Canad. J. Math. Vol. 69 (2), 2017 pp. 284–320
http://dx.doi.org/10.4153/CJM-2016-019-6

Convolution Powers of Salem Measures With Applications

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Abstract. We study the regularity of convolution powers for measures supported on Salem sets, and prove related results on Fourier restriction and Fourier multipliers. In particular we show that for α of the form d/n, $n = 2, 3, \dots$ there exist α-Salem measures for which the L^2 Fourier restriction theorem holds in the range $p \le \frac{2d}{2d-\alpha}$. The results rely on ideas of Körner. We extend some of his constructions to obtain upper regular α-Salem measures, with sharp regularity results for n-fold convolutions for all $n \in \mathbb{N}$.

1 Introduction

Given a finite positive Borel measure μ on $\mathbb{R}^d$ satisfying the condition

$$|\widehat{\mu}(\xi)| = O(|\xi|^{-b})$$

for some $b > 0$, the Fourier transform maps $L^p(\mathbb{R}^d)$ to $L^2(d\mu)$ for some $p > 1$. This is the Fourier restriction phenomenon discovered by Stein in the 1960s. Much research on Fourier analysis has been done regarding the case of μ being a surface measure on the sphere where sharp results are due to Tomas and Stein [34, 35]. A general version of Tomas' theorem is due to Mockenhaupt [25] and also Mitsis [24]. These authors showed that under the above assumption and the additional regularity condition

$$\mu(B) = O(\operatorname{diam}(B)^a),$$

for all balls B, the Fourier transform maps $L^p(\mathbb{R}^d)$ to $L^2(d\mu)$ for $1 \le p < p_{a,b} = \frac{2(d-a+b)}{2(d-a)+b}$. It was shown in [1] that the result is also valid for $p = p_{a,b}$. The Fourier decay assumption implies that the regularity condition holds for $a = b$. Moreover, if the support of μ is contained in a set of Hausdorff dimension α, then $b \le \alpha/2$ and $a \le \alpha$. See [37, Chapter 8], [24] for these facts. Of particular interest are measures supported on sets E of Hausdorff dimension α for which the Fourier decay condition holds for all $b < \alpha/2$; such sets are commonly called *Salem sets*. The existence of Salem sets is due to Salem [28]; for other constructions we refer to [2, 3, 11, 15, 16, 22].

Here we are also interested in the special Salem sets E that carry probability measures for which the endpoint bound $|\widehat{\mu}(\xi)| = O(|\xi|^{-\dim(E)/2})$ holds for large ξ, and make the following definition.

Received by the editors September 2, 2015; revised April 27, 2016.
Published electronically September 14, 2016.
Supported in part by the National Science Foundation.
AMS subject classification: **42A85**, 42B99, 42B15, 42A61.
Keywords: convolution powers, Fourier restriction, Salem sets, Salem measures, random sparse sets, Fourier multipliers of Bochner–Riesz type.

Definition (i) A Borel probability measure μ is called an α-*Salem measure* if it is compactly supported, the support of μ is contained in a set of Hausdorff dimension α, and if

$$\sup_{\xi\in\mathbb{R}^d}(1+|\xi|)^{\alpha/2}|\widehat{\mu}(\xi)|<\infty. \tag{1.1}$$

(ii) An α-Salem measure is called upper regular (or α-upper regular) if

$$\sup_B \frac{\mu(B)}{\operatorname{diam}(B)^\alpha}<\infty, \tag{1.2}$$

where the sup is taken over all balls.

Körner constructed examples of upper regular α-Salem measures [17]; see also [8] for various refinements.

By the result in [1], if μ is an upper regular α-Salem measure, then the Fourier transform maps $L^p(\mathbb{R}^d)$ to $L^2(\mu)$ for $1\le p\le \frac{4d-2\alpha}{4d-3\alpha}$. By analogy with results and conjectures for surface measure on the sphere, Mockenhaupt conjectured that the Fourier transform should map $L^p(\mathbb{R}^d)$ to $L^1(\mu)$ for the larger range $1\le p<\frac{2d}{2d-\alpha}$. By [24, Proposition 3.1], such an $L^p\to L^2$ result cannot hold for $p>\frac{2d}{2d-\alpha}$. Furthermore, Mockenhaupt remarked that for suitable examples there is a possibility that even the stronger Stein–Tomas $L^p\to L^2(\mu)$ bound could hold in this range. For a dense set of α's (and $d=1$), Hambrook and Łaba [12] recently gave examples of Salem sets of dimension α showing that the p range for the $L^p\to L^2(\mu)$ bound in [1] cannot be improved in general. Their examples carry randomness and arithmetic structures at different scales. Chen [8] extended this idea to provide, among other things, for all $\alpha\in[0,1]$, examples of upper regular α-Salem measures on the real line for which $\mathcal{F}$ does not map L^p to $L^2(\mu)$ for any $p>\frac{4d-2\alpha}{4d-3\alpha}$. Recently, Hambrook and Łaba [13] obtained related sharpness results in higher dimensions which involve examples of measures whose supports have Hausdorff dimension greater than $d-1$.

All these examples still do not exclude the Mockenhaupt scenario of a larger p-range for the L^2 restriction estimate for other types of Salem measures. The question was explicitly posed in a recent survey paper by Łaba [21]. We show an optimal result when α is of the form d/n with some integer n.

Theorem A *Given $\alpha=d/n$ where $n\in\mathbb{N}$, $n\ge 2$, there exists an upper regular α-Salem measure such that $\mathcal{F}\colon L^p(\mathbb{R}^d)\to L^2(\mu)$ is bounded in the optimal range $1\le p\le\frac{2d}{2d-\alpha}$.*

Remark Shmerkin and Suomala [30] have independently obtained a similar result for $d=1$, $\alpha>1/2$. Their method also covers the cases $d=2,3$, $d/2<\alpha\le 2$. Their approach is quite different from the methods used here.

It would be of great interest to find Ahlfors–David regular α-Salem measures, *i.e.*, besides (1.1) and (1.2), we would also have a lower bound $\mu(B)\gtrsim \operatorname{rad}(B)^\alpha$ for all balls B with radius ≤ 1 that are centered in the support of μ. This question has been raised by Mitsis [24]; see also the list of problems in Mattila [23]. We remark that the examples by Shmerkin and Suomala [30] for the non-endpoint $L^2\to L^4$ restriction estimate (with $\alpha>1/2$) are Ahlfors–David regular. However the measures satisfying Theorem A are necessarily not Ahlfors–David α-regular; see §4.

A variant of Theorem A can be used to derive new results on a class of Fourier multipliers of Bochner–Riesz type as considered by Mockenhaupt [25]. In what follows we let M_p^q to be the space of all $m \in \mathcal{S}'(\mathbb{R}^d)$ for which $f \mapsto \mathcal{F}^{-1}[m\widehat{f}]$ extends to a bounded operator from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$. The norm on M_p^q is the operator norm, *i.e.*,

$$\|m\|_{M_p^q} = \sup_{\substack{f \in \mathcal{S}(\mathbb{R}^d) \\ \|f\|_p \le 1}} \|\mathcal{F}^{-1}[m\widehat{f}]\|_q.$$

Mockenhaupt [25] introduced a class of Fourier multipliers associated with general measures that reflect the properties of Bochner–Riesz multipliers in the case when μ is the surface measure on a smooth hypersurface.

Given a compactly supported α-upper regular Borel measure, $\lambda > \alpha - d$, and $\chi \in C_c^\infty(\mathbb{R}^d)$, the function

$$m_\lambda(\xi) = \int_{\mathbb{R}^d} \chi(\xi - \eta)|\xi - \eta|^{\lambda - \alpha}\, d\mu(\eta) \tag{1.3}$$

is well defined as an L^1 function. Mockenhaupt showed that in the range $1 \le p \le \frac{4d-2\alpha}{4d-3\alpha}$ (the Fourier restriction range in [1, 25]), the multiplier m_λ belongs to M_p^q if $\lambda > d(\frac{1}{q} - \frac{1}{2}) - \frac{d-\alpha}{2}$ (the case $p = q$ was explicitly stated in [25]). Theorem A enables us to prove a better range for certain α-regular Salem measures, and an endpoint result in some cases. This endpoint result relies on special properties of our Salem measures and fails, for example, for the surface measure of the unit circle in $\mathbb{R}^2$, for any $q < 2$. In §4 we prove the following theorem.

