Hints to selected problems

Chapter 2

2.1 $a'_\mu = g_{\mu\rho}a'^\rho = g_{\mu\rho}L^\rho{}_\lambda a^\lambda = g_{\mu\rho}L^\rho{}_\lambda g^{\lambda\nu}a_\nu$. Hence $a'_\mu = L_\mu{}^\nu a_\nu$ where $L_\mu{}^\nu = g_{\mu\rho}L^\rho{}_\lambda g^{\lambda\nu}$. In particular, $L_0{}^1 = g_{00}L^0{}_1 g^{11} = -L^0{}_1$.

2.2 $a'^\mu = L^\mu{}_\nu a^\nu$. Multiply on the left by $L_\mu{}^\rho \cdot L_\mu{}^\rho a'^\mu = L_\mu{}^\rho L^\mu{}_\nu a^\nu = a^\rho$, or $a^\mu = a'^\nu L_\nu{}^\mu$. Similarly, $a_\mu = a'_\nu L^\nu{}_\mu$.

2.3 $\mathrm{d}\phi = \dfrac{\partial\phi}{\partial x^\mu}\mathrm{d}x^\mu = \dfrac{\partial\phi}{\partial x'^\nu}\mathrm{d}x'^\nu = \dfrac{\partial\phi}{\partial x'^\nu}L^\nu{}_\mu \mathrm{d}x^\mu$. Since the $\mathrm{d}x^\mu$ are arbitrary,

$$\frac{\partial\phi}{\partial x^\mu} = \frac{\partial\phi}{\partial x'^\nu}L^\nu{}_\mu.$$

This is a covariant vector field transformation (Problem 2.2).

2.4

$$\begin{aligned}\det\left(L_\mu{}^\nu\right) &= \det(g_{\mu\rho})\det\left(L^\rho{}_\lambda\right)\det(g^{\lambda\nu})\\ &= (-1)^2\det\left(L^\rho{}_\lambda\right).\end{aligned}$$

From (2.14), $\det(L_\mu{}^\nu)\det(L^\mu{}_\rho)\det\left(\delta^\nu{}_\rho\right) = 1$. The result follows.

2.6 Note that if $\det\mathbf{L}_1 = 1$ and $\det\mathbf{L}_2 = 1$ then $\det\mathbf{L}_1\det\mathbf{L}_2 = 1$.

2.7 $\delta'^\mu_\nu = L^\mu{}_\rho L_\nu{}^\lambda \delta^\rho_\lambda = L^\mu{}_\rho L_\nu{}^\rho = \delta^\mu_\nu$ using Problem 2.2.

2.8 Using (2.3),
$$\begin{aligned}\omega' &= \omega\cosh\theta - k\sinh\theta\\ &= \omega(\cosh\theta - \sinh\theta)\ \text{since}\ \omega = k\\ &= e^{-\theta}\omega.\end{aligned}$$
Since $\upsilon/c = \tanh\theta$, the result follows.

2.9 Jacobian is $\det(\partial x'^\mu/\partial x^\nu) = \det(L^\mu{}_\nu) = 1$.

2.10 The operation of space inversion can be written as $x_\mu{}' = P^\nu_\mu x_\nu$. Then the tensor $\varepsilon_{\mu\nu\lambda\rho}$, transforms as

$$\begin{aligned}\varepsilon'_{\mu\nu\lambda\rho} &= P^\alpha_\mu P^\beta_\nu P^\gamma_\lambda P^\delta_\rho \varepsilon_{\alpha\beta\gamma\delta}\\ &= \varepsilon_{\mu\nu\lambda\rho}\det\mathbf{P} = -\varepsilon_{\mu\nu\lambda\rho}.\end{aligned}$$

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Chapter 3

3.1 Let $x_i (i = 1, ..., 3N)$ be the Cartesian coordinates of the particles. Since $x_i = x_i(q)$, $\dot{x}_i = (\partial x_i/\partial q_j)\,\dot{q}_j$. Then $T = (m/2)\dot{x}_i\dot{x}_i = (m/2)(\partial x_i/\partial q_j)(\partial x_i/\partial q_k)\dot{q}_j\dot{q}_k$.

3.2
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int \left[\dot{\phi}\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\ddot{\phi} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\ddot{\phi} - \frac{\partial \mathcal{L}}{\partial \phi'}\dot{\phi}'\right]\mathrm{d}x.$$

Integrate by parts the term $-(\partial\mathcal{L}/\partial\phi')(\partial\dot{\phi}/\partial x)$ and use (3.12).

3.4 Use orthogonality and the dispersion relation (3.20). Note that H and P^i form a contravariant four-vector $(H, \mathbf{P})$.

3.5 Varying ψ^*,

$$\begin{aligned}\delta S &= \int \delta\mathcal{L}\,\mathrm{d}t\,\mathrm{d}^3\mathbf{x}\\ &= \int \Big[-(1/2\mathrm{i})\left(\delta\psi^*\frac{\partial\psi}{\partial t} - \frac{\partial(\delta\psi^*)}{\partial t}\psi\right)\\ &\qquad -(1/2m)\nabla(\delta\psi^*)\cdot\nabla\psi - \delta\psi^* V\psi\Big]\,\mathrm{d}t\,\mathrm{d}^3\mathbf{x}.\end{aligned}$$

Integrating by parts the terms involving $\partial(\delta\psi^*)/\partial t$ and $\nabla(\delta\psi^*)$ gives

$$\delta S = \int\left[-(1/\mathrm{i})\frac{\partial\psi}{\partial t} + (1/2m)\nabla^2\psi - V\psi\right]\delta\psi^*\mathrm{d}t\,\mathrm{d}^3\mathbf{x}.$$

Since this is true for any $\delta\psi^*$, the integrand must vanish. Hence

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -(1/2m)\nabla^2\psi + V\psi.$$

Chapter 4

4.1 $\mathcal{L} = -(1/4)F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu$. From (4.16), $F^{01} = -E_x = -F_{01}$, $F^{12} = -B_2 = F_{12}$, etc.

4.2 $\mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla\chi$. We require $\nabla\cdot\mathbf{A}' = \nabla\cdot(\mathbf{A} - \nabla\chi) = f - \nabla^2\chi = 0$. The solution is

$$\chi(\mathbf{r}, t) = -\frac{1}{4\pi}\int\frac{f(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}\mathrm{d}^3\mathbf{r}'.$$

4.3
$$\begin{aligned}\tilde{F}_{01} &= (\varepsilon_{0123}F^{23} + \varepsilon_{0132}F^{32})/2\\ &= (F^{23} - F^{32})/2 = (-B_x - B_x)/2 = -B_x,\ \text{etc.}\end{aligned}$$

4.4
$$\begin{aligned}\mathbf{A} &= \frac{1}{\sqrt{2\omega V}}\left[(\varepsilon_x + \mathrm{i}\varepsilon_y)\mathrm{e}^{\mathrm{i}(kz-\omega t)} + (\varepsilon_x - \mathrm{i}\varepsilon_y)\mathrm{e}^{-\mathrm{i}(kz-\omega t)}\right]\\ &= \frac{1}{\sqrt{2\omega V}}\left[2\cos(kz-\omega t), -2\sin(kz-\omega t), 0\right],\\ \mathbf{E} &= -\frac{\partial\mathbf{A}}{\partial t} = \sqrt{\frac{2\omega}{V}}\left[\sin(\omega t - kz), -\cos(\omega t - kz), 0\right].\end{aligned}$$

By inspection, on any plane of fixed z, $\mathbf{E}$ rotates in a positive sense about the z-axis.

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4.5 If the fields vanish at infinity, a term $\partial_i(A_0 F^{0i}) = \partial_\mu(A_0 F^{0\mu})$ does not contribute to the energy. Thus the energy density is not unique, and we may take

$$\begin{aligned} T^0_0 &= -F^{0\mu}\partial_0 A_\mu + \partial_\mu(A_0 F^{0\mu}) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &= -F^{0\mu}(\partial_0 A_\mu - \partial_\mu A_0) + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \end{aligned}$$

since in free space $\partial_\mu F^{0\mu} = 0$ by (4.8),

$$= -F^{0\mu}F_{0\mu} + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

4.6 $L = \frac{1}{2}m\dot{\mathbf{x}}^2 - q\phi + q\dot{\mathbf{x}}\cdot\mathbf{A}$, $p^i = (\partial L/\partial \dot{x}^i) = m\dot{x}^i + qA^i$ are the generalised momenta. The equation of motion $(\mathrm{d}p^i/\mathrm{d}t) = (\partial L/\partial x^i)$ is

$$m\ddot{x}^i + q(\partial A^i/\partial t) + q(\partial A^i/\partial x^j)\dot{x}^j = -q(\partial\phi/\partial x^i) + q\dot{x}^j(\partial A^j/\partial x^i),$$

giving

$$m\ddot{x}^i = q[-(\partial\phi/\partial x^i) - q(\partial A^i/\partial t)] - qF^{ij}\dot{x}^j$$

(noting $\partial^i = -\partial/\partial x^i$, and definition (4.6)). Taking $i = 1$,

$$\begin{aligned} m\ddot{x} &= q(E_x - F^{12}\dot{y} - F^{13}\dot{z}) \\ &= q(E_x + \dot{y}B_z - \dot{z}B_y), \end{aligned}$$

and similarly for the other components.

$$\begin{aligned} H(\mathbf{p},\mathbf{x}) &= p^i\dot{x}^i - L \\ &= \mathbf{p}\cdot(\mathbf{p} - q\mathbf{A})/m - [(\mathbf{p} - q\mathbf{A})^2/2m - q\phi + q(\mathbf{p} - q\mathbf{A})\cdot\mathbf{A}/m] \\ &= (\mathbf{p} - q\mathbf{A})^2/2m + q\phi. \end{aligned}$$

4.7 $\int L\,\mathrm{d}t = \int(\gamma L)\,\mathrm{d}\tau$, where $\mathrm{d}\tau = \mathrm{d}t/\gamma$ is Lorentz invariant (see (2.5); τ is the 'proper time'). Hence the result.

