ON THE INVERSION OF GENERAL TRANSFORMATIONS*

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Let k be the kernel of a "general transformation"; that is, $k(x)/x \in L_2(0,\infty)$, and if x and y are positive

$$(1) \qquad \int_0^\infty k(xu)k(yu)u^{-2}\,du = \min(x,y).$$

Then it is well known (see for example [8; Theorems 129 and 131]) that if the transform of $f \in L_2(0,\infty)$ is g, that is, if

$$(2) \qquad g(x) = (d/dx)\int_0^\infty k(xy)f(y)dy/y,$$

then the inverse transform is given by

$$(3) \qquad f(x) = (d/dx)\int_0^\infty k(xy)g(y)dy/y.$$

In practice, the inversion formula (3) is often hard to use. For example, the integral may be too difficult to evaluate; moreover, since (2) requires a differentiation, it is not well suited for numerical calculation. Hence it seems worthwhile to find other methods for inverting the transformation.

Here we shall give a technique for finding a large number of inversion formulae, and will illustrate the technique by a number of examples. It should be noted that, since the relation between f and g is reciprocal, we can calculate the transform of f by applying the inversion methods developed here to f rather than to g.

The essence of the inversion technique to be developed here is the conversion, by a suitable operation, of the general transformation into some other transformation. For this second transformation we chose the Laplace transformation since it has a particularly rich inversion theory.

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To this end, suppose the Laplace transform of f is F, that is,

$$(4) \qquad F(s) = \int_0^\infty e^{-sx} f(x)dx, \qquad s > o,$$

and let

$$(5) \qquad K(s) = s\int_0^\infty e^{-sx} k(x)dx, \qquad s > o.$$

Then since (by [2; Chapter 3, § 2, Theorem 1]) we may differentiate a Laplace transform as often as we like under the integral sign, we have, if s and x are positive,

$$x^{-1}K(s/x) = sx^{-2}\int_0^\infty e^{-sy/x}\, k(y)dy$$

$$= (d/dx)\int_0^\infty e^{-sy/x}k(y)dy/y = (d/dx)\int_0^\infty e^{-sy}k(xy)dy/y,$$

that is, $x^{-1}K(s/x)$ is the k-transform of e^{-sx}. Hence, by the Parseval formula for general transformations [8; theorem 129],

$$(6) \qquad \int_0^\infty K(s/x)g(x)dx/x = \int_0^\infty e^{-sx}f(x)dx = F(s),$$

that is, $\int_0^\infty K(s/x)g(x)dx/x$ is the Laplace transform of f. Hence any inversion technique for the Laplace transformation, when applied to $\int_0^\infty K(s/x)g(x)dx/x$ will serve to invert the general transformation as well.

As a first example, let $L_{n,x}$ denote the Widder Post inversion operator for the Laplace transformation, that is

$$L_{n,x}[F] = [(-1)^n/n!]\,(n/x)^{n+1}F^{(n)}(n/x).$$

Then from (6), $L_{n,x}[F]$ formally is equal to

$$(7) \quad \mathcal{K}_{n,x}[g] = [(-1)^n/n!]\,(n/x)^{n+1}\int_0^\infty K^{(n)}(n/xy)g(y)dy/y^{n+1}.$$

Thus we would suspect that $\mathcal{K}_{n,x}[g]$ should yield f in the limit as $n \to \infty$. We prove this in the following theorem.

THEOREM 1. If $f \in L_2(0,\infty)$ and g is the k-transform of f then

$$\lim_{n\to\infty} \mathcal{K}_{n,x}[g] = f(x)$$

at every point $x > 0$ in the Lebesgue set of f.