Theorem B *Let $\alpha = d/n$ where $n \in \mathbb{N}$, $n \ge 2$, and $\lambda > 0$. There exists an upper regular α-Salem measure on $\mathbb{R}^d$ so that for $1 \le p < \frac{2d}{2d-\alpha}$ and $p \le q \le 2$ we have $m_\lambda \in M_p^q$ if and only if $\lambda \ge d(\frac{1}{q} - \frac{1}{2}) - \frac{d-\alpha}{2}$.*

We now discuss estimates for self-convolutions of certain Salem measures and how they are used in the proof of Theorem A. Let μ^{*n} be the convolution of n copies of μ; more precisely, we set $\mu^{*0} = \delta_0$ (the Dirac measure at 0), $\mu^{*1} = \mu$, and $\mu^{*n} = \mu * \mu^{*(n-1)}$ for $n \ge 2$. The proof of the Fourier restriction result of Theorem A for $\alpha = d/n$ is based on a regularity result for self-convolutions of suitable Salem measures as stated in Theorem C below, and the inequality

$$\int |\widehat{g\mu}|^{2n}\, d\xi \lesssim \|\mu^{*n}\|_\infty \Big(\int |g(x)|^2\, d\mu \Big)^n, \tag{1.4}$$

is a special case of an inequality in [6], closely related to a result by Rudin [26].

For $n = 2$, Körner [18] proved the existence of a compactly supported probability measure on $\mathbb{R}$, supported on a set of Hausdorff dimension 1/2 for which $\mu * \mu$ is a continuous function. Moreover, given $\frac{1}{2} \le \alpha < 1$, there exists a Borel probability measure μ on $\mathbb{R}$ supported on a compact set of Hausdorff dimension α such that $\mu * \mu \in C_c^{\alpha-1/2}(\mathbb{R})$; here $C^{\alpha-1/2}$ is the standard Hölder class and $C_c^{\alpha-1/2}$ is the subspace consisting of compactly supported $C^{\alpha-1/2}$ functions; see the discussion of related classes in the next paragraph. Körner thus substantially improved and extended previous results by Wiener–Wintner [36] and Saeki [27] on convolution squares for singular measures. Note that by taking adjoints, inequality (1.4) for $n = 2$ shows that $\mathcal{F}: L^{4/3} \to L^2(\mu)$; for $\alpha < 2/3$ this yields a range larger than $[1, \frac{4-2\alpha}{4-3\alpha}]$, the largest range that could be proved from [1]. It is not stated in Körner's paper that the measures constructed there have the appropriate Fourier decay properties but as we shall see this is not hard to accomplish.

For integers $k \geq 0$, let $C^k(\mathbb{R}^d)$ be the space of functions whose derivatives up to order k are continuous and bounded; the norm is given by $\|f\|_{C^k} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_\infty$. Let $\psi: [0, \infty) \to [0, \infty)$ be a nondecreasing bounded function satisfying

$$\lim_{t \to 0} t^{-\varepsilon} \psi(t) = \infty, \quad \varepsilon > 0$$

and, for some $C_\psi > 0$,

$$\psi(t) \leq C_\psi \psi(t/2), \quad t > 0. \tag{1.5}$$

For a function f on $\mathbb{R}^d$, define

$$\omega_{\rho,\psi}(f) = \sup_{\substack{x,y \in \mathbb{R}^d \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\rho \psi(|x-y|)} \tag{1.6}$$

and $C^{\rho,\psi}(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) : \omega_{\rho,\psi}(f) < \infty\}$. If $\rho \geq 1$, define

$$C^{\rho,\psi}(\mathbb{R}^d) = \left\{ f \in C^{\lfloor \rho \rfloor}(\mathbb{R}^d) : \partial^\beta f \in C^{\rho - \lfloor \rho \rfloor, \psi}(\mathbb{R}^d), |\beta| = \lfloor \rho \rfloor \right\}.$$

For $0 < \rho < 1$, the choice of $\psi(t) = 1$ yields the usual Hölder spaces $C^\rho(\mathbb{R}^d)$. Only the definition of ψ for small t is relevant. Other suitable choices for ψ are

(i) $\psi(t) = \exp(-\sqrt{\log t^{-1}})$ for $t \leq e^{-1}$,
(ii) $\psi(t) = 1/(\log t^{-1})$ for $t \leq e^{-1}$, or
(iii) $\psi(t) = 1/(\log \log t^{-1})$ for $t \leq e^{-e}$.

We extend Körner's constructions to prove the following result for higher convolution powers of upper regular α-Salem measures.

Theorem C *Given $d \geq 1$ and $0 < \alpha < d$, there exists a Borel probability measure μ on $\mathbb{R}^d$ satisfying the following properties.*

(i) *μ is supported on a compact set of Hausdorff and lower Minkowski dimension α.*
(ii) *For all $\xi \in \mathbb{R}^d, |\xi| \geq 1, |\widehat{\mu}(\xi)| \lesssim \psi(|\xi|^{-1})|\xi|^{-\alpha/2}$.*
(iii) *For all $x \in \mathbb{R}^d, 0 < r < 1, 1 \leq n < d/\alpha$, $\mu^{*n}(B(x,r)) \lesssim \psi(r) r^{n\alpha}$.*
(iv) *For $n \geq d/\alpha$, $\mu^{*n} \in C_c^{\frac{n\alpha - d}{2}, \psi}(\mathbb{R}^d)$.*

Note that under the dimensional restriction, the Fourier decay exponent, the upper regularity exponents $n\alpha$, and the Hölder exponent $\frac{n\alpha - d}{2}$ for μ^{*n} are all optimal (*cf.* §2.6 for the latter).

Notation We write $\square_1 \lesssim \square_2$ to indicate that $\square_1 \leq C \square_2$ for some constant $0 < C < \infty$ that is independent of the testing inputs, which will usually be clear from the context. For a measurable subset E of $\mathbb{R}^d$ or $\mathbb{T}^d$ we let $|E|$ denote the Lebesgue measure of E.

Structure of the paper The proof of Theorem C is given in the next two sections. The restriction and multiplier theorems are considered in §4.

2 Körner's Baire Category Approach

This section contains the extensions of Körner's arguments adapted and extended to yield Theorem C. The results will be stated in the periodic setting and followed by a relatively straightforward transference argument.

To fix notations, we write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $\mathbb{T}^d = \mathbb{T} \times \cdots \times \mathbb{T}$. We occasionally denote by λ the uniform probability measure on $\mathbb{T}^d$. While λ is usually identified with the function 1, we shall also identify a continuous function g with the measure $g\lambda$. A subset $J \subset \mathbb{T}$ is called an *interval* if it is connected. A rectangle is of the form $R = J_1 \times \cdots \times J_d$ where J_i are intervals; R is called a *cube* if these intervals have the same length. If μ is a finite Borel measure on $\mathbb{T}^d$, the Fourier transform of μ is defined as

$$\widehat{\mu}(r) = \int_{\mathbb{T}^d} e^{-2\pi i r\cdot t}\, d\mu(t),$$

where $r \in \mathbb{Z}^d$. Here as usual we have identified $\mathbb{T}^d$ with $[0,1)^d$. Note that $\widehat{\mu}(0) = \mu(\mathbb{T}^d)$ and $\widehat{\lambda}(r) = \delta_0(r)$. Let μ and ν be two finite Borel measures on $\mathbb{T}^d$, $\mu * \nu$ is the finite Borel measure on $\mathbb{T}^d$ with Fourier transform $\widehat{\mu}(r)\widehat{\nu}(r)$. Finally, we equip $\mathbb{T}^d$ with the usual group structure and the intrinsic metric which will be denoted by

$$|x-y| := \Big(\sum_{i=1}^{d} |x_i - y_i|^2\Big)^{1/2},$$

where $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$, and $|x_i - y_i|$ denotes the intrinsic metric on $\mathbb{T}$. We will also fix an orientation of $\mathbb{T}$ so that derivatives are uniquely defined. With this distance the expression $\omega_{\rho,\psi}(f)$ in (1.6) and the spaces $C^{\rho,\psi}$ can be defined in the same way on $\mathbb{T}^d$.

For each integer $n \geq d/\alpha$ we fix a finite smooth partition of unity on $\mathbb{T}^d$, indexed by $\imath \in \mathfrak{I}_n$

$$\mathfrak{O}^{(n)} = \{\chi_\imath^{(n)}\}_{\imath\in\mathfrak{I}_n} \tag{2.1}$$

so that each $\chi_\imath^{(n)}$ is supported on a cube of side length smaller than $(2n)^{-1}$.

