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LETTERS TO THE EDITOR

STRONG UNIMODALITY

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Abstract

It is proved that a distribution F is strongly unimodal iff any two quantiles of the convolution of F with any other distribution are further apart than the corresponding quantiles of F itself. Some related characterizations of strong unimodality are also given.

SPREAD; CONVOLUTION; JENSEN'S INEQUALITY

1.1 Strong unimodality and spread

A distribution F on $\mathbb{R}$ is called strongly unimodal iff the convolution of F with any unimodal distribution is again unimodal. We define F^{-1} by

$$F^{-1}(t)=\inf\{x\in\mathbb{R}\mid F(x)\geqq t\},\qquad 0\leqq t\leqq 1, \tag{1.1}$$

and we call a distribution F more spread out than a distribution G, notation $F\geqq_1 G$, iff

$$F^{-1}(v)-F^{-1}(u)\geqq G^{-1}(v)-G^{-1}(u),\qquad 0\leqq u\leqq v\leqq 1, \tag{1.2}$$

holds. Furthermore we denote the convolution of F and G by $F*G$, the set of distributions on $\mathbb{R}$ by $\mathscr{G}$ and the set of distributions with mass $\frac{1}{2}$ at 0 and mass $\frac{1}{2}$ at some $a\in\mathbb{R}$ by $\mathscr{G}_0$.

Theorem 1.1 *For $F\in\mathscr{G}$ the following statements are equivalent*:

(1.3) *F is strongly unimodal,*

(1.4) *$F*G_1\geqq_1 F*G_2$ holds for all $G_1, G_2\in\mathscr{G}$ with $G_1\geqq_1 G_2$,*

(1.5) *$F*G\geqq_1 F$ holds for all $G\in\mathscr{G}$,*

(1.6) *$F*G\geqq_1 F$ holds for all $G\in\mathscr{G}_0$.*

The equivalence of (1.3) and (1.4) has been proved by Lewis and Thompson (1981) and Lynch et al. (1983). Under the assumption, that F has a Lebesgue density which is positive on $\mathbb{R}$, the equivalence of (1.3), (1.5) and (1.6), with $\mathscr{G}_0$ replaced by the set of all two-point distributions, has been proved by Droste and Wefelmeyer (1985). Since the implications (1.4) ⇒ (1.5) ⇒ (1.6) are trivial, it suffices to show (1.6) ⇒ (1.3) in order to

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complete a proof of Theorem 1.1. Our proof of this implication is based on Jensen's inequality.

2. Intermediate results and proofs

Let I_F be the interior of the smallest interval containing the support of F. From the classical paper of Ibragimov (1956) we obtain the following characterization.

Lemma 2.1. *F is strongly unimodal iff F is degenerate or F is absolutely continuous with a density f, which is continuous and positive on I_F, vanishing outside I_F and such that $f(F^{-1}(\cdot))$ is concave on* $[0, 1]$.

Proof. Without loss of generality we assume that F is non-degenerate. Ibragimov (1956) proves that F is strongly unimodal iff F is absolutely continuous and differentiable on I_F with derivative f such that $\log f(\cdot)$ is concave on I_F. By, for example, 18.43 of Hewitt and Stromberg (1965) this concavity shows that $\log f(\cdot)$ is absolutely continuous on I_F with a non-increasing Radon–Nikodym derivative $f'(\cdot)/f(\cdot)$. Consequently $f'(F^{-1}(\cdot))/f(F^{-1}(\cdot))$ is non-increasing on $(0, 1)$ and $f(F^{-1}(\cdot))$ is concave on $(0, 1)$. In the same way concavity of $f(F^{-1}(\cdot))$ on $(0, 1)$ implies concavity of $\log f(\cdot)$ on I_F.

The gist of our approach is Jensen's inequality, which provides a simple proof of the following result.

Lemma 2.2. *Let F be absolutely continuous with a density f, which is continuous and positive on I_F and vanishing outside I_F. The function $f(F^{-1}(\cdot))$ is concave on* $[0, 1]$, *iff* (1.5) *holds, iff* (1.6) *holds.*

Proof. Let h be the density of $H = F * G$. Note that $F * G \geqq_1 F$ iff $h(H^{-1}(s)) \leqq f(F^{-1}(s))$, for all $s, 0 < s < 1$, iff $h(x) \leqq f(F^{-1}(H(x)))$, for all $x \in \mathbb{R}$, iff

$$\int f(F^{-1}(F(x-y)))\, dG(y) \leqq f\left(F^{-1}\left(\int F(x-y)\, dG(y)\right)\right), \quad \text{for all} \quad x \in \mathbb{R}. \tag{2.1}$$

Consequently $F * G \geqq_1 F$ holds for all $G \in \mathcal{G}$ iff

$$Ef(F^{-1}(X)) \leqq f(F^{-1}(EX)) \tag{2.2}$$

is valid for every random variable taking values in $[0, 1]$. In view of Jensen's inequality this is equivalent to the concavity of $f(F^{-1}(\cdot))$ on $[0, 1]$. The equivalence of (1.6) and the concavity of $f(F^{-1}(\cdot))$ can be shown in the same way.

Note that Lemma 2.1 and 2.2 give a direct proof of the implications (1.3) $\Rightarrow$ (1.5) $\Rightarrow$ (1.6) and show the equivalence of (1.3), (1.5) and (1.6) under the assumption of existence of a continuous positive density on I_F.

Proof of Theorem 1.1. In view of the remarks at the end of Section 1, it suffices to show the implication (1.6) $\Rightarrow$ (1.3). Let Φ_n denote the normal distribution with mean 0 and variance n^{-1}. Since Φ_n is strongly unimodal the implication (1.3) $\Rightarrow$ (1.4) shows that (1.6) implies

$$F * \Phi_n * G \geqq_1 F * \Phi_n, \quad \text{for all} \quad G \in \mathcal{G}_0. \tag{2.3}$$

Since $F * \Phi_n$ has a continuous and positive density on $\mathbb{R}$, Lemmas 2.2 and 2.1 imply that $F * \Phi_n$ is strongly unimodal for all n. Taking the limit for $n \to \infty$ we see that F itself is strongly unimodal (cf. Lemma 2 of Ibragimov (1956)).

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