Chapter 5

5.3 Under the transformations (5.19) and (5.20),

$$\begin{aligned} \psi_R'^\dagger\psi_L' &= \psi_R^\dagger\mathbf{N}^\dagger\mathbf{M}\psi_L = \psi_R^\dagger\psi_L, \\ \psi_L'^\dagger\psi_R' &= \psi_L^\dagger\mathbf{M}^\dagger\mathbf{N}\psi_R = \psi_L'^\dagger\psi_R, \\ \psi_R'^\dagger\sigma^\mu\psi_R' &= \psi_R^\dagger\mathbf{N}^\dagger\sigma^\mu\mathbf{N}\psi_R = L^\mu{}_\nu\psi_R\sigma^\nu\psi_R, \\ \psi_L'^\dagger\tilde{\sigma}^\mu\psi_L' &= \psi_L^\dagger\mathbf{M}^\dagger\tilde{\sigma}^\mu\mathbf{M}\psi_L = L^\mu{}_\nu\psi_L^\dagger\tilde{\sigma}^\nu\psi_L, \\ \psi_R'\sigma^\mu\tilde{\sigma}^\nu\psi_L' &= \psi_R^\dagger\mathrm{M}^\dagger\sigma^\mu\mathbf{M}\mathbf{N}^\dagger\sigma^\nu\mathbf{N}\psi_L \ \left(\text{since}\,\mathbf{M}\mathbf{N}^\dagger = \mathbf{I}\right) \\ &= L^\mu{}_\lambda L^\mu{}_\rho\psi_R^\dagger\sigma^\lambda\tilde{\sigma}^\rho\psi_L, \text{ etc.} \end{aligned}$$

5.4 Using (5.28), (5.31) becomes

$$\psi^\dagger\beta\,(\mathrm{i}\beta\partial_0 + \mathrm{i}\beta\alpha_i\partial_i - m)\,\psi = \psi^\dagger\,(\mathrm{i}\partial_0 + \mathrm{i}\alpha_i\partial_i - \beta m)\,\psi \text{ since } \beta^2 = \mathbf{I}.$$

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5.6

$$\begin{aligned} \mathrm{i}\bar{\psi}\gamma^5\psi &= \mathrm{i}\left(\psi_{\mathrm{L}}^{\dagger}, \psi_{\mathrm{R}}^{\dagger}\right)\begin{pmatrix} \mathbf{0} & \sigma^0 \\ \sigma^0 & \mathbf{0} \end{pmatrix}\begin{pmatrix} -\sigma^0 & \mathbf{0} \\ \mathbf{0} & \sigma^0 \end{pmatrix}\begin{pmatrix} \psi_{\mathrm{L}} \\ \psi_{\mathrm{R}} \end{pmatrix} \\ &= \mathrm{i}\left(\psi_{\mathrm{L}}^{\dagger}\psi_R - \psi_{\mathrm{R}}^{\dagger}\psi_{\mathrm{L}}\right). \end{aligned}$$

This is invariant under proper Lorentz transformations, but changes sign under the parity operation (5.27).

5.7 The results follow from the definitions (5.30) and (5.4).

Chapter 6

6.1

$$\begin{aligned} \psi_+^{\dagger}\psi_+ &= \frac{1}{2}(\langle + |\, \mathrm{e}^{-\theta/2}, \langle + |\, \mathrm{e}^{\theta/2}) \begin{pmatrix} \mathrm{e}^{-\theta/2}| + \rangle \\ \mathrm{e}^{\theta/2}| + \rangle \end{pmatrix} \\ &= \frac{1}{2}[\mathrm{e}^{-\theta}\, \langle + | + \rangle \, + \, \mathrm{e}^{\theta}\, \langle + | + \rangle] \\ &= \cosh\theta \, = \, \gamma \, = \, E/m. \end{aligned}$$

From (6.14), probability of right-handed mode

$$= \frac{e^{\theta}}{e^{\theta} + e^{-\theta}} = \frac{e^{\theta}}{2\cosh\theta} = \frac{1}{2}\left(1 + \frac{v}{c}\right), \text{ since } \tanh\theta = \frac{v}{c}.$$

6.3

$$\begin{aligned} u_+^{\dagger}(\mathbf{p})u_+(\mathbf{p}) &= \tfrac{1}{2}(e^{\theta} + e^{-\theta}) = \cosh\theta = E/m, \text{ etc.} \\ u_+^{\dagger}(\mathbf{p})u_-(\mathbf{p}) &= 0 \text{ since } \langle + | - \rangle = 0. \end{aligned}$$

Note that

$$\sigma\cdot\hat{\mathbf{p}}| + \rangle = | + \rangle \quad \text{and} \quad \boldsymbol{\sigma}\cdot\hat{\mathbf{p}}| - \rangle = -\, | - \rangle$$

implies

$$\boldsymbol{\sigma}\cdot(-\hat{\mathbf{p}})\,| + \rangle = -\,| + \rangle \quad \text{and} \quad \boldsymbol{\sigma}\cdot(-\hat{\mathbf{p}})\,| - \rangle = | - \rangle.$$

6.5 $|+\rangle$ and $|-\rangle$ are evidently normalised, and by direct substitution and the use of trigonometric identities, $\boldsymbol{\sigma}\cdot\mathbf{p}\,|+\rangle = |+\rangle$, $\boldsymbol{\sigma}\cdot\mathbf{p}\,|-\rangle = -|-\rangle$.

Chapter 7

7.1 This follows using the orthogonality properties of plane waves and those derived in Problem 6.3.

7.2 For example,

$$\psi_+^c = -\mathrm{i}\gamma^2\psi_+^* = \left(\mathrm{i}/\sqrt{2}\right)e^{-\mathrm{i}(pz-Et)}\begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-\theta/2} & |+\rangle \\ \mathrm{e}^{\theta/2} & |+\rangle \end{pmatrix}$$

and $\sigma^2\,|+\rangle = \mathrm{i}\,|-\rangle$, giving

$$\psi_+^c = \left(1/\sqrt{2}\right)\mathrm{e}^{-\mathrm{i}(pz-Et)}\begin{pmatrix} \mathrm{e}^{\theta/2} & |-\rangle \\ -\mathrm{e}^{-\theta/2} & |-\rangle \end{pmatrix}.$$

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7.3 Under the parity operation,

$$\psi_{\rm L} \to \psi_{\rm R}, \tilde{\sigma}^{\mu}\partial_{\mu} \to \sigma^{\mu}\partial_{\mu},$$

from (5.26) and (5.27). Under charge conjugation,

$$\psi_{\rm R} \to {\rm i}\sigma^2\psi_{\rm L}^*.$$

Hence under the combined operations,

$$
{\rm i}\psi_{\rm L}^{\dagger}\tilde{\sigma}^{\mu}\partial_{\mu}\psi_{\rm L} \to {\rm i}\psi_{\rm L}^{\rm T}\sigma^2\sigma^{\mu}\sigma^2\partial_{\mu}\psi_{\rm L}^* = -{\rm i}\partial_{\mu}\psi_{\rm L}^{\dagger}(\sigma^2\sigma^{\mu}\sigma^2)^{\rm T}\psi_{\rm L}
$$

(recall the – sign that must be introduced when spinor fields are interchanged). But $\left(\sigma^2\sigma^{\mu}\sigma^2\right)^{\rm T} = \tilde{\sigma}^{\mu}$.

Finally, integrating by parts in the action yields the Lagrangian density ${\rm i}\psi_{\rm L}^{\dagger}\tilde{\sigma}^{\mu}\partial_{\mu}\psi_{\rm L}$.

7.4 $\psi_{\rm R} \to \psi_{\rm R}' = \mathbf{N}\psi_{\rm R}$ by (5.20).

$$
{\rm i}\sigma^2\psi_{\rm R}^* \to {\rm i}\sigma^2\mathbf{N}^*\psi_{\rm R}^*.
$$

But $\sigma^2\mathbf{N}^* = \mathbf{M}\sigma^2$. This is true for $\mathbf{M}$ and $\mathbf{N}$ given by (5.24), and holds in general.