2.1 A Metric Space

Let $\mathfrak{K}$ be the collection of closed subsets of $\mathbb{T}^d$ which form a complete metric space with respect to the Hausdorff distance

$$d_{\mathfrak{K}}(K_1, K_2) = \sup_{x\in K_1} \operatorname{dist}(x, K_2) + \sup_{y\in K_2} \operatorname{dist}(y, K_1) = \sup_{x\in K_1} \inf_{y\in K_2} |x-y| + \sup_{y\in K_2} \inf_{x\in K_1} |x-y|;$$

see [31]. We now consider metric spaces of pairs (K,μ) where K is a compact subset of $\mathbb{T}^d$ and μ is a nonnegative Borel measure supported on E. These measures are assumed to satisfy

$$\lim_{|r|\to\infty} \frac{|r|^{\alpha/2}|\widehat{\mu}(r)|}{\psi(\frac{1}{|r|})} = 0. \tag{2.2}$$

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Moreover, for $n \geq d/\alpha$ and for each n-tuple $i = (\imath_1, \ldots, \imath_n) \in \mathfrak{I}_n^n$, the n-fold convolution $(\chi_{\imath_1}^{(n)}\mu) * \cdots * (\chi_{\imath_n}^{(n)}\mu)$ is absolutely continuous and we have

$$(\chi_{\imath_1}^{(n)}\mu) * \cdots * (\chi_{\imath_n}^{(n)}\mu) = g_{\mu,i}^{(n)}\lambda, \text{ with } g_{\mu,i}^{(n)} \in C^{\frac{n\alpha-d}{2},\psi}. \tag{2.3}$$

We let $\mathfrak{W}$ be the set of all (K,μ) where $K \subset \mathbb{T}^d$ is closed, and μ is a nonnegative Borel measure supported in K satisfying (2.2) and (2.3). A metric on $\mathfrak{W}$ is given by

$$\begin{aligned} d_{\mathfrak{W}}&((K_1,\mu_1),(K_2,\mu_2)) \\ &= d_{\mathfrak{K}}(K_1,K_2) + |\widehat{\mu}_1(0) - \widehat{\mu}_2(0)| + \sup_{r \in \mathbb{Z}^d \setminus \{0\}} \frac{|r|^{\alpha/2}|\widehat{\mu}_1(r) - \widehat{\mu}_2(r)|}{\psi(|r|^{-1})} \\ &\quad + \sum_{n \geq d/\alpha} 2^{-n} \min\Big\{1, \sum_{i \in \mathfrak{I}_n^n} \|g_{\mu_1,i}^{(n)} - g_{\mu_2,i}^{(n)}\|_{C^{\frac{n\alpha-d}{2},\psi}}\Big\}. \end{aligned}$$

Lemma 2.1 (i) *$(\mathfrak{W}, d_{\mathfrak{W}})$ is a complete metric space.*

(ii) *For every nonnegative C^∞ function f and every compact set K containing $\operatorname{supp}(f)$, the pair (K,f) belongs to $\mathfrak{W}$.*

(iii) *Let $\mathfrak{V}$ be the subspace of $\mathfrak{W}$ consisting of (K,μ) satisfying*

$$\mu^{*n}(Q) \leq \psi(|Q|)|Q|^{n\alpha/d} \tag{2.4}$$

for all cubes Q and $1 \leq n < d/\alpha$. Then $\mathfrak{V}$ (with the metric inherited from $\mathfrak{W}$) is a closed subspace of $\mathfrak{W}$.

(iv) *Let $\mathfrak{V}_0$ be the subset of $\mathfrak{V}$ consisting of pairs $(K,g) \in \mathfrak{V}$ with $g \in C^\infty(\mathbb{T}^d)$ and let $\overline{\mathfrak{V}_0}$ be the closure of $\mathfrak{V}_0$ in $\mathfrak{V}$ with respect to the metric $d_{\mathfrak{W}}$. Then $\overline{\mathfrak{V}_0}$ is a complete metric space and for every nonnegative $g \in C^\infty(\mathbb{T}^d)$ there is a $C > 0$ so that for all compact $K \supset \operatorname{supp}(g)$ the pair $(K, g/C)$ belongs to $\overline{\mathfrak{V}_0}$.*

Proof The theorem of Banach–Alaoglu is used to identify a limit measure of a Cauchy sequence. The proof is a straightforward modification of the arguments in [18, 20]; see also [5, 19, 31]. ∎

In order to prove a version of Theorem C, we wish to show that there are pairs $(K,\mu) \in \overline{\mathfrak{V}_0}$ such that μ is supported in a set of lower Minkowski dimension and Hausdorff dimension α. This will be deduced from a Baire category argument, as follows.

Theorem 2.2 *Suppose $\alpha < \gamma < d$ and $\varepsilon > 0$. Let $\mathfrak{V}^{\gamma,\varepsilon}$ be the subset of $\overline{\mathfrak{V}_0}$ consisting of pairs (K,μ) for which there are cubes $Q_1, \ldots, Q_M$ with*

$$K \subset \bigcup_{j=1}^{M} Q_j \quad \text{and} \quad |Q_1| = \cdots = |Q_M| < \varepsilon M^{-d/\gamma}. \tag{2.5}$$

Then $\mathfrak{V}^{\gamma,\varepsilon}$ is open and dense in $\overline{\mathfrak{V}_0}$.

The Baire category theorem gives the following.

Corollary 2.3 *$\bigcap_{N=1}^{\infty} \mathfrak{V}^{\alpha+1/N,1/N}$ is a dense G_δ set in $\overline{\mathfrak{V}_0}$.*

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Let $\underline{\dim}_{\mathrm{M}}(K)$, $\dim_{\mathrm{H}}(K)$ denote the lower Minkowski dimension and Hausdorff dimension, respectively. Then $\dim_{\mathrm{H}}(K) \le \underline{\dim}_{\mathrm{M}}(K)$. If $(K,\mu) \in \bigcap_{N=1}^{\infty} \mathfrak{V}^{\alpha+1/N,1/N}$, then $\underline{\dim}_{\mathrm{M}}(K) \le \alpha$ and hence also $\dim_{\mathrm{H}}(K) \le \alpha$. On the other hand, (2.2) implies $\dim_{\mathrm{H}}(K) \ge \alpha$ (see [37, Corollary 8.7]). Thus we obtain the following corollary.

Corollary 2.4 *The set of $(K,\mu) \in \overline{\mathfrak{V}}_0$ satisfying $\underline{\dim}_{\mathrm{M}}(K) = \dim_{\mathrm{H}}(K) = \alpha$ is of second category in $\overline{\mathfrak{V}_0}$.*

Concerning the proof of Theorem 2.2, it is easy to see that the sets $\mathfrak{V}_{\gamma,\varepsilon}$ are open subsets of $\overline{\mathfrak{V}_0}$. The remainder of this section is devoted to proving that they are dense.

2.2 Averages of Point Masses

For large N let Γ_N be the finite subgroup of $\mathbb{T}$ of order N, consisting of $\{k/N : k = 0,1,\dots,N-1\}$. Let Γ_N^d be the d-fold product, a subgroup of $\mathbb{T}^d$.

The following result yields measures on $\mathbb{T}^d$ which are sums of point masses supported on points in Γ_N^d and satisfy properties analogous to (2.2), (2.3), and (2.5).

Proposition 2.5 *Given $0 < \beta < d$ and an integer $\mathfrak{n} \ge 2$, there exist $N_0(\beta,\mathfrak{n}) \ge 1$, $C_1 = C_1(d)$, $C_2 = C_2(\beta,d)$, and $C_3 = C_3(\beta,d,\mathfrak{n})$ such that for all $N \ge N_0(\beta,\mathfrak{n})$ with $\gcd(\mathfrak{n}!,N) = 1$, $P := \lfloor N^\beta \rfloor$ there is a choice of $x_1,\dots,x_P$ with $x_j \in \Gamma_N^d$, such that the following properties hold for the measure $\mu = \frac{1}{P}\sum_{j=1}^{P} \delta_{x_j}$.*

(i) *For all $r \in \Gamma_N^d \setminus \{0\}$,*

$$|\widehat{\mu}(Nr)| \le C_1 N^{-\beta/2} (\log N)^{1/2}. \tag{2.6}$$

(ii) *For $1 \le \ell \le d/\beta$ and for all cubes Q with $|Q| \le N^{-\ell\beta}$,*

$$\mu^{*\ell}(Q) \le C_2 N^{-\ell\beta} \log N. \tag{2.7}$$

(iii) *For $d/\beta \le \ell \le \mathfrak{n}$,*

$$\max_{u\in\Gamma_N^d} \big| \mu^{*\ell}(\{u\}) - N^{-d} \big| \le C_3 \frac{N^{-d} (\log N)^{\frac{\ell+1}{2}}}{N^{(\ell\beta-d)/2}}. \tag{2.8}$$

While this result is not optimal (in particular with respect to the powers of the logarithm), it is all we need for the proof of Theorem 2.2. The proof of Proposition 2.5 will be given in §3.

2.3 Transference

For $N \ge 1$, we will write $\sqcap_N = N^d \mathbb{1}_{[-1/2,1/2)^d}(Nt)dt$ and

$$\tau_N = \frac{1}{N^d} \sum_{j\in\Gamma_N^d} \delta_{j/N}.$$

Recall that λ is the uniform probability measure, *i.e.*, normalized Lebesgue measure, on $\mathbb{T}^d$.

We start with some simple observations.

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Lemma 2.6 *The following holds true for $N \geq 1$.*

(i) $\sqcap_N^{*\ell} * \tau_N = \lambda$ *for* $\ell = 1, 2, \ldots$.
(ii) $\widehat{\tau_N}(r) = 1$ *for* $r \in (N\mathbb{Z})^d$, *and* $\widehat{\tau_N}(r) = 0$ *otherwise.*
(iii) $\widehat{\sqcap_N}(r) = 0$ *for* $r \in (N\mathbb{Z})^d, r \neq 0$.

