding real quadratic fields $\boldsymbol{Q}(\sqrt{p})$[6].

$$p = ((2n+1)q)^2 - 2$$

p	n	q	$h(p)$	p	n	q	$h(p)$
7#	0	3	1#	357	0	19	3
79	1	3	3	1,087*	1	11	7
223	2	3	3	1,847	0	43	3
439	3	3	5	3,023	2	11	3
727	4	3	5	5,927	3	11	5
1,087	5	3	7	7,919	0	89	7
3,967	10	3	5	11,447	0	107	7
4,759	11	3	13	14,159	3	17	9
5,623	12	3	9	14,639	5	11	17
8,647	15	3	13	17,159	0	131	15
13,687	19	3	21	19,319	0	139	11
18,223	22	3	17	31,327*	1	59	27
31,327	29	3	27	42,023	2	41	15
33,487	30	3	19	44,519	0	211	11
53,359	38	3	37	53,359*	10	11	37
56,167	39	3	27	54.287	0	233	15
71,287	44	3	19	61,007	6	19	15
74,527	45	3	23	64,007	11	11	11
77,839	46	3	37	66,047	0	257	13
81,223	47	3	33	71,287*	1	89	19
91,807	50	3	45	81,223*	7	19	33
95,479	51	3	33	90,599	3	43	19
99,223	52	3	29	97,967	0	313	25

$$p = ((2n+1)q)^2 + 2$$

p	n	q	$h(p)$	p	n	q	$h(p)$
443	1	7	3	56,171	1	79	11
11,027	7	7	9	65,027	7	17	21
15,131	1	41	15	74,531	19	7	17
21,611	10	7	15	95,483	1	103	11
47,963	1	73	9				

# indicates only one exceptional case with class number $h(p) = 1$.
* indicates that the prime has appeared in the case of $q = 3$

6) For this purpose we referred to Wada's table of class numbers of real quadratic fields in [7].

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