7.5 Varying $\mathbf{\Phi}^*$ in the action gives

$$
\begin{aligned}
\delta S &= \int \{-[({\rm i}\partial_{\mu} + qA_{\mu})\delta\mathbf{\Phi}^*][({\rm i}\partial^{\mu} - qA^{\mu})\mathbf{\Phi}] - m^2\delta\mathbf{\Phi}^*\mathbf{\Phi}\}\,{\rm d}t\,{\rm d}^3\mathbf{x} \\
&= \int \delta\mathbf{\Phi}^*\{({\rm i}\partial_{\mu} - qA_{\mu})({\rm i}\partial^{\mu} - qA^{\mu})\mathbf{\Phi} - m^2\mathbf{\Phi}\}\,{\rm d}t\,{\rm d}^3\mathbf{x},
\end{aligned}
$$

after integrating by parts. Since this holds for any $\delta\Phi^*$, the Klein–Gordon equation follows.

7.6 If $\mathbf{\Phi} \to e^{{\rm i}\alpha}\mathbf{\Phi}$ with $\alpha = \alpha(x)$ small,

$$
\begin{aligned}
&({\rm i}\partial_{\mu} + qA_{\mu})({\rm e}^{{\rm i}\alpha}\mathbf{\Phi}) = {\rm e}^{{\rm i}\alpha}({\rm i}\partial_{\mu} + qA_{\mu})\mathbf{\Phi} - (\partial_{\mu}\alpha){\rm e}^{{\rm i}\alpha}\mathbf{\Phi} \\
&\delta S = \int \{-(\partial_{\mu}\alpha)\mathbf{\Phi}^*[({\rm i}\partial^{\mu} - qA^{\mu})\mathbf{\Phi}] + [({\rm i}\partial^{\mu} + qA^{\mu})\mathbf{\Phi}^*](\delta_{\mu}\alpha)\mathbf{\Phi}\}\,{\rm d}t\,{\rm d}^3\mathbf{x} \\
&\quad = \int \alpha(x)\partial_{\mu}\{\mathbf{\Phi}^*[({\rm i}\partial^{\mu} - qA^{\mu})\mathbf{\Phi}] - [({\rm i}\partial^{\mu} + qA^{\mu})\mathbf{\Phi}^*]\mathbf{\Phi}\}\,{\rm d}t\,{\rm d}^3\mathbf{x},
\end{aligned}
$$

after integrating by parts. Hence the current

$$
j^{\mu} = {\rm i}[\mathbf{\Phi}^*(\partial^{\mu}\mathbf{\Phi}) - (\partial^{\mu}\mathbf{\Phi}^*)\mathbf{\Phi}] - 2qA^{\mu}\mathbf{\Phi}^*\mathbf{\Phi}
$$

is conserved, as is also qj^{μ}. (Note that $qj^{\mu} = -\partial\mathcal{L}/\partial A_{\mu}$ is the electromagnetic current.)

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7.7 Verify by direct calculation, e.g. for positive helicity and taking $\mu = 3$,

$$\begin{aligned} qj^3 &= -e\psi^+\gamma^0\gamma^3\psi \\ &= -(e/2)\left(e^{-\theta/2}\langle +|, e^{\theta/2}\langle +|\right)\begin{pmatrix} -\sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}\begin{pmatrix} e^{-\theta/2} & |+\rangle \\ e^{\theta/2} & |+\rangle \end{pmatrix} \\ &= -e \sinh\theta, \text{ since } \sigma^3 |+\rangle = |+\rangle. \end{aligned}$$

7.8 This follows since the electric field lines are reversed in direction, $\mathbf{E} \to \mathbf{E}' = -\mathbf{E}$.

7.9 Assuming $\rho(\mathrm{t}) \to \rho'(\mathrm{t}') = \rho(-\mathrm{t})$, Maxwell's equations retain the same form if $\mathbf{E} \to \mathbf{E}' = \mathbf{E}$, $\mathbf{B} \to \mathbf{B}' = -\mathbf{B}$, $\mathbf{J} \to \mathbf{J}' = -\mathbf{J}$, or equivalently

$$\phi \to \phi' = \phi, \ \mathbf{A} \to \mathbf{A}' = -\mathbf{A}.$$

Taking the complex conjugate of (7.6) and multiplying on the left by $\gamma^1\gamma^3$ gives

$$\gamma^1\gamma^3[\gamma^{\mu *}(-\mathrm{i}\partial_\mu - qA_\mu) - m]\psi^* = 0.$$

Now

$$\begin{aligned} \gamma^1\gamma^3\left(\gamma^0\right)^* &= \gamma^1\gamma^3\gamma^0 = \gamma^0\gamma^1\gamma^3, \\ \gamma^1\gamma^3\left(\gamma^i\right)^* &= -\gamma^i\gamma^1\gamma^3 \quad \text{for } i = 1, 2, 3, \end{aligned}$$

and the result follows.

Chapter 8

8.3 If an e^+e^- pair is created there is a frame of reference (the centre of mass frame) in which the total momentum of the pair is zero. The photon would also have zero momentum in this frame and hence zero energy: energy conservation would be violated.

Chapter 9

9.1 Conservation of energy gives $m_\pi = E_e + E_\nu$. Conservation of momentum gives $p_e = p_\nu$. Also

$$E_\nu = p_\nu, \ {E_e}^2 = {p_e}^2 + {m_e}^2, \ \upsilon_e = p_e/E_e.$$

Hence

$$(m_\pi - p_e)^2 = {E_e}^2 = {p_e}^2 + {m_e}^2, \quad p_e = \frac{{m_\pi}^2 - {m_e}^2}{2m_\pi}.$$

Then

$$\begin{aligned} E_e &= m_\pi - p_e = \frac{{m_\pi}^2 + {m_e}^2}{2m_\pi}, \\ \frac{1}{2}\left(1 - \frac{\upsilon}{c}\right) &= \frac{1}{2}\left(1 - \frac{{m_\pi}^2 - {m_e}^2}{{m_\pi}^2 + {m_e}^2}\right) = \frac{{m_e}^2}{{m_\pi}^2 + {m_e}^2}. \end{aligned}$$

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9.2 Final energy $E = E_e + E_\nu = E_e + p_e$

$$\frac{dE}{dp_e} = \frac{dE_e}{dp_e} + 1 = \frac{P_e}{E_e} + 1 = \frac{E_e + P_e}{E_e} = \frac{M_\pi}{E_e}.$$

9.3 Using Problem (9.1),

$$\left(1 - \frac{\upsilon_e}{c}\right) p_e{}^2 E_e = \frac{m_e{}^2}{4m_\pi{}^3} = \left(m_\pi{}^2 - m_e{}^2\right)^2,$$

with a similar expression for the μ leptons.

9.4 Since the pion is at rest, only the term $\partial\Phi/\partial t$ contributes. From (3.35), there is a factor in $\mathcal{L}_{\text{int}}$ arising from this:

$$\frac{1}{\sqrt{V}} \frac{(-\mathrm{i}m_\pi)}{\sqrt{2m_\pi}} a_0.$$

From Problem 6.5, the $\bar{\nu}$ factor is

$$\frac{1}{\sqrt{V}} d^\dagger_{\mathbf{p}'} \mathrm{e}^{\mathrm{i}(-\mathbf{p}'\cdot\mathbf{r})} \left|-\right\rangle_{\mathbf{p}'}.$$

From (6.24), the $e^\dagger_L$ factor is

$$\frac{1}{\sqrt{V}} \sqrt{\frac{m_e}{E_p}} b^\dagger_p \mathrm{e}^{\mathrm{i}(-\mathbf{p}\cdot\mathbf{r})} \frac{1}{\sqrt{2}} e^{-\theta/2} \left\langle+\right|_{\mathbf{p}}.$$

(Only this helicity term contributes.)

Integrating over volume gives $\mathbf{p}' = -\mathbf{p}$ and a volume factor V, so that, for a given $\mathbf{p}$,

$$\left\langle e_{\mathbf{p}}, \bar{\nu}_{-\mathbf{p}} \left| V(0) \right| \pi^- \right\rangle = \frac{(-\mathrm{i})}{\sqrt{V}} \sqrt{\frac{m_\pi}{2}} \sqrt{\frac{m_e}{E_e}} \frac{\alpha_\pi}{\sqrt{2}} \mathrm{e}^{-\theta/2}.$$

(Note that $\left|-\right\rangle_{-\mathbf{p}} = \left|+\right\rangle_{\mathbf{p}}$.)

Hence the transition rate s is obtained. The factor 4π in the density of states comes from summing over all directions of $\mathbf{p}$. Also $(E_e/m_e) = \cosh\theta$ and $\mathrm{e}^{-\theta}/\cosh\theta = (1 - \tanh\theta) = (1 - \upsilon/c)$.