Proof (i) follows by direct computation of the convolution (it is also a consequence of (ii) and (iii)). For (ii) notice that if $r \notin (N\mathbb{Z})^d$,

$$\widehat{\tau_N}(r) = \frac{1}{N^d} \sum_{j \in [N]^d} e^{-2\pi i r \cdot j/N} = \prod_{k=1}^{d} \frac{1}{N} \frac{e^{-2\pi i r_k} - 1}{e^{-2\pi i r_k/N} - 1} = 0.$$

Otherwise $\widehat{\tau_N}(r) = 1$. For (iii) just notice that $\widehat{\sqcap_N}(r) = \prod_{k=1}^{d} \frac{\sin(\pi r_k/N)}{\pi r_k/N}$. ∎

In what follows we let v be a nonnegative smooth function with support in $(-1/2, 1/2)^d$ such that $\int v(t)\,dt = 1$, and let $v_N = N^d v(N\cdot)$. Thus v_N generate a standard smooth approximation of the identity. We now convolve the point masses obtained in Proposition 2.5 with $\sqcap_N$ and the mollifier v_N.

Lemma 2.7 *Let μ be as in Proposition 2.5 and let $f = v_N * \sqcap_N * \mu$. Then f is a smooth function satisfying the following properties.*

(i) *For* $l = 0, 1, \ldots$, $\|\nabla^l f\|_\infty \leq C(l) N^{d+l}$. *There are cubes* Q_j, $j = 1, \ldots, \lfloor N^\beta \rfloor$ *with side length* $2/N$ *such that* $\operatorname{supp}(f) \subset \bigcup_{j=1}^{\lfloor N^\beta \rfloor} Q_j$.
(ii) *For* $r \in \mathbb{Z}^d \backslash \{0\}$, $\Lambda \geq 0$

$$|\widehat{f}(r)| \leq C(\log N)^{-1/2} N^{-\beta/2} \min\Big(\frac{C_0(\Lambda) N^\Lambda}{|r|^\Lambda}, 1\Big). \tag{2.9}$$

(iii) *For all cubes* Q

$$\int_Q f^{*n}(t)\,dt \leq 2^d |Q|^{n\beta/d} \log N, \quad 1 \leq n \leq d/\beta. \tag{2.10}$$

(iv) *For* $l = 0, 1, 2, \ldots$,

$$\|\nabla^l (f^{*n} - 1)\|_\infty \leq C(l) C(\beta, \mathfrak{n}) \frac{(\log N)^{\frac{n+1}{2}}}{N^{(n\beta - d)/2}} N^l, \quad d/\beta \leq n \leq \mathfrak{n}. \tag{2.11}$$

Proof The assertion about the support follows immediately from the definition. Let $g(t) = \sqcap_N * \mu(t) = \int_{\mathbb{T}^d} \sqcap_N(t - s)\,d\mu(s)$. The mollifiers satisfy

$$v_N(r) \leq N^d \max\{1, C(\Lambda)(N/|r|)^\Lambda\}$$

for any $\Lambda \geq 0$. We thus observe that the estimates for f are implied by the following estimates for g.

$$\sup_{r \in \mathbb{Z}^d \backslash \{0\}} |\widehat{g}(r)| \leq C(\log N)^{1/2} N^{-\beta/2}, \tag{2.12}$$

$$\int_Q g^{*n}(t)\,dt \leq 2^d |Q|^{n\beta/d} \log N, \quad n \leq d/\beta, \tag{2.13}$$

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for all cubes Q,

$$\sup_{t\in\mathbb{T}^d} |g^{*n}(t) - 1| \le C(\beta, \mathfrak{n}) \frac{(\log N)^{\frac{n+1}{2}}}{N^{(n\beta-d)/2}}, \quad d/\beta \le n \le \mathfrak{n}, \tag{2.14}$$

and

$$\sup_{t\in\mathbb{T}^d} |g(t)| \le N^d. \tag{2.15}$$

To show (2.12), notice that $\widehat{g}(r) = \widehat{\sqcap_N}(r)\widehat{\mu}(r)$. If $r \in (N\mathbb{Z})^d$, then $\widehat{g}(r) = 0$, by Lemma 2.6 (ii). Otherwise use the trivial bound $|\widehat{\sqcap_N}(r)| \le 1$ and (2.6) together with the observation that $\widehat{\mu}$ is N-periodic.

To show (2.13), we consider separately the three cases $|Q| \le N^{-d}$, $N^{-d} \le |Q| \le N^{-n\beta}$, and $|Q| \ge N^{-n\beta}$.

Case 1: $|Q| \le N^{-d}$. Notice that, as in the proof of (2.15), we have

$$\sqcap_N * \mu^{*n}(t) = N^d \mu^{*n}(\{u\}) \le N^d M(\beta) \frac{\log N}{N^{n\beta}}.$$

Thus

$$\begin{aligned}\sqcap_N * \mu^{*n}(Q) &\le |Q| N^d M(\beta) \frac{\log N}{N^{n\beta}} \\ &= |Q|^{n\beta/d} (|Q| N^d)^{1-n\beta/d} M(\beta) \log N \le M(\beta) |Q|^{n\beta/d} \log N\end{aligned}$$

by our assumption on $|Q|$.

Case 2: $N^{-d} \le |Q| \le N^{-n\beta}$. In this case, by (2.7)

$$\begin{aligned}\int_Q g^{*n}(t)\, dt = \int_Q \sqcap_N^n * \mu^{*n}(t)\, dt &\le \max_{Q:|Q|=N^{-n\beta}} \mu^{*n}(Q) \\ &\le M(\beta) N^{-n\beta} \log N \le M(\beta) |Q|^{n\beta/d} \log N.\end{aligned}$$

Case 3: $|Q| \ge N^{-n\beta}$. In this case we can split Q into no more than $2^d N^{n\beta} |Q|$ cubes of size at most $N^{-n\beta}$. Applying (2.7) to each cube, we may bound $\mu^{*n}(Q)$ by

$$(2^d N^{n\beta} |Q|) M(\beta) \frac{\log N}{N^{n\beta}} = 2^d M(\beta) |Q| \log N \le 2^d M(\beta) |Q|^{n\beta} \log N.$$

Since $g = \sqcap_N^n * \mu^{*n}$, (2.13) follows also in Case 3.

To show (2.14), notice that by Lemma 2.6 (i) and (2.8),

$$g^{*n} = \sqcap_N^{*n} * \tau_N + \sqcap_N^{*n} * (\mu^{*n} - \tau_N) = \lambda + \sqcap_N^{*n} * (\mu^{*n} - \tau_N)$$

and

$$|\mu^{*n} - \tau_N| \le C(\beta, \mathfrak{n}) \frac{(\log N)^{\frac{n+1}{2}}}{N^{(n\beta-d)/2}} \tau_N.$$

Now g^{*n} is continuous and we get

$$|g^{*n} - 1| \le C(\beta, \mathfrak{n}) \frac{(\log N)^{\frac{n+1}{2}}}{N^{(n\beta-d)/2}}$$

and thus (2.14).

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To show (2.15), notice that for any $t \in \mathbb{T}$, $g(t) = N^d \mu(\{u\})$ where u is the unique point in Γ_N^d contained in the cube $(t - 1/(2N), t + 1/(2N)]^d$. Now (2.15) follows from (2.7) with $n = 1$ and Q containing u. ∎

Definition Let f be a smooth function on $\mathbb{T}^d$ and let $p \in \mathbb{N}$. We let the *p-periodization* $\operatorname{Per}_p f$ be the unique smooth function on $\mathbb{T}^d$ which is $1/p$-periodic in each of the d variables and satisfies $\operatorname{Per}_p f(t) = f(pt)$ for $0 \le t_i < p^{-1}$, $i = 1, \dots, d$.

The following lemma is analogous to a crucial observation about periodized functions in [18].