Proof. As already remarked, we can differentiate $K(s) = s\int_0^\infty e^{-su}k(u)du$, as often as we like under the integral

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sign. Using the Leibnitz rule, we obtain

$$K^{(n)}(s) = (-1)^n s \int_0^\infty e^{-su} u^n k(u)du + n(-1)^{n-1} \int_0^\infty e^{-su} u^{n-1} k(u)du,$$

so that if $x > 0$,

$$[(-1)^n/y^{n+1}]K^{(n)}(n/xy) = (n/y^{n+2}) \int_0^\infty e^{-nu/xy}[x^{-1}u-y]u^{n-1}k(u)du$$

$$= (d/dy)y^{-n} \int_0^\infty e^{-nu/xy} u^{n-1} k(u)du = (d/dy) \int_0^\infty e^{-nu/x} u^{n-1} k(yu)du.$$

Hence if $x > 0$, as a function of y, $[(-1)^n/y^{n+1}]K^{(n)}(n/xy)$ is the k-transform of $e^{-ny/x}y^n$. By the Parseval theorem for general transformations [8; theorem 129],

$$\mathcal{K}_{n,x}[g] = [(-1)^n/n!](n/x)^{n+1} \int_0^\infty K^{(n)}(n/xy)g(y)dy/y^{n+1}$$

$$= (1/n!)(n/x)^{n+1} \int_0^\infty e^{-ny/x} y^n f(y)dy$$

$$= [(-1)^n/n!](n/x)^{n+1} F^{(n)}(n/x) = L_{n,x}[F].$$

By [9; chapter 7, theorem 6a], $\lim_{n\to\infty} L_{n,x}[F] = f(x)$ at every point $x > 0$ in the Lebesgue set of f, and hence $\lim_{n\to\infty} \mathcal{K}_{n,x}[g] = f(x)$ at every such point.

Let us see what $\mathcal{K}_{n,x}$ looks like for certain particular general transformations. If, for example, $k(x) = (2/\pi)^{\frac{1}{2}} \sin x$, then the equation of the transformation is

$$g(x) = (d/dx)(2/\pi)^{\frac{1}{2}} \int_0^\infty y^{-1} \sin xy\, f(y)dy$$

that is, we obtain the Fourier cosine transformation in this case. Then from [3; § 4.7(1)], $K(s) = (2/\pi)^{\frac{1}{2}}(s/s^2+1) = (2\pi)^{-\frac{1}{2}}\mathrm{Re}\{1/s-i\}$, and

$$\mathcal{K}_{n,x}[g] = n^{n+1}(2\pi)^{-\frac{1}{2}} \int_0^\infty \mathrm{Re}\{1/(n-ixy)^{n+1}\}\, g(y)dy.$$

Similarly for the Fourier sine transformation; that is if $k(x) = (2/\pi)^{\frac{1}{2}}(1-\cos x)$, we obtain

$$\mathcal{K}_{n,x}[g] = n^{n+1}(2\pi)^{-\frac{1}{2}} \int_0^\infty \mathrm{Im}\{1/(n-ixy)^{n+1}\} g(y)dy.$$

From these two formulae it is easy to find an inversion formula for the complex Fourier transformation, defined for $f \in L_2(-\infty,\infty)$ by

$$g(x) = (d/dx)(2\pi)^{-\frac{1}{2}} \int_0^\infty [(e^{ixy}-1)/iy]\, f(y)dy.$$

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The even part of g is the cosine transform of the even part of f, and the odd part of g is i - times the sine transform of the odd part of f. Putting things together in this manner we see that, if

$$\mathcal{F}_{n,x}[g] = (-in/x)^{n+1}(2\pi)^{-\frac{1}{2}}\int_0^{\infty}(y-in/x)^{-(n+1)}\, g(x)dx$$

and if $f\in L_2(-\infty,\infty)$ and g is the complex Fourier transform of f, then $\lim_{n\to\infty}\mathcal{F}_{n,x}[g] = f(x)$ at every point $x \neq 0$ of the Lebesgue set of f. (For other uses of this inversion operator for the Fourier transform see [7].)