9.7

$$G_F \approx \left(\frac{192\pi^3}{\tau m_\mu{}^5}\right)^{1/2} = 1.164 \times 10^{-5} (\text{GeV})^{-2}.$$

9.8 The square of the centre of mass energy

$$s = (E_e + E_\nu)^2 - (\mathbf{p}_e + \mathbf{p}_\nu)^2$$

is Lorentz invariant. In the electron's rest frame

$$s = (m_e + E_\nu)^2 - p_\nu^2 = m_e^2 + 2m_e E_\nu.$$

9.9 The expression (9.8) contains the term

$$-2\sqrt{2} G_F g_{\mu\nu} e^\dagger_L \tilde{\sigma}^\mu \nu_{eL} \nu^\dagger_{eL} \tilde{\sigma}^\nu e_L.$$

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The expression (9.15) contains the term

$$\left(G_F/\sqrt{2}\right)g_{\mu\nu}\nu^\dagger_{\mu L}\tilde{\sigma}^\mu\nu_{\mu L}\bar{\psi}_e\gamma^\nu(c_v - c_A\gamma^5)\psi_e.$$

9.10

$$\frac{\tau\,(K\to\mu\bar{\nu}_\mu)}{\tau\,(K\to e\bar{\nu}_e)} = \frac{m_e^2(m_K^2 - m_e^2)^2}{m_\mu^2(m_K^2 - m_\mu^2)^2} = 2.57\times10^{-5}$$

$$\frac{1}{\tau\,(K\to\mu\bar{\nu}_\mu)} = \frac{{\alpha_K}^2}{4\pi}\left(1-\frac{\upsilon_\mu}{c}\right){p_\mu}^2E_\mu \qquad \text{(cf. (9.3))},$$

where $$\left(1-\frac{\upsilon_\mu}{c}\right)p_\mu^2E_\mu = \frac{{m_\mu}^2}{4{m_K}^2}({m_K}^2 - {m_\mu}^2)^2$$

(cf. Problem 9.3).

This gives $\alpha_K = 5.82\times10^{-10}$ MeV^{-1}, and $\alpha_\pi = 2.09\times10^{-9}$ (text), giving $\alpha_K/\alpha_\pi = 0.28$.

9.11 Consider the decay $\tau^- \to \pi^- + \nu_\tau$. The term in $\mathcal{L}_{\text{int}}$ that generates the decay is

$$\nu^\dagger_{\tau L}\tilde{\sigma}^\mu\tau_L\partial_\mu\mathbf{\Phi}^\dagger.$$

Consider the τ to be at rest with its spin aligned along the z-axis, and the neutrino momentum to be $\mathbf{p}$. The pion momentum is then $(-\mathbf{p})$, and the interaction energy contains a term

$$\frac{\alpha_\pi}{\sqrt{V}}\frac{i}{\sqrt{2E_\pi}}a^\dagger_\pi(-\mathbf{p})\,b^\dagger_\nu(\mathbf{p})\,b_\tau(0)\,\langle-|_p\left(\sigma^0E_\pi - \boldsymbol{\sigma}\cdot\mathbf{p}\right)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}.$$

Now $\langle-|_\mathbf{p}(\sigma^0E_\pi - \boldsymbol{\sigma}\cdot\mathbf{p}) = \langle-|_\mathbf{p}(E_\pi + p_\nu) = \langle-|_\mathbf{p}\,m_\tau$, and from Problem 6.5, $\langle-|_\mathbf{p} = (-\sin(\theta/2)e^{i\phi},\ \cos(\theta/2))$ where θ and ϕ are the polar angles of $\mathbf{p}$. Hence

$$\left\langle\pi_{-\mathbf{p}},\nu_\mathbf{p}\,|V|\,\tau\right\rangle = -\frac{\alpha_\pi}{\sqrt{V}}\frac{i}{\sqrt{2E_\pi}}m_\tau\frac{1}{\sqrt{2}}\sin(\theta/2)\,e^{i\phi}.$$

The decay rate is

$$\frac{1}{\tau} = 2\pi\int|\langle f|\,V\,|i\rangle|^2\,p\,(m_\tau)\,d\Omega$$

where

$$p\,(m_\tau) = \frac{V}{(2\pi)^3}\frac{\left({m_\tau}^2 - {m_\pi}^2\right)^2}{4{m_\tau}^2}\frac{E_\pi}{m_\tau},$$

and the angular integration gives a factor 2π.

Chapter 10

10.1 The term $-(m^2/2{\phi_0}^2)\sqrt{2}\phi_0\chi\psi^2$ links the χ and ψ fields, and $m = m_\chi/\sqrt{2}$. Since the ψ particles are massless, the final energy $E = 2p$, and the density of states factor

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for the decay is

$$\rho(E) = \frac{V}{(2\pi)^3} 4\pi p^2 \frac{\mathrm{d}p}{\mathrm{d}E} \quad \text{where} \quad \frac{\mathrm{d}p}{\mathrm{d}E} = \frac{1}{2},$$

and the factor 4π comes from the angular integration.

In the matrix element $\langle \mathbf{p}, -\mathbf{p}|V|\chi \text{ at rest}\rangle$, the χ field gives a factor $1/\sqrt{2m_\chi}$ from the expansion (3.21), and each of the ψ fields gives a factor $1/\sqrt{2p}$. Hence

$$2\pi|\langle \mathbf{p}|V|i\rangle|^2 \rho(E) = 2\pi \frac{{m_\chi}^4}{8{\phi_0}^2} \frac{1}{2m_\chi} \frac{1}{4p^2} \frac{4\pi p^2}{(2\pi)^3} \frac{1}{2}$$
$$= \frac{m_\chi}{128\pi} \left(\frac{m_\chi}{\phi_0}\right)^2.$$

10.2 The decay of an isolated vector boson requires a term in $\mathcal{L}_{\text{int}}$ linear in A_μ. There is a term $(\sqrt{2}\phi_0 q^2) A_\mu A^\mu h$ that allows the decay of the scalar boson if energy conservation can be satisfied, i.e. $m_h = \sqrt{2}m > 2\left(\sqrt{2}q\phi_0\right)$.

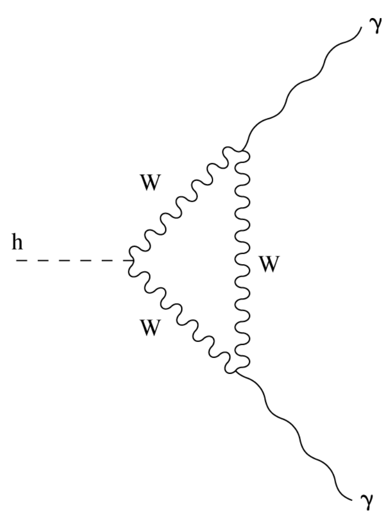
Chapter 11

11.1 The term $\mathbf{UWU}^\dagger$ satisfies $(\mathbf{UWU}^\dagger)^\dagger = \mathbf{UWU}^\dagger$ and $\text{Tr}(\mathbf{UWU}^\dagger) = \text{Tr}(\mathbf{U}^\dagger\mathbf{UW}) = \text{Tr}(\mathbf{W}) = 0$.

Noting that $(\hat{\boldsymbol{\alpha}} \cdot \boldsymbol{\tau})^2 = \mathbf{I}$ and $(\partial_\mu \alpha^j)\alpha^j = 0$ since $\alpha^j \alpha^j = 1$, the term $(2i/g_2)(\partial_\mu \mathbf{U})\mathbf{U}^\dagger$ may be written as a linear combination of the matrices τ^j with real coefficients. Each τ^j is Hermitian and has zero trace.

11.3 The last term may be written as $({g_2}^2{\phi_0}^2/4)({W_\mu}^1 W^{1\mu} + {W_\mu}^2 W^{2\mu})$, and in the absence of electromagnetic fields the term that precedes it can be handled similarly. There are therefore two independent fields each with mass $g_2\phi_0/\sqrt{2}$ (cf. Section 4.9).

11.4 The interaction Lagrangian density (11.32) contains a term ${g_2}^2/\sqrt{2})hW_\mu^- W^{+\mu}$ coupling the h field and the charged W fields.



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11.5 Consider

$$\mathbf{U} = \cos\alpha \mathbf{I} + \mathrm{i}\sin\alpha \boldsymbol{\tau}\cdot\hat{\boldsymbol{\alpha}} \text{ (see B.9)}.$$

Then

$$\mathbf{U}^{*} = \cos\alpha \mathbf{I} - \mathrm{i}\sin\alpha(\tau^1\hat{\alpha}^1 - \tau^2\hat{\alpha}^2 + \tau^3\hat{\alpha}^3)$$

and

$$\tau^2\mathbf{U}^{*} = [\cos\alpha \mathbf{I} + \mathrm{i}\sin\alpha(\tau^1\hat{\alpha}^1 + \tau^2\hat{\alpha}^2 + \tau^3\hat{\alpha}^3)]\tau^2$$

using

$$\tau^2\tau^1 = -\tau^1\tau^2,\ \tau^2\tau^3 = -\tau^3\tau^2.$$

Hence

$$\mathrm{i}\tau^2\mathbf{U}^{*} = \mathbf{U}(\mathbf{i}\tau^2) \quad \text{and} \quad \mathrm{i}\tau^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The result follows.