Lemma 2.8 *Let $p \in \mathbb{N}$.*

(i) *Let $f \in C^\infty(\mathbb{T}^d)$. Then*

$$\widehat{\operatorname{Per}_p f}(r) = \begin{cases} \widehat{f}(k) & \text{if } r = kp, k \in \mathbb{Z}^d, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *Let $\mathcal{R} = [a_1, a_1+p) \times \cdots \times [a_d, a_d+p)$, for some $a \in \mathbb{R}^d$, and for $v = 1, \dots n$, let P_v be a trigonometric polynomial with frequencies in $\mathcal{R}$, i.e., P_v is a linear combination of the functions $x \mapsto \exp(2\pi i \langle k, x\rangle)$ with $k \in \mathcal{R} \cap \mathbb{Z}^d$. Let $f_1, \dots, f_n$ be smooth functions on $\mathbb{T}^d$ and let $G_v = \operatorname{Per}_p f_v$. Then*

$$(G_1 P_1) * \cdots * (G_n P_n) = (G_1 * \cdots * G_n)(P_1 * \cdots * P_n).$$

Proof This follows easily by Fourier expansion using the fact that every $k \in \mathbb{Z}^d$ can be written in a unique way as $k = pl + k'$, where $l \in \mathbb{Z}^d$ and $k' \in \mathcal{R}$. ∎

Lemma 2.9 *Let $\eta > 0$, $\beta > \alpha$ and let k be an integer with $k > \frac{\alpha+1}{\beta-\alpha}$. Then there exists $m_0 = m_0(\alpha, \beta, \mathfrak{n}, \eta, \psi, k) \ge N_0(\beta, \mathfrak{n})$ such that for all $m \ge m_0$ with $\gcd(\mathfrak{n}!, m) = 1$, the following hold with $N = m^k$ and f as in Lemma 2.7.*

(i) *The $(2m+1)$-periodization of f, $F_m = \operatorname{Per}_{2m+1} f$, is smooth with $\int_{\mathbb{T}^d} F_m(t)\,dt = 1$, and for $l = 0, 1, \dots, L$*

$$\|\nabla^l(F_m)\|_\infty \le C(L) m^{kd+(k+1)l}. \tag{2.16}$$

Moreover, there are cubes Q_j $j = 1, \dots, (2m+1)^d \lfloor m^{k\beta} \rfloor$, of side length m^{-k-1}, such that

$$\operatorname{supp}(F_m) \subset \bigcup_{j=1}^{(2m+1)^d \lfloor m^{k\beta} \rfloor} Q_j. \tag{2.17}$$

(ii) *For $r \in \mathbb{Z}^d \backslash \{0\}$,*

$$\frac{|r|^{\alpha/2} |\widehat{F_m}(r)|}{\psi(1/|r|)} \le \eta. \tag{2.18}$$

(iii) *For all cubes Q with side length at most $2/\sqrt{m}$,*

$$\int_Q F_m^{*n}(t)\,dt \le \eta \psi(|Q|)|Q|^{n\alpha/d}, \quad 1 \le n < d/\alpha. \tag{2.19}$$

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(iv) *For* $n \ge d/\alpha$, *let* $\rho_n = \frac{n\alpha - d}{2}$. *Then*

$$\|F_m^{*n} - 1\|_{C^{\rho_n,\psi}} \le \eta, \quad d/\alpha \le n \le \mathfrak{n}. \tag{2.20}$$

(v) *For all rectangles* R *of side lengths at least* $1/\sqrt{m}$.

$$\int_R F_m^{*n}(t)\,dt \le (1+\eta)|R|, \; n < d/\alpha. \tag{2.21}$$

Proof Part (i) is straightforward given Lemma 2.7. We only need to give the proof of (ii).

We first recall from Lemma 2.8 that $\widehat{F_m}((2m+1)k) = \widehat{f}(k)$ for $k \in \mathbb{Z}^d$, and $\widehat{F_m}(r) = 0$ for r not of this form. Thus for $r \ne 0$, by (2.9)

$$\begin{aligned}
|\widehat{F_m}(r)| &\le \frac{\sqrt{8}\log^{1/2}(8m^{kd})}{m^{k\beta/2}} \min\Big(\frac{C(\Lambda)m^{k\Lambda}(2m+1)^{\Lambda}}{|r|^{\Lambda}}, 1\Big) \\
&\le C_{\Lambda,k}\frac{\log^{1/2} m}{m^{k\beta/2}} \min\Big(\frac{m^{(k+1)\Lambda}}{|r|^{\Lambda}}, 1\Big) \\
&= C_{\Lambda,k}\frac{\log^{1/2} m}{m^{(k(\beta-\alpha)-\alpha)/4}} \frac{\psi(m^{-k-1})^{-1}}{m^{(k(\beta-\alpha)-\alpha)/4}} \frac{\psi(m^{-k-1})}{m^{(k+1)\alpha/2}} \min\Big(\frac{m^{(k+1)\Lambda}}{|r|^{\Lambda}}, 1\Big) \\
&\le \eta \frac{\psi(m^{-k-1})^{-1}}{m^{(k(\beta-\alpha)-\alpha)/4}} \frac{\psi(m^{-k-1})}{m^{(k+1)\alpha/2}} \min\Big(\frac{m^{(k+1)\Lambda}}{|r|^{\Lambda}}, 1\Big) \qquad (2.22)
\end{aligned}$$

provided that $\ge m \ge m_0$ and m_0 is chosen large enough. We separately consider the cases $0 < |r| \le m^{k+1}$ and $|r| \ge m^{k+1}$. In the first case we obtain (2.18) directly from (2.22), provided that m_0 is large enough. Now let $2^l \le r/m^{k+1} < 2^{l+1}$ with $l \ge 0$. Then by the monotonicity of ψ and the doubling condition (1.5),

$$\psi(m^{-k-1}) \le \psi(2^{l+1}|r|^{-1}) \le C_\psi^{l+1}\psi(|r|^{-1}),$$

and we see in this case that (2.22) is estimated by

$$\eta \frac{\psi(m^{-k-1})^{-1}}{m^{(k(\beta-\alpha)-\alpha)/4}} 2^{(l+1)\alpha} C_\psi^{l+1} 2^{-l\Lambda} \psi(|r|^{-1})|r|^{-\alpha/2}.$$

Thus if we choose Λ so large that $2^{\alpha+2-\Lambda}C_\psi \le 1$, we can sum in l. Then by choosing m_0 large, we obtain (2.18) for all $r \ne 0$.

Proof of (iv). Notice that by (2.11) and our assumption on k,

$$\|F_m^{*n} - 1\|_{C^{\lfloor \rho_n \rfloor}} \le N^{-(\rho_n - \lfloor \rho_n \rfloor) - \epsilon},$$

for some $\epsilon > 0$ and sufficiently large m. Setting $G = \nabla^{\lfloor \rho_n \rfloor}(F_m^{*n} - 1)$, it remains to show $\omega_{\rho_n - \lfloor \rho_n \rfloor, \psi}(G) \le \eta/2$ for $m \ge m_0$ and large enough m_0.

Again by (2.11), we have $\|G\|_\infty + N^{-1}\|\nabla G\|_\infty \le N^{-(\rho_n - \lfloor \rho_n \rfloor) - \epsilon}$ for some $\epsilon > 0$ and sufficiently large m. Now if $0 < |h| \le 1/N$, then by the mean value theorem, for any

$x \in \mathbb{T}^d$,

$$\begin{aligned}\frac{|G(x+h)-G(x)|}{|h|^{\rho_n-\lfloor\rho_n\rfloor}\psi(|h|)} &= \frac{|G(x+h)-G(x)|}{|h|}\frac{|h|^{1-(\rho_n-\lfloor\rho_n\rfloor)}}{\psi(|h|)}\\ &\le N^{1-(\rho_n-\lfloor\rho_n\rfloor)-\epsilon}C_{\psi,\epsilon}|h|^{1-(\rho_n-\lfloor\rho_n\rfloor)-\epsilon/2}\\ &\le C_{\psi,\epsilon}N^{-\epsilon/2}\le \eta/2,\end{aligned}$$

provided that m_0 is chosen large enough. If $|h|\ge 1/N$, then

$$\frac{|G(x+h)-G(x)|}{|h|^{\rho_n-\lfloor\rho_n\rfloor}\psi(|h|)} \le \frac{2\|G\|_\infty}{|h|^{\rho_n-\lfloor\rho_n\rfloor}\psi(|h|)} \le 2N^{-(\rho_n-\lfloor\rho_n\rfloor)-\epsilon}\frac{N^{\rho_n-\lfloor\rho_n\rfloor}}{\psi(1/N)} \le \eta/2$$

provided that m_0 is chosen large enough. This proves (2.20).

Proofs of (iii) and (v). In what follows we say that a *fundamental cube* is a cube of the form $\prod_{i=1}^d[\frac{v_i}{2m+1},\frac{v_i+1}{2m+1})$ where $v_i\in\{0,\dots 2m\}$ for each $i=1,\dots,d$.

We first consider the claim (v). Let R be a rectangle with side lengths $l_1\ge\cdots\ge l_d$, and assume that $l_d\ge m^{-1/2}$. Notice that R is contained in a union of no more than

$$(2m+1)^d l_1\cdots l_d + C_d(2m+1)^{d-1}l_1\cdots l_{d-1}$$

many fundamental cubes of size $1/(2m+1)^d$. Since the integral of F_m^{*n} over any fundamental cube is equal to $(2m+1)^{-d}$, we see that

$$\begin{aligned}\int_I F_m^{*n}(x)\,dx &\le l_1\cdots l_d + C_d(2m+1)^{-1}l_1\cdots l_{d-1}\\ &= |R| + \frac{C_d}{(2m+1)l_d}|R| \le |R| + \frac{C_d}{2\sqrt{m}}|R|.\end{aligned}$$

Thus (2.21) is satisfied if m_0 is chosen large enough.