To obtain other inversion formulae for a general transformation we select other inversion operators for the Laplace transformation. For example let $M_{n,x}$ denote Phragmén's inversion operator, that is

$$M_{n,x}[F] = \sum_{r=1}^{\infty} (-1)^{r+1}(r!)^{-1}e^{rnx}F(rn) .$$

Applying this to (6) and using [2; chapter 8, § 1, theorem 1] we see that if $x > 0$,

$$\int_0^x f(y)dy = \lim_{n\to\infty}\sum_{r=1}^{\infty}(-1)^{r+1}(r!)^{-1}e^{rnx}\int_0^{\infty} K(nr/y)g(y)dy/y .$$

We can put this in a somewhat simpler form:

THEOREM 2. If $f\in L_2(0,\infty)$ and g is the k-transform of f then for $x > 0$,

$$\int_0^x f(y)dy = \lim_{n\to\infty}\int_0^{\infty} H(n,x,y)g(y)dy,$$

where

$$H(n,x,y) = (d/dy)\int_0^{\infty}(1-e^{-e^{n(x-u)}})k(yu)du/u.$$

Proof. From the Parseval formula for general transforms [8; theorem 129] we have

$$\int_0^{\infty} H(n,x,y)g(y)dy = \int_0^{\infty}(1-e^{-e^{n(x-y)}})f(y)dy,$$

and expanding the exponential we obtain

$$\int_0^{\infty} H(n,x,y)g(y)dy = \sum_{r=1}^{n}(-1)^{r+1}(r!)^{-1}e^{rnx}\int_0^{\infty}e^{-rny}f(y)dy$$

$$= \sum_{r=0}^{\infty}(-1)^{r+1}(r!)^{-1}e^{rnx}F(rn) = M_{n,x}[F] ,$$

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provided we justify the interchange of integration and summation. For this it suffices to show that

$$\sum_{r=0}^{\infty} (e^{rnx}/r!) \int_0^{\infty} e^{-rny} |f(y)| \, dy < \infty ,$$

and this is easy since by [2; Chapter 3, § 6, Theorem 1],

$$\int_0^{\infty} e^{-rny} \, | f(y) | \, dy \to 0 \text{ as } r \to \infty .$$

By using other inversion operators for the Laplace transformation, such as the Widder-Boas operator [1], or Hirschmann's operator [4], or the Erdélyi-Rooney operators [5,6] a variety of other inversion formulae all of much the same type, can be found. A rather different inversion method comes about by making use of a technique using Laguerre polynomials. We state this as a theorem.

THEOREM 3. If $f \in L_2(0,\infty)$ and g is the k-transform of f, then

$$f(x) = \text{l. i. m.}_{n\to\infty} \, e^{-\frac{1}{2}x} \sum_{r=0}^{n} q_r L_r(x)$$

where

$$q_n = \sum_{r=0}^{n} \binom{n}{r} (r!)^{-1} \int_0^{\infty} K^{(r)} (1/2x) g(x) dx / x^{r+1} .$$

Proof. The theorem follows from [2; Chapter 8, § 3, Theorem 1] once we have show that

$$F^{(r)}(s) = \int_0^{\infty} K^{(r)} (s/x) g(x) dx / x^{r+1}.$$

As in the proof of Theorem 1,

$$K^{(r)}(s) = (-1)^r s \int_0^{\infty} e^{-sy} y^r k(y) dy + r(-1)^{r-1} \int_0^{\infty} e^{-sy} y^{r-1} k(y) dy,$$

so that

$$x^{-(r+1)} K^{(r)}(s/x) = (-1)^r s x^{-(r+2)} \int_0^{\infty} e^{-sy/x} (y-rx) y^{r-1} k(y) dy$$

$$= (-1)^r (d/dx) x^{-r} \int_0^{\infty} e^{-sy/x} y^{r-1} k(y) dy$$

$$= (-1)^r (d/dx) \int_0^{\infty} e^{-sy} y^{r-1} k(xy) dy ;$$

that is $x^{-(r+1)} K^{(r)}(s/x)$ is the k-transform of $(-y)^r e^{-sy}$. Hence by the Parseval theorem for general transformations

$$\int_0^\infty K^{(r)}(s/x)g(x)dx/x^{r+1} = \int_0^\infty e^{-sy}(-y)^r f(y)dy = F^{(r)}(s),$$

on using [2; Chapter 3, § 2, Theorem 1], and our theorem is proved.

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