11.6 Using (B.9).

$$\begin{aligned}\mathbf{U} &= \cos\alpha \mathbf{I} + \sin\alpha(\sin\phi\tau^1 + \cos\phi\tau^2) \\ &= \begin{pmatrix} \cos\alpha & \mathrm{i}\sin\alpha(\sin\phi - \mathrm{i}\cos\phi) \\ \mathrm{i}\sin\alpha(\sin\phi + \mathrm{i}\cos\phi) & \cos\alpha \end{pmatrix}.\end{aligned}$$

Chapter 12

12.2 Take the two fields to be

$$\mathbf{L} = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}.$$

To maintain local gauge invariance, the dynamical term in the Lagrangian density must be $\mathbf{L}^{\dagger}\tilde{\sigma}^{\mu}\mathrm{i}(\partial_{\mu} + \mathrm{i}(\mathrm{g}_2/2)\mathbf{W}_{\mu})\mathbf{L}$.
There are terms which mix L_1 and L_2, for example,

$$\begin{aligned}&-(g_2/2){L_1}^{\dagger}\tilde{\sigma}^{\mu}({W_{\mu}}^1 - \mathrm{i}{W_{\mu}}^2)L_2 \\ &= -(g_2/2){L_1}^{\dagger}\tilde{\sigma}^{\mu}L_2{W_{\mu}}^{\dagger}.\end{aligned}$$

The operator ${W_{\mu}}^{\dagger}$ destroys electric charge e, so that to conserve charge ${L_1}^{\dagger}\tilde{\sigma}^{\mu}L_2$, must create charge e.

12.3 The Higgs particle at rest has zero momentum and zero angular momentum. Hence the e^+ and e^- have opposite momentum. If they had opposite helicities, they would have to carry orbital angular momentum with a component $+1$ or -1 along their direction of motion, to conserve angular momentum. This is not possible since $\mathbf{p}\cdot(\mathbf{r}\times\mathbf{p}) = 0$.

The final density of momentum states is

$$\rho(E) = \frac{V}{(2\pi)^3}4\pi {p_{\mathrm{e}}}^2\frac{\mathrm{d}p_{\mathrm{e}}}{\mathrm{d}E}.$$

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The final energy $E = 2E_e$,where $E_e{}^2 = m_e{}^2 p_e{}^2$. Hence

$$\frac{dp_e}{dE} = \frac{1}{2}\frac{dp_e}{dE_e} = \frac{E_e}{2p_e}, \quad \text{and} \quad p(E) = \frac{V}{(2\pi)^2} p_e E_e.$$

The interaction term in (12.9) is $-(c_e\sqrt{2})h\bar{\psi}\psi$. From (6.24) and (3.21), this gives

$$\langle f|V|i\rangle = \frac{1}{\sqrt{V}}\frac{1}{\sqrt{2m_H}}\frac{m_e}{E_e}[\bar{\mu}_+(\mathbf{p})v_+(-\mathbf{p})]$$

or

$$[\bar{\mu}_-(\mathbf{p})v_-(-\mathbf{p})].$$

Now $\bar{\mu}_\pm(\mathbf{p})v_\pm(-\mathbf{p}) = \sinh\theta$, and $E_e/m_e = \cosh\theta$. Hence the decay rate to positive helicities is

$$2\pi|\langle f|V|i\rangle|^2\rho(E) = 2\pi\frac{c_e{}^2}{2}\frac{1}{2m_H}\tanh^2\theta\frac{1}{(2\pi)^2}p_e E_e.$$

Also $\tan\theta = v_e/c = p_e/E_e$ and $E_e = m_H/2$. The decay rate to negative helicities is the same, and the result follows.

12.4 Since $c_\tau > c_\mu > c_e$ (see (12.13)) the decay to $\tau^+\tau^-$ dominates in the leptonic partial width. Also, since the Higgs mass is much greater than the τ mass, $v_\tau \approx c$. Hence

$$\frac{\Gamma}{m_H} \approx \frac{c_\tau^2}{16\pi} = \frac{1}{16\pi}\left(\frac{m_\tau}{\phi_0}\right)^2.$$

Chapter 13

13.1 In the rest frame of the W, and neglecting the lepton mass, $\mathbf{p}_l = -\mathbf{p}_v$, $E_l = p_l = M_w/2$, and $p_i{}^2 = M_w{}^2/4 = p_x{}^2 + p_y{}^2 + p_z{}^2$. Taking the x-axis to be the beam direction, the mean square transverse momentum is

$$\overline{p_x{}^2} + \overline{p_y{}^2} = (2/3)p_l{}^2 = M_w{}^2/6.$$

13.2 From (12.23), the Z_μ is produced by right-handed electron fields with a coupling $e\tan\theta_w = 2e\sin^2\theta_w/\sin(2\theta_w)$ and by left-handed fields with a coupling $-e\cos(2\theta_w)/\sin(2\theta_w)$. In head-on collisions at high energies the right-handed component of the electron (positron) has positive (negative) helicity. Hence the total spin is $+1$ along the electron beam direction. The spin of the left-handed components is opposite. For unpolarised beams the left-handed and right-handed components are equally populated, and the result follows.

13.3 Consider the decay $W^- \to e^- + \bar{\nu}_e$ in the W^- rest frame. With no loss of generality we may take the W^- to have $J = 1$, $J_z = 0$ (see Section 4.9). The interaction Lagrangian density responsible for the decay is (from (12.15) and (12.16))

$$\mathcal{L} = -(g_2/\sqrt{2})j^3 W_3^-.$$

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If the electron has momentum $\mathbf{p}$, the neutrino has momentum $-\mathbf{p}$. Neglecting the electron mass (see Problem 6.5) the matrix element for the decay is

$$\langle f|\, V\, |i\rangle = \frac{g_2}{\sqrt{2}}\frac{1}{\sqrt{2M_{\mathrm{w}}V}}\,\langle -|\,\sigma^3\,|+\rangle\,.$$

(Recall $\boldsymbol{\sigma}\cdot\mathbf{p}\,|-\rangle = -\,|-\rangle\,, \boldsymbol{\sigma}\cdot(-\mathbf{p})\,|+\rangle = -\,|+\rangle$.) Also, from Problem 6.6, $\langle -|\,\sigma^3\,|+\rangle = -\sin\theta \mathrm{e}^{\mathrm{i}\phi}$. The decay rate is

$$\Gamma = 2\pi\int |\,\langle f|\, V\, |i\rangle\,|^2 \mathrm{d}\Omega \frac{V}{(2\pi)}^3 {p_e}^2 \frac{\mathrm{d}p_{\mathrm{e}}}{\mathrm{d}E}$$

where $\mathrm{d}p_{\mathrm{e}}/\mathrm{d}E = 1/2$, $p_{\mathrm{e}} = M_{\mathrm{w}}/2$, giving

$$\Gamma = \frac{{g_2}^2}{48\pi}M_{\mathrm{w}} = \frac{G_{\mathrm{F}}{M_{\mathrm{w}}}^3}{6\pi\sqrt{2}},\ \text{by (12.22).}$$

The decay rate for $Z \to \nu\bar{\nu}$ requires a similar calculation, with M_{w} replaced by M_{z} and the coupling constant $g_2/\sqrt{2}$ replaced by $e/\sin 2\theta_{\mathrm{w}} = g_2/2\cos\theta_{\mathrm{w}} = g^2 M_z/2M_{\mathrm{w}}$. (We have used (12.23), (11.38) and (11.37a).) Then

$$\Gamma(Z \to \nu\bar{\nu}) = \frac{G_{\mathrm{F}}M_{\mathrm{z}}^3}{12\pi\sqrt{2}}.$$

There are two terms in (12.23) contributing to $\Gamma(Z \to \mathrm{e}^+\mathrm{e}^-)$, yielding

$$\Gamma(Z \to e^+e^-) = \Gamma(Z \to \nu\bar{\nu})[(2\sin^2\theta_{\mathrm{w}})^2 + (\cos 2\theta_{\mathrm{w}})^2].$$

13.4 83.86 MeV.