In order to show (iii), we separately consider the two cases where the side length of Q is larger or smaller than $(2m+1)^{-1}$.

Case 1: $(2m+1)^{-1}\le|Q|^{1/d}\le 2m^{-1/2}$. In this case the argument above shows

$$\int_Q F_m^{*n}(x)\,dx \le (1+C_d)|Q|$$

and (2.19) will follow if $(1+C_d)|Q|\le\eta\psi(|Q|)|Q|^{n\alpha/d}$. But this is indeed the case if $|Q|\le 2/\sqrt{m}\le 2/\sqrt{m_0}$ and m_0 is large enough.

Case 2: $|Q|^{1/d}<(2m+1)^{-1}$. We first assume that Q is contained in a $[0,(2m+1)^{-1})^d$. Then by (2.10)

$$\begin{aligned}\int_Q F_m^{*n}(x)\,dx &= \frac{1}{(2m+1)^d}\int_{(2m+1)Q} f^{*n}(t)\,dt\\ &\le \frac{1}{(2m+1)^d}2^d\Big((2m+1)^d|Q|\Big)^{n\beta/d}\log N\\ &= \frac{2^d k\log m}{(2m+1)^{d-n\beta}}|Q|^{n\beta/d}\le|Q|^{n\beta/d}\end{aligned}$$

provided that m_0 is chosen large enough. If $|Q|^{n\beta/d}\le\eta\psi(|Q|)|Q|^{n\alpha/d}$, (2.19) will follow. But this is the case if $|Q|^{1/d}\le 1/(2m+1)\le 1/m_0$ is small enough. By periodicity the above argument holds true if Q is contained in any fundamental cube of size

$(2m+1)^{-d}$. Moreover if Q is any cube of size $\le (2m+1)^{-d}$, then we can split Q into 2^d rectangles supported in fundamental cubes and apply the same argument to each such rectangle.

This finishes the proof of (iii). ∎

2.4 Approximation

We are now ready to prove Theorem 2.2. It remains to show that for every $\gamma \in (\alpha, d)$ and every $\varepsilon_1 > 0$, the set $\mathfrak{V}^{\gamma,\varepsilon_1}$ is dense in $\overline{\mathfrak{V}_0}$. This reduces to approximating $(K, g) \in \mathfrak{V}_0$ where g is smooth. We may further assume that there exists a small constant $c > 0$ such that g satisfies

$$\int_Q g^{*n}(x)\,dx \le (1-c)\psi(|Q|)|Q|^{n\alpha/d} \tag{2.23}$$

for all cubes Q and $1 \le n < d/\alpha$. This is because otherwise we can approximate (K, g) by $(K, (1-c)g)$ and let $c \to 0$.

Lemma 2.10 *Suppose $\alpha < \gamma < d$, $\varepsilon_1 > 0$, $c \in (0,1)$, $(K, g) \in \mathfrak{V}_0$ where g is a smooth function satisfying (2.23). Let $\epsilon > 0$. Then there exists a compact set F and a smooth function f such that $(F, fg) \in \mathfrak{V}^{\gamma,\varepsilon_1}$ and $d_{\mathfrak{V}}((K,g),(F,fg)) < \epsilon$.*

Proof We let $\epsilon' = \epsilon/100$. Fix β with $\alpha < \beta < \gamma$. Choose $\mathfrak{n} \equiv \mathfrak{n}(\epsilon) = 1 + \lceil \log_2 \frac{1}{\epsilon'} \rceil$ so that

$$\sum_{n>\mathfrak{n}} 2^{-n} < \epsilon'. \tag{2.24}$$

Fix an integer k such that $k > \frac{d-\gamma}{\gamma-\beta}$.

With these parameters we consider the functions F_m as constructed in Lemma 2.9. We let $A_{\epsilon'}$ be a finite ϵ'-net of K, *i.e.*, a finite set of points in K such that K is contained in the union of balls of radius ϵ' centered at points in $A_{\epsilon'}$. We shall show that if $\eta > 0$ is chosen small enough and if $m \ge m_0(\alpha, \beta, \eta, \psi, k)$ is chosen large enough, then the choice $(H, F_m g)$ with $H = \operatorname{supp}(F_m g) \cup A_{\epsilon'}$ will give the desired approximation of (K, g).

Notation In this proof we shall write $B_1 \lesssim B_2$ for two nonnegative quantities B_1, B_2 if $B_1 \le CB_2$ where C may only depend on α, β, γ, ε_1, k, d, and ϵ and on the function g (so C will not depend on η or m). We shall call such a C an *admissible constant*.

To show that $(H, F_m g\lambda) \in \mathfrak{V}^{\gamma,\varepsilon_1}$, we only need to verify (2.4) and (2.5). We postpone (2.4) to a later part of the proof and now verify (2.5). By (2.17)

$$\operatorname{supp}(F_m g) \subset \bigcup_j Q_j,$$

where Q_j, $j = 1, \dots, (2m+1)^d \lfloor m^{k\beta} \rfloor$, are cubes with side length m^{-k-1}. Thus $H = \operatorname{supp}(F_m g) \cup A_{\epsilon'}$ can be covered by $M = (2m+1)^d \lfloor m^{k\beta} \rfloor + (\# A_{\epsilon'})$ cubes of side length m^{-k-1}. To verify (2.5), it now suffices to show $m^{-k-1} < \varepsilon_1 M^{-1/\gamma}$, which follows

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from $3^d m^{k\beta+d} + (\#A) < \varepsilon_1{}^\gamma m^{k\gamma+\gamma}$. Since $k > \frac{d-\gamma}{\gamma-\beta}$, the last inequality holds provided that m is large enough.

We need to show that for sufficiently large m $d_{\mathfrak{W}}((K, g\lambda), (H, F_m g\lambda)) < \epsilon$. Since $\mathrm{supp}(F_m g) \subset \mathrm{supp}(g) \subset K$, we have $H = \mathrm{supp}(F_m g) \cup A_{\epsilon'} \subset K$. Thus the Hausdorff distance of H and K satisfies $d_{\mathfrak{K}}(H, K) \le \epsilon'$.

To handle the other components of $d_{\mathfrak{W}}$, we set $L = 10\mathfrak{n}kd$ and we will use the fact that, since g is smooth, there exists an admissible constant $C > 0$ such that

$$\sum_{|r|_\infty \ge m} |r|^L |\widehat{g}(r)| \le Cm^{-(k+2)L} \tag{2.25}$$

for all $m \ge 1$. By the periodicity of F_m, we have

$$|\widehat{g}(0) - \widehat{F_m g}(0)| = \Big|\sum_{u \ne 0} \widehat{F_m}(-u)\widehat{g}(u)\Big| \le \sum_{|u|_\infty \ge m} |\widehat{g}(u)| \le Cm^{-1}$$

and hence $|\widehat{g}(0) - \widehat{F_m g}(0)| \le \epsilon'$ provided that m is large enough.

For the nonzero Fourier coefficients we have

$$\begin{aligned} |\widehat{g}(r) - \widehat{F_m g}(r)| &= \Big|\sum_{u \ne r} \widehat{F_m}(r-u)\widehat{g}(u)\Big| \\ &\le \sum_{|u| \le |r|/2} |\widehat{F_m}(r-u)\widehat{g}(u)| + \sum_{\substack{|u|>|r|/2 \\ u \ne r}} |\widehat{F_m}(r-u)\widehat{g}(u)|. \end{aligned}$$

By (2.18), this is estimated by

$$\begin{aligned} &\eta C_\psi \psi(|r|^{-1}) 2^{\alpha/2} |r|^{-\alpha/2} \sum_{|u| \le |r|/2} |\widehat{g}(u)| + \eta\psi(1) \sum_{|u|>|r|/2} |\widehat{g}(u)| \\ &\qquad\qquad \lesssim \big(\eta\psi(|r|^{-1})|r|^{-\alpha/2} + \eta|r|^{-d}\big) \lesssim \eta\psi(|r|^{-1})|r|^{-\alpha/2}, \end{aligned}$$

and this is $< \psi(|r|^{-1})|r|^{-\alpha/2}\epsilon'$ provided that $\eta > 0$ is chosen small enough. With this choice of η we have proved

$$\sup_{r \in \mathbb{Z}^d \setminus \{0\}} \frac{|r|^{\alpha/2}}{\psi(|r|^{-1})} |\widehat{g}(r) - \widehat{F_m g}(r)| < \epsilon',$$

if η is sufficiently small and m is sufficiently large.

It remains to show that (2.4) holds for $\mu = F_m g\lambda$, *i.e.*,

$$\int_Q (F_m g)^{*n}(x)\,dx \le \psi(|Q|)|Q|^{n\alpha/d}, \quad 1 \le n < d/\alpha \tag{2.26}$$

and that for $d/\alpha \le n \le \mathfrak{n}$,

$$\sum_{i_1,\dots,i_n} \big\|(\chi_{i_1}^{(n)} g) * \cdots * (\chi_{i_n}^{(n)} g) - (\chi_{i_1}^{(n)} F_m g) * \cdots * (\chi_{i_n}^{(n)} F_m g)\big\|_{C^{\rho_n,\psi}} < \epsilon', \tag{2.27}$$

provided that η is small enough and m is large enough. Notice that by the definition of the metric $d_{\mathfrak{W}}$ and by (2.24), the corresponding terms for $n > \mathfrak{n}$ can be ignored.