Chapter 14

14.3 Under an $SU(2)$ transformation, and from Appendix A.2

$$(\Phi^{\mathrm{T}}\varepsilon L) \to (\Phi^{\mathrm{T}}U^{\mathrm{T}}\varepsilon UL)$$

$$U^{\mathrm{T}}\varepsilon U = \begin{bmatrix} U_{AA} & U_{BA} \\ U_{AB} & U_{BB}\end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}\begin{bmatrix} U_{AA} & U_{AB} \\ U_{BA} & U_{BB}\end{bmatrix} = \begin{bmatrix} 0 & \mathrm{Det}(U) \\ -\mathrm{Det}(U) & 0\end{bmatrix}$$

$$= (\mathrm{Det}(U))\varepsilon$$

$$= \varepsilon,\ \text{since Det(U)} = 1.\ \text{Hence}\ (\boldsymbol{\Phi}^{\mathrm{T}}U^{\mathrm{T}}\varepsilon UL) = (\boldsymbol{\Phi}^{\mathrm{T}}\varepsilon L)$$

14.4 From (11.23),

$$\boldsymbol{\Phi} = \begin{pmatrix} 0 \\ \phi_0 + h/\sqrt{2}\end{pmatrix}.$$

Inserting this in (14.6) gives the coupling terms

$$-(1/\sqrt{2})\sum[G^d_{ij}d^\dagger_{\mathrm{L}i}d_{\mathrm{R}j}h + \text{Hermitian conjugate.}$$

Similar terms arise from (14.9) and (14.10). Using the true quark masses these become

$$-(1/\sqrt{2}\phi_0)\sum[m^d_i(d^\dagger_{\mathrm{L}i}d_{\mathrm{R}i} + d^\dagger_{\mathrm{R}i}d_{\mathrm{L}i}) + m^u_i(u^\dagger_{\mathrm{L}i}u_{\mathrm{R}i} + u^\dagger_{\mathrm{R}i}u_{\mathrm{L}i})]h.$$

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The coupling to the top quark is

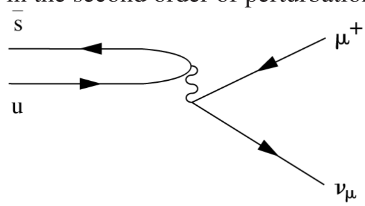
$$c_t = \frac{m_t}{\sqrt{2}\phi_0} \approx \frac{180\,\text{GeV}}{\sqrt{2} \times 180\,\text{GeV}} \approx 0.7.$$

14.5 For $\text{K}^+ \to \mu^+ + \nu_\mu$, the terms

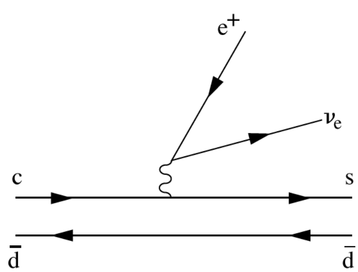
$$s_\text{L}{}^\dagger \tilde{\sigma}^\mu u_\text{L} V_{us}^* \text{ from } j^\mu,\; \nu_{\mu\text{L}}{}^\dagger \tilde{\sigma}^\mu \mu_\text{L} \text{ from } j^{\mu\dagger}$$

contribute in the second order of perturbation theory. (See (a).)

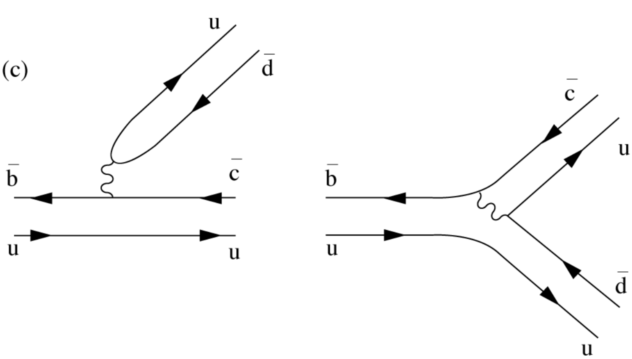
(a)



(b)



(c)



For $\text{D}^+ \to K^0 + e^+ + \nu_\text{e}$,

$$s_\text{L}^\dagger \tilde{\sigma}^\mu c_\text{L} V_{cs}^* \text{ from } j^\mu,\; \nu_{\text{eL}}^\dagger \tilde{\sigma}^\mu e_\text{L} \text{ from } j^{\mu\dagger}. \text{ (See (b).)}$$

For $\text{B}^\dagger \to \bar{\text{D}}^0 + \pi^\dagger$,

$$b_\text{L}^\dagger \tilde{\sigma}^\mu c_\text{L} V_{cb}^* \text{ from } j^\mu,\; u_\text{L}^\dagger \tilde{\sigma}^\mu d_\text{L} V_{ud} \text{ from } j^{\mu\dagger}. \text{ (See (c).)}$$

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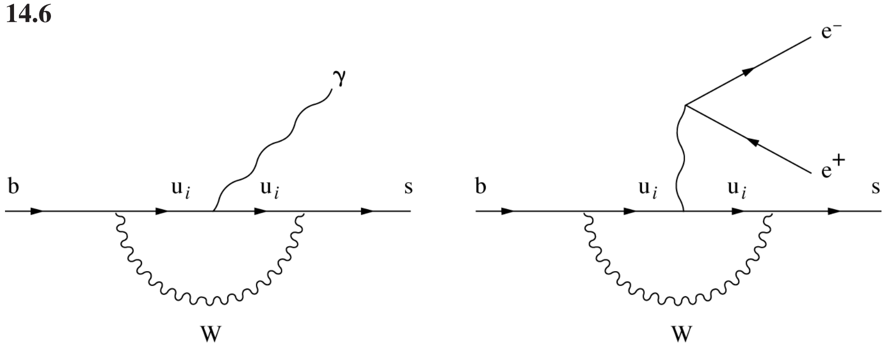
14.6



The quark labelled u_i can be u, c or t.

Chapter 15

15.1 The decay rate for $Z \to d\bar{d}$ of (15.3) can be compared with the decay rate for $Z \to e^+e^-$ of (13.3), calculated in the answer to Problem 13.3. Comparing the interaction Lagrangian densities (12.23) and (14.14), the term in the left-handed coupling $\cos 2\theta_w = 1 - 2\sin^2\theta_w$ is replaced by $(1 - (2/3)\sin^2\theta_w)$, and in the right-handed coupling $2\sin^2\theta_w$ is replaced by $(2/3)\sin^2\theta_w$. Including a colour factor of 3 and replacing $\sin^2\theta_w$ by $(1/3)\sin^2\theta_w$ in the rate (13.3) gives the rate (15.3).

Similarly for $Z \to u\bar{u}$. Comparing (12.23) with (14.14), $\sin^2\theta_w$ is replaced by $(2/3)\sin^2\theta_w$.

The decay rate $W^+ \to u_i\bar{d}_j$ of (15.6) can be compared with the rate $W^+ \to e^+\nu_e$ of (13.2) calculated in the answer to Problem 13.3. Comparing the interactions (12.18) and (14.20), $g_2/\sqrt{2}$ is replaced by $eV_{ij}/\sqrt{2}\sin\theta_w = g_2V_{ij}/\sqrt{2}$. Including the colour factor of 3, the rate (15.6) follows from the rate (13.2).

Chapter 16

16.1

$$\begin{aligned} \mathrm{G}_{\mu\nu} &= \partial_\mu \mathbf{G}_\nu - \partial_\nu \mathbf{G}_\mu + \mathrm{i}g(\mathbf{G}_\mu\mathbf{G}_\nu - \mathbf{G}_\nu\mathbf{G}_\mu) \\ &= (\partial_\mu \mathbf{G}^a_\nu - \partial_\nu \mathbf{G}^a_\mu)(\lambda_a/2) \\ &\quad + \mathrm{i}(g/4)\Big(G^b_\mu G^c_\nu \lambda_b\lambda_c - G^c_\nu G^b_\mu \lambda_c\lambda_b\Big), \end{aligned}$$

and

$$(\lambda_b\lambda_c - \lambda_c\lambda_b) = 2\mathrm{i}f_{bca}\lambda_a \quad \text{(see (B.27))}.$$

Hence

$$G_{\mu\nu} = [(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu) - gf_{abc}G^b_\mu G^c_\nu](\lambda_a/2).$$

16.2 These are the terms in (16.9) cubic and quadratic in the G fields.

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16.3 Variation of G^a_ν gives

$$\delta S = \int \left[-(1/2)G^{a\mu\nu}\delta G^a_{\mu\nu} - g\sum_f \bar{\mathbf{q}}_f \gamma^\nu \delta G^a_\nu (\lambda_a/2)\mathbf{q}_f \right] \mathrm{d}^4 x,$$

and

$$-(1/2)G^{a\mu\nu}\delta G^a_{\mu\nu} = -G^{a\mu\nu}\partial_\mu(\delta G^a_\nu) + gG^{c\mu\nu}G^b_\mu \delta G^a_\nu f_{cba}.$$

(There are two equal contributions to the right-hand side.) Integrating by parts gives

$$\delta S = \int \left[\partial_\mu G^{a\mu\nu} - gG^{c\mu\nu}G^b_\mu f_{abc} - g\sum_f \bar{\mathbf{q}}_f \gamma^\nu (\lambda_a/2)\mathbf{q}_f \right] \delta G^a_\nu \, \mathrm{d}^4 x$$

$(f_{cba} = -f_{abc})$.

Since the δG^a_ν are arbitrary (16.14) is obtained.