Proof of (2.26). Following [18], we write $P_m(x) = \sum_{|r|_\infty \le m} \widehat{g}(r) e^{2\pi i \langle r, x\rangle}$. By (2.25) we have, for sufficiently large m,

$$\|g - P_m\|_{C^L} \le Cm^{-(k+2)L} \le 1. \tag{2.28}$$

We first verify that for every $n = 1, \dots, \mathfrak{n}$,

$$\|g^{*n} - (P_m)^{*n}\|_{C^L} \le m^{-1}, \tag{2.29}$$

$$\|(F_m g)^{*n} - (F_m P_m)^{*n}\|_\infty \le m^{-1}, \tag{2.30}$$

provided that m is chosen large enough. To see this we write

$$\begin{aligned} g^{*n} - (P_m)^{*n} &= ((g - P_m) + P_m)^{*n} - (P_m)^{*n} \\ &= (g - P_m)^{*n} + \sum_{\nu=1}^{n-1} \binom{n}{\nu} (g - P_m)^{*(n-\nu)} * (P_m)^{*\nu}. \end{aligned}$$

Therefore, using $\binom{n}{\nu} = \frac{n}{n-\nu}\binom{n-1}{\nu}$ for $1 \le \nu \le n-1$ and (2.28),

$$\begin{aligned} \|g^{*n} - (P_m)^{*n}\|_{C^L} &\le \|g - P_m\|_{C^L} \sum_{\nu=0}^{n-1} \binom{n}{\nu} \|P_m\|_\infty^\nu \\ &\le \|g - P_m\|_{C^L} \big(1 + \|P_m\|_\infty\big)^{n-1} n \lesssim m^{-2} n \big(2 + \|g\|_\infty\big)^{n-1}, \end{aligned}$$

and this gives (2.29) provided that m is large enough.

By (2.16) and the first estimate in (2.28) we have

$$\begin{aligned} \|F_m(P_m - g)\|_{C^L} &\lesssim \|F_m\|_{C^L} \|P_m - g\|_{C^L} \\ &\lesssim m^{kd+(k+1)L} m^{-(k+2)L} \le C m^{kd-L} \le 1 \end{aligned}$$

for sufficiently large m The same argument as above then gives

$$\begin{aligned} \|(F_m P_m)^{*n} - (F_m g)^{*n}\|_{C^L} &\le \|F_m(P_m - g)\|_{C^L} n \big(1 + \|F_m g\|_\infty\big)^{n-1} n \\ &\lesssim m^{kd-L} (1 + m^{kd} \|g\|_\infty)^{n-1} n \lesssim m^{nkd-L} (1 + \|g\|_\infty)^{n-1} n, \end{aligned}$$

and this gives (2.30) provided that m is large enough.

As a consequence of Lemma 2.8 (ii) we have

$$(F_m P_m)^{*n} = (F_m)^{*n} (P_m)^{*n}. \tag{2.31}$$

Now for fixed $n < d/\alpha$ and a cube Q, we have by (2.31) and (2.30)

$$\begin{aligned} \int_Q (F_m g)^{*n}(x)\,dx &\le \Big| \int_Q (F_m P_m)^{*n}(x)\,dx \Big| \\ &\qquad + \Big| \int_Q ((F_m g)^{*n}(x) - (F_m P_m)^{*n}(x))\,dx \Big| \\ &\le \Big| \int_Q (F_m)^{*n}(x) (P_m)^{*n}(x)\,dx \Big| + m^{-1}|Q| \\ &\le \Big| \int_Q (F_m)^{*n}(x) (P_m)^{*n}(x)\,dx \Big| + C m^{-1} \psi(|Q|) |Q|^{n\alpha/d} \\ &\le \Big| \int_Q (F_m)^{*n}(x) (P_m)^{*n}(x)\,dx \Big| + \frac{c}{2} \psi(|Q|) |Q|^{n\alpha/d} \end{aligned}$$

for sufficiently large m. Thus, in order to finish the proof of (2.26), we must show that

$$\int_Q (F_m)^{*n}(x) (P_m)^{*n}(x)\,dx \le \Big(1 - \frac{c}{2}\Big) \psi(|Q|) |Q|^{n\alpha/d}. \tag{2.32}$$

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If the side length of Q is $\leq 2/\sqrt{m}$, then

$$\Big|\int_Q (F_m)^{*n}(x)(P_m)^{*n}(x)\,dx\Big| \leq \|(P_m)^{*n}\|_\infty \int_Q (F_m)^{*n}(x)\,dx$$
$$\leq (1+\|g^{*n}\|_\infty)\eta\psi(|Q|)|Q|^{n\alpha/d}$$
$$\leq (1-\frac{c}{2})\psi(|Q|)|Q|^{n\alpha/d},$$

where in the last inequality η is chosen sufficiently large (the second inequality follows from (2.19)).

If the side length of Q is $> 2/\sqrt{m}$, then Q can be split into rectangles R of side lengths between $1/\sqrt{m}$ and $2/\sqrt{m}$. Writing

$$a_R = ⨍_R g^{*n}(x)\,dx \quad\text{and}\quad b_R = ⨍_R (P_m)^{*n}(x)\,dx,$$

we then have

$$\|(P_m)^{*n} - b_R\|_{L^\infty(R)} \lesssim m^{-1/2}\|(P_m)^{*n}\|_{C^1} \lesssim m^{-1/2}$$

and

$$|b_R - a_R| \leq \|g^{*n} - (P_m)^{*n}\|_\infty \leq m^{-1},$$

by (2.29). Now

$$\Big|\int_Q (F_m)^{*n}(x)(P_m)^{*n}(x)\,dx\Big|$$
$$\leq \Big|\sum_R \int_R (F_m)^{*n}(x)a_R\,dx\Big|$$
$$+\Big|\sum_R \int_R (F_m)^{*n}(x)(b_R - a_R)\,dx\Big|$$
$$+\Big|\sum_R \int_R (F_m)^{*n}(x)(b_R - (P_m)^{*n}(x))\,dx\Big|$$
$$\leq \sum_R a_R(1+\eta)|R| + \Big(\frac{1}{m} + \frac{C}{\sqrt{m}}\Big)\sum_R \int_R (F_m)^{*n}(x)\,dx$$
$$\leq (1+\eta)\int_Q g^{*n}(x)\,dx + \frac{C'}{\sqrt{m}}\int_Q (F_m)^{*n}(x)\,dx,$$

where C' is admissible. By (2.21) and (2.23) the last expression is less than or equal to

$$(1+\eta)(1-c)\psi(|Q|)|Q|^{n\alpha/d} + \frac{C'}{\sqrt{m}}(1+\eta)|Q|$$
$$\leq \Big(\Big(1-\frac{3}{4}c\Big) + \frac{C''}{\sqrt{m}}\Big)\psi(|Q|)|Q|^{n\alpha/d} \leq \Big(1-\frac{c}{2}\Big)\psi(|Q|)|Q|^{n\alpha/d}$$

provided that η is small enough and m is large enough.

In either case we have verified (2.32), and this concludes the proof of (2.26).

Proof of (2.27). Fix n with $d/\alpha \leq n \leq \mathfrak{n}$ and $i = (\imath_1,\dots,\imath_n) \in (\mathfrak{I}_n)^n$. Write

$$g_j = \chi^{(n)}_{\imath_j} g,$$

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for $j = 1, \dots, n$, and $P_{j,m}(x) = \sum_{|r|_\infty \le m} \widehat{g_j}(r) e^{2\pi i \langle r,x\rangle}$. Equation (2.27) reduces to estimating

$$\begin{aligned}\|g_1 * \cdots * g_n - (F_m g_1) * \cdots * (F_m g_n)\|_{C^{\rho_n,\psi}} \\ \le \|g_1 * \cdots * g_n - P_{1,m} * \cdots * P_{n,m}\|_{C^{\rho_n,\psi}} \\ + \|P_{1,m} * \cdots * P_{n,m} - (F_m P_{1,m}) * \cdots * (F_m P_{n,m})\|_{C^{\rho_n,\psi}} \\ + \|(F_m P_{1,m}) * \cdots * (F_m P_{n,m}) - (F_m g_1) * \cdots * (F_m g_n)\|_{C^{\rho_n,\psi}}.\end{aligned}$$

Arguing as before (*cf.* (2.25)), we have, for sufficiently large m,

$$\|P_{j,m} - g_j\|_{C^L} \le C m^{-(k+2)L} \le 1 \quad \text{and} \quad \|F_m(P_{m,j} - g_j)\|_{C^L} \le C m^{kd-L} \le 1.$$