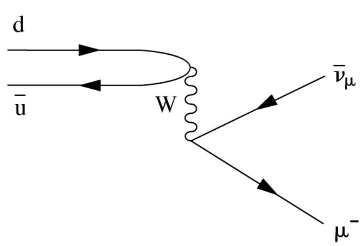
16.4

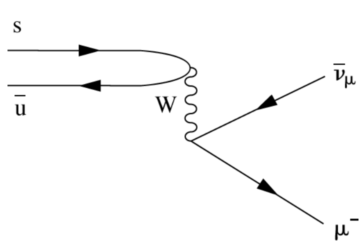
$$Q^2/4m^2 = \mathrm{e}^{12x^2/e^2} = \mathrm{e}^{3\pi/\alpha} = 10^{560}.$$
$$2m \sim 1\,\text{MeV},\ Q^2 \sim 10^{560}\,(\text{MeV})^2.$$

16.5 Take $\mathbf{Q}\cdot\mathbf{r} = Qr\cos\theta$ and $\mathrm{d}^3\mathbf{Q} = Q^2\mathrm{d}Q\, d(\cos\theta)\mathrm{d}\phi$ where (Q, θ, ϕ) are the polar coordinates of $\mathbf{Q}$, with $\mathbf{r}$ taken to be (0, 0, r).

Chapter 18

18.1





From (14.15), the interaction terms in $\bar{\mathrm{u}}\mathrm{d}\mathrm{W}^+$ and $\bar{\mathrm{u}}\mathrm{s}\mathrm{W}^+$ contain factors V_{ud} and V_{us}, respectively. Problem (9.10) shows $\alpha_{\mathrm{K}}/\alpha_\pi \approx 0.28$. Setting this equal to $V_{\mathrm{us}}/V_{\mathrm{ud}}$ gives $\sin\theta_{12} \approx 0.27$.

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18.2 The internal wave function of two pions at $\mathbf{r}_1$ and $\mathbf{r}_2$ in an S state is a function of only $|\mathbf{r}_1-\mathbf{r}_2|$ and $|\mathbf{r}_1-\mathbf{r}_2|$ is invariant under both C and P. Hence

$$CP\left|\pi^0\pi^0\right\rangle = \left|\pi^0\pi^0\right\rangle \quad \text{and} \quad CP\left|\pi^+\pi^-\right\rangle = \left|\pi^+\pi^-\right\rangle.$$

18.3 The internal wave function of three pions at $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, depends only on two relative coordinates, say $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{r}_{23} = \mathbf{r}_3 - \mathbf{r}_2$. To be invariant under rotations ($J=0$) the internal wave function can be a function of only three scalars: $\mathbf{r}_{12}\cdot\mathbf{r}_{12}$, $\mathbf{r}_{12}\cdot\mathbf{r}_{23}$, and $\mathbf{r}_{23}\cdot\mathbf{r}_{23}$. These are invariant under C and P. Since the intrinsic parity of the π^0 is negative,

$$CP\left|\pi^0\pi^0\pi^0\right\rangle = -\left|\pi^0\pi^0\pi^0\right\rangle.$$

18.4 The area of the triangle formed by the origin and the points $\mathbf{r}_1 = (x_1, y_1, 0)$ and $\mathbf{r}_2 = (\mathrm{x}_2, \mathrm{y}_2, 0)$ is

$$\begin{aligned}(1/2)|\mathbf{r}_1 \times \mathbf{r}_2| &= (1/2)|x_1 y_2 - x_2 y_2)| \\ &= (1/2)|\mathrm{Im}(z_1^* z_2)|,\end{aligned}$$

where $z_1 = x_1 + \mathrm{i}y_1$, $z_2 = x_2 + \mathrm{i}y_2$. Hence the area of the unitary triangle is

$$(1/2)|\mathrm{Im}(\mathrm{V}_{ud}^* V_{ub} V_{cd} V_{cb}^*)| = J/2.$$

18.5 All the complex numbers z_i are transformed to $z_i^1 = \mathrm{e}^{\mathrm{i}(\theta_d-\theta_b)} z_i$ and the triangle is rotated through an angle $(\theta_d - \theta_b)$.

Chapter 19

19.2 (a) $(U_{\beta j}^* U_{\alpha j} U_{\beta i} U_{\alpha i}^*) = (U_{\beta i}^* U_{\alpha i} U_{\beta j} U_{\alpha j}^*)^*$ hence
$\mathrm{Im}(U_{\beta j}^* U_{\alpha j} U_{\beta i} U_{\alpha i}^*) = -\mathrm{Im}(U_{\beta i}^* U_{\alpha i} U_{\beta j} U_{\alpha j}^*)$.

(b) Since $\mathbf{U}$ is unitary,
$\sum_i F_{\beta\alpha ij} = \mathrm{Im}(\partial_{\alpha\beta} U_{\beta j} U_{\alpha j}^*) = \mathrm{Im}(|U_{\alpha j}|^2) = 0.$
As two examples $F_{\beta\alpha 12} + F_{\beta\alpha 32} = 0$ and $F_{\beta\alpha 13} + F_{\beta\alpha 23} = 0$.
Hence $F_{\beta\alpha 12} + F_{\beta\alpha 23} = F_{\beta\alpha 31}$.

(c)

$$\sum_{i>j} F_{\mu e i j} \sin(\frac{\Delta m_{ij}^2 L}{2E}) = -J\left[\sin(\frac{\Delta m_{21}^2 L}{2E}) + \sin(\frac{\Delta m_{32}^2 L}{2E}) - \sin(\frac{(\Delta m_{21}^2 + \Delta m_{32}^2)L}{2E})\right]$$

and the result follows.

Chapter 21

21.1 Let $\left(\mathrm{i}\sigma^2 \nu^*\right)^\dagger \sigma^\mu \partial_\mu \left(\mathrm{i}\sigma^2 \nu^*\right) = E$
Inserting explicit spinor indices

$$E = \nu_i \sigma_{ij}^2 \sigma_{jk}^\mu \sigma_{kl}^2 \partial_\mu \nu_l^*, \quad \text{(repeated indices summed).}$$

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But from the algebra of Pauli matrices $\sigma^2_{ij}\sigma^\mu_{jk}\sigma^2_{kl} = \tilde{\sigma}^\mu_{li}$. Taking account of the anticommuting spinor fields $E = -\partial_\mu \nu^*_l \tilde{\sigma}^\mu_{li}\nu_i$. and discarding a total derivative that makes no contribution to the action

$$E = \nu^*_l \tilde{\sigma}^\mu_{li}\partial_\mu \nu_i = \nu^\dagger \tilde{\sigma}^\mu \partial_\mu \nu.$$

21.2 Inserting explicit spinor indices

$$\nu^{\mathrm{T}}_\alpha \sigma^2 \nu_\beta = \nu_{\alpha i}\sigma^2_{ij}\nu_{\beta j} = -\nu_{\alpha i}\sigma^2_{ji}\nu_{\beta j} = \nu_{\beta j}\sigma^2_{ji}\nu_{\alpha i} = \nu^{\mathrm{T}}_\beta \sigma^2 \nu_\alpha.$$

21.3 From (21.15)

$$U^M_{\beta j}\, U^{M*}_{\alpha j} = U^D_{\beta j} \mathrm{e}^{\mathrm{i}\Delta j}\, U^{D*}_{\alpha j} \mathrm{e}^{-\mathrm{i}\Delta j} = U^D_{\beta j}\, U^{D*}_{\alpha j}.$$

Appendix A

A.1 The equation holds for $\alpha\beta \ldots \nu = 1, 2, \ldots, n$. Interchanging, say, α and β is equivalent to interchanging column i with column j, and gives the same sign change.

A.3 $\mathbf{M} = (\mathbf{M} + \mathbf{M}^\dagger)/2 + \mathrm{i}(\mathbf{M} - \mathbf{M}^\dagger)/2\mathrm{i}$. $(\mathbf{M} + \mathbf{M}^\dagger)/2$ is Hermitian, as is $(\mathbf{M} - \mathbf{M}^\dagger)/2\mathrm{i}$. $\mathbf{A}$ and $\mathbf{B}$, and hence $\mathbf{M}$, can be diagonalised by the same transformation if and only if

$$\mathbf{AB} - \mathbf{BA} = \mathbf{0}, \text{ i.e. } (\mathbf{M} + \mathbf{M}^\dagger)(\mathbf{M} - \mathbf{M}^\dagger) - (\mathbf{M} - \mathbf{M}^\dagger)(\mathbf{M} + \mathbf{M}^\dagger) = \mathbf{0}$$

or

$$\mathbf{M}^\dagger\mathbf{M} - \mathbf{M}\mathbf{M}^\dagger = \mathbf{0}.$$

(This condition is satisfied if $\mathbf{M}$ is unitary.)

A.4 Since $(\mathbf{MM}^\dagger)^\dagger = \mathbf{MM}^\dagger$, we can find $\mathbf{U}_1$ such that $\mathbf{U}_1(\mathbf{MM}^\dagger)\mathbf{U}_1^\dagger = \mathbf{M}_D{}^2$. $\mathbf{M}_D{}^2$ has diagonal elements ≥ 0, since $\mathbf{M}_D{}^2 = \mathbf{U}_1\mathbf{M}(\mathbf{U}_1\mathbf{M})^\dagger$. Thus we can choose $\mathbf{M}_D$ with real diagonal elements ≥ 0. If none are zero, $\mathbf{M}_D$ can be inverted. We may then define

$$\mathbf{H} = \mathbf{U}_1{}^\dagger \mathbf{M}_D \mathbf{U}_1 = \mathbf{H}^\dagger, \quad \text{and} \quad \mathbf{V} = \mathbf{H}^{-1}\mathbf{M}.$$

Hence

$$\begin{aligned}
\mathbf{VV}^\dagger &= \mathbf{H}^{-1}\mathbf{MM}^\dagger\mathbf{H}^{-1} \text{ since } (\mathbf{H}^{-1})^\dagger = \mathbf{H}^{-1} \\
&= \mathbf{H}^{-1}\mathbf{U}_1{}^\dagger\mathbf{M}_D{}^2\mathbf{U}_1\mathbf{H}^{-1} \\
&= \mathbf{U}_1{}^\dagger\mathbf{M}_D{}^{-1}\mathbf{U}_1\mathbf{U}_1{}^\dagger\mathbf{M}_D{}^2\mathbf{U}_1\mathbf{U}_1{}^\dagger\mathbf{M}_D{}^{-1}\mathbf{U}_1 \\
&= \mathbf{I}, \text{ since } \mathrm{U}_1\mathbf{U}_1{}^\dagger = \mathbf{I}.
\end{aligned}$$

Thus $\mathbf{V}$ is unitary, as is $\mathbf{U}_1\mathbf{V} = \mathbf{U}_2$.
Finally, $\mathbf{M} = \mathbf{HV} = \mathbf{U}_1{}^\dagger\mathbf{M}_D\mathbf{U}_1\mathbf{V} = \mathbf{U}_1^\dagger\mathbf{M}_D\mathbf{U}_2$.

Appendix B

B.1 A unitary transformation, $\mathbf{H} \to \mathbf{H}' = \mathbf{VHV}^\dagger = \mathbf{H}_\mathrm{D}$, say, also diagonalises each term of $\mathbf{U}$ and hence

$$\mathbf{U} \to \mathbf{U}' = \mathbf{VUV}^\dagger = \mathbf{U}_\mathrm{D} = \exp(\mathrm{i}\mathbf{H}_\mathrm{D}).$$

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$$\det \mathbf{U} = \det \mathbf{U}_{\mathrm{D}} = \prod_n \exp \mathrm{i}(\mathbf{H}_{\mathrm{D}})_{nn}$$
$$= \exp\left[\mathrm{i}\sum_n (\mathbf{H}_{\mathrm{D}})_{nn}\right] = \exp[\mathrm{i}\mathrm{Tr}\,\mathbf{H}_{\mathrm{D}}].$$

But $\mathrm{Tr}\mathbf{H}_D = \mathrm{Tr}\mathbf{H}$. Hence if $\mathrm{Tr}\,\mathbf{H} = 0$, $\det \mathbf{U} = 1$.

B.2 The $SU(2)$ matrices corresponding to $R_{01}(\theta)$ and $R_{02}(\theta)$ are respectively

$$\begin{pmatrix} \cos(\theta/2) & \mathrm{i}\sin(\theta/2) \\ \mathrm{i}\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

and the correspondence can be checked directly.

B.3 From equation (B.5), using (B.12) and Problem B.2, $R(\psi, \theta, \phi)$ corresponds to the product

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\psi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\psi/2} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix}.$$

B.4 Under a Lorentz transformation, $\mathbf{l} \to \mathbf{l}' = \mathbf{Ml}$, $\mathbf{r} \to \mathbf{r}' = \mathbf{Nr}$.
Hence

$$\begin{aligned} \mathbf{l}^\dagger \tilde{\sigma}^\mu \sigma^\nu \mathbf{r} &\to \mathbf{l}^\dagger \mathbf{M}^\dagger \tilde{\sigma}^\mu \sigma^\nu \mathbf{N}\mathbf{r} \\ &= \mathbf{l}^\dagger \mathbf{M}^\dagger \tilde{\sigma}^\mu \mathbf{M}\mathbf{N}^\dagger \sigma^\nu \mathbf{N}\mathbf{r} \quad \text{since } \mathrm{MN}^\dagger = \mathbf{I} \\ &= \mathbf{l}^\dagger L^\mu{}_\lambda \tilde{\sigma}^\lambda \mathbf{L}^\nu{}_\rho \sigma^\rho \mathbf{r} \quad \text{from (B.17) and (B.18)} \\ &= L^\mu{}_\lambda L^\nu{}_\rho (\mathbf{l}^\dagger \tilde{\sigma}^\lambda \sigma^\rho \mathbf{r}). \end{aligned}$$

It is easy to verify that

$$\tilde{\sigma}^\mu \sigma^\nu + \tilde{\sigma}^\nu \sigma^\mu = \begin{cases} 0 & \text{if } \mu \neq \nu, \\ 2 & \text{if } \mu = \nu = 0, \\ -2 & \text{if } \mu\nu = i;\ i = 1, 2, 3. \end{cases}$$

B.5 Equation (B.10) gives

$$\mathbf{X}(x) = x^i \sigma^i$$
$$\mathbf{X}'(x') = x'^i \sigma^i = R^i_j x^j \sigma^i.$$

Also $\mathbf{X}' = \mathbf{UXU}^\dagger = \mathbf{U}x^j\sigma^j\mathbf{U}^\dagger$. The x^j are arbitrary. Hence $\mathbf{U}\sigma^j\mathbf{U}^\dagger = \mathrm{R}^i_j\sigma^i$. Multiplying on the left by σ^k and taking the trace,

$$\mathrm{Tr}(\sigma^k \mathbf{U}\sigma^j \mathbf{U}^\dagger) = R^i{}_j \mathrm{Tr}(\sigma^k \sigma^i).$$

Now

$$\mathrm{Tr}(\sigma^k \sigma^i) = \begin{cases} 2 & \text{if } k = i, \\ 0 & \text{if } k \neq i. \end{cases}$$

Hence the result.

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B.6 From (B.17), $\mathbf{M}^\dagger \tilde{\sigma}^\mu \mathbf{M} = L^\mu{}_\lambda \, \tilde{\sigma}^\lambda$. Multiplying on the left by $\tilde{\sigma}^\nu$ and taking the trace, the result follows, since

$$\mathrm{Tr}(\tilde{\sigma}^\nu \tilde{\sigma}^\lambda) = \begin{cases} 2 & \text{if } \lambda = \nu, \\ 0 & \text{if } \lambda \neq \nu. \end{cases}$$

Appendix C

C.2 The ground state is given by $a|0\rangle = 0$, or $(X + \mathrm{i}P)|0\rangle = 0$. In the Schrödinger representation. $P = -\mathrm{i}\,\mathrm{d}/\mathrm{d}X$, so that $(X + \mathrm{d}/\mathrm{d}X)\psi_0 = 0$, giving $\psi_0 = A\mathrm{e}^{-X^2}/2$, where the constant A is determined by normalisation.

C.3

$$\begin{aligned} N_i b_i{}^\dagger|0\rangle &= b_i{}^\dagger b_i b_i{}^\dagger|0\rangle \\ &= b_i{}^\dagger(1 - b_i{}^\dagger b_i)|0\rangle = b_i{}^\dagger|0\rangle. \end{aligned}$$

Appendix D

D.1

$$\begin{aligned} Q^2 &= (\mathbf{p} - \mathbf{p}')^2 - (E - E')^2 \\ &= (p^2 - E^2) + (p'^2 - E'^2) - 2\mathbf{p}\cdot\mathbf{p}' + 2EE'. \end{aligned}$$

But $E^2 = p^2 + m^2$, $E'^2 = p'^2 + m^2$, so that, neglecting electron masses,

$$Q^2 = -2pp' \cos\theta + 2EE' = 2EE'(1 - \cos\theta) = 4EE' \sin^2(\theta/2).$$

The energy and momentum of the recoil proton are given by $E_p = M + E - E'$, $\mathbf{P} = \mathbf{p} - \mathbf{p}'$; also $E_\mathrm{p}{}^2 = M^2 + P^2$. Hence

$$\begin{aligned} Q^2 &= p^2 - (E - E')^2 \\ &= (M + E - E')^2 - M^2 - (E - E')^2 \\ &= 2M(E - E') \end{aligned}$$

so that (D.3) follows.

D.3

$$\begin{aligned} Q^2 &= 2EE'(1 - \cos\theta) \\ \nu &= E - E' \\ \mathrm{d}Q^2\mathrm{d}\nu &= \frac{\partial(Q^2, \nu)}{\partial(\cos\theta, E')}\mathrm{d}(\cos\theta)\mathrm{d}E' \end{aligned}$$

where the Jacobian of the transformation is

$$\begin{vmatrix} -2EE' & 2E(1 - \cos\theta) \\ 0 & -1 \end{vmatrix} = 2EE'.$$

Hence the result.

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