Using the continuous embedding $C^L \rightsquigarrow C^{\rho_n,\psi}$, we get, for sufficiently large m,

$$\|g_1 * \cdots * g_n - P_{1,m} * \cdots * P_{n,m}\|_{C^{\rho_n,\psi}} \le C m^{-(k+2)L} n \prod_{j=1}^{n} (1 + \|g_j\|_\infty) \le m^{-1}$$

and

$$\begin{aligned}|(F_m P_{1,m}) * \cdots * (F_m P_{n,m}) - (F_m g_1) * \cdots * (F_m g_n)\|_{C^{\rho_n,\psi}} \\ \le C m^{nkd-L} n \prod_{j=1}^{n} (1 + \|g_j\|_\infty) \le m^{-1}.\end{aligned}$$

On the other hand, using Lemma 2.8 (ii), again we have

$$(F_m P_{1,m}) * \cdots * (F_m P_{n,m}) = (F_m)^{*n} (P_{1,m} * \cdots * P_{n,m}).$$

Thus, by (2.20)

$$\begin{aligned}\|P_{1,m} * \cdots * P_{n,m} - (F_m P_{1,m}) * \cdots * (F_m P_{n,m})\|_{C^{\rho_n,\psi}} \\ = C\|(1 - F_m^{*n})(P_{1,m} * \cdots * P_{n,m})\|_{C^{\rho_n,\psi}} \\ \lesssim \|1 - F_m^{*n}\|_{C^{\rho_n,\psi}} \|P_{1,m} * \cdots * P_{n,m}\|_{C^{\rho_n,\psi}} \\ \lesssim \eta \|P_{1,m} * \cdots * P_{n,m}\|_{C^L} \lesssim \eta(1 + \|g_1 * \cdots * g_n\|_{C^L}) \lesssim \eta,\end{aligned}$$

provided that m is sufficiently large.

Combining the above estimates, we get

$$\|g_1 * \cdots * g_n - (F_m g_1) * \cdots * (F_m g_n)\|_{C^{\rho_n,\psi}} \lesssim m^{-1} + \eta.$$

This guarantees (2.27) if η is chosen sufficiently small and m is chosen sufficiently large. This completes the proof of Lemma 2.10. ∎

2.5 Conclusion of the Proof of Theorem C

The result is about measures on $\mathbb{R}^d$ rather than $\mathbb{T}^d$. We use that every measure on $\mathbb{T}^d$ that is supported on a cube of sidelength < 1 can be identified with a measure that is supported on a cube of diameter < 1 in $\mathbb{R}^d$. We take a measure μ as in Corollary 2.4. After multiplying it with a suitable C_c^∞ function, we may assume that it is supported on a cube of diameter < 1. For each n we may decompose μ using the partition of unity (2.1). The regularity properties (iii) and (iv) in Theorem C follow immediately from (2.3) and (2.4). The compact support of μ and the decay property (2.2) on $\mathbb{Z}^d$

imply the decay property in (ii). This is a standard argument (see [15, p. 252] with slightly different notation). ∎

2.6 Optimality of the Hölder Continuity

Following the argument in [18], we show that the Hölder continuity obtained in Theorem C is the best possible.

Proposition 2.11 *Let μ be a Borel probability measure on $\mathbb{R}^d$ supported on a compact set of Hausdorff dimension $0 < \alpha < d$. Suppose $\mu^{*n} \in C^{\lambda}(\mathbb{R}^d)$, where $n \in \mathbb{N}$, $n \geq 2$, and $0 \leq \lambda < \infty$. Then $\lambda \leq \frac{n\alpha - d}{2}$.*

Proof Define by $\mathcal{E}_\gamma(\mu) = \iint |x-y|^{-\gamma}\, d\mu(x) d\mu(y) = c \int |\widehat{\mu}(\xi)|^2 |\xi|^{\gamma-d} d\xi$, the γ-dimensional energy of μ. Recall from [37, p. 62] that the Hausdorff dimension of E is equal to the supremum over all γ for which there is a probability measure ν supported on E with $\mathcal{E}_\gamma(\nu) < \infty$. Thus it suffices to show that $\mathcal{E}_\gamma(\mu)$ is finite for $\gamma < (d+2\lambda)/n$.

Since μ^{*n} is compactly supported it also belongs to the Besov space $B^2_{\lambda,\infty}$ and thus, by Plancherel, we have, for $R > 1$, $\int_{|\xi|\approx R} |\widehat{\mu}(\xi)|^{2n}\, d\xi \lesssim R^{-2\lambda}$. Now let $0 < \gamma < d$. By Hölder's inequality,

$$\int_{|\xi|\approx R} \frac{|\widehat{\mu}(\xi)|^2}{|\xi|^{d-\gamma}}\, d\xi \lesssim R^{\gamma-d}\Big(\int_{A_R} |\widehat{\mu}(\xi)|^{2n} d\xi\Big)^{1/n} R^{d(1-1/n)} \lesssim R^{\gamma-d/n-2\lambda/n}.$$

Letting $R = 2^j$, $j = 0, 1, \dots$, we see that $\mathcal{E}_\gamma(\mu)$ is finite if $\gamma < (d+2\lambda)/n$, and the proof is complete. ∎

3 Random Sparse Subsets

The purpose of this section is to prove Proposition 2.5 and, in fact, to establish a better quantitative version of this proposition.

3.1 Assumptions and Notations

In this section $x_1, x_2, \dots$ will be independent random variables uniformly distributed on Γ_N^d. That is, for any $m \in \mathbb{N}$ and subsets $A_1, \dots, A_m$ of Γ_N^d, the probability of the event that $x_\nu \in A_\nu$ for $\nu = 1, \dots, m$ is equal to $N^{-dm} \prod_{\nu=1}^m \operatorname{card}(A_\nu \cap \Gamma_N^d)$. We denote by $\mathcal{F}_0$ the trivial σ-algebra and by $\mathcal{F}_j$ the σ-algebra generated by the (inverse images) of the random variables $x_1, \dots, x_j$.

Given random Dirac masses δ_{x_ν}, $\nu = 1, \dots, m$, we define the random measures μ_m and σ_m by $\sigma_0 = \mu_0 = 0$, $\sigma_m = \sum_{\nu=1}^m \delta_{x_\nu}$, $\mu_m = m^{-1}\sigma_m$, $m = 1, 2, \dots$.

3.2 A Fourier Decay Estimate

The Fourier transform $\widehat{\mu}$ is defined on $\mathbb{Z}_N^d$ or, after scaling, on Γ_N^d, and we have

$$\widehat{\mu}_m(Nu) = \frac{1}{m} \sum_{j=1}^m e^{-2\pi i N\langle u, x_j\rangle}, \quad u \in \Gamma_N^d.$$

Lemma 3.1 *Let $h \geq 1$. The event*

$$\Big\{ \max_{u \in \Gamma_N^d \setminus \{0\}} |\widehat{\mu}_m(Nu)| \leq \frac{4 \log^{1/2}(8N^{d+h})}{m^{1/2}} \Big\} \tag{3.1}$$

has probability at least $1 - N^{-h}$.

Proof The proof is essentially the same as in the classical paper by Erdős and Rényi [9]. Fix $u \in \Gamma_N^d \setminus \{0\}$, and consider the random variables $X_\nu = e^{-2\pi i N \langle u, x_\nu \rangle}$. Then X_ν, $\nu = 1, \dots, m$ are independent with $|X_\nu| \leq 1$ and $\mathbb{E} X_j = 0$. Thus by Bernstein's inequality (see Corollary A.4) for all $t > 0$, $\mathbb{P}(|\widehat{\mu}(Nu)| \geq t) \leq 4e^{-mt^2/4}$. Setting $t = 2m^{-1/2} \log^{1/2}(4N^{d+h})$, we get $\mathbb{P}\{|\widehat{\mu}(Nu)| \geq t\} \leq N^{-d-h}$. Allowing $u \in \Gamma_N^d$ to vary, we see that $\mathbb{P}\{(3.1) \text{ fails}\} \leq N^{-h}$. ∎

3.3 Regularity of Self-convolutions

We begin with a few elementary observations. Let $\Delta_{j,\ell} = \sigma_j^{*\ell} - \sigma_{j-1}^{*\ell}$, so that

$$\sigma_m^{*\ell} = \sum_{j=1}^{m} \Delta_{j,\ell}. \tag{3.2}$$

Lemma 3.2 (i) *For $j \geq 1$, $\Delta_{j,\ell}$ is a positive measure, and we have for $\ell \geq 2$,*

$$\Delta_{j,\ell} = \delta_{\ell x_j} + \sum_{k=1}^{\ell-1} \binom{\ell}{k} \delta_{(\ell-k)x_j} * \sigma_{j-1}^{*k} \tag{3.3}$$

$$= \delta_{\ell x_j} + \sum_{k=1}^{\ell-1} \binom{\ell}{k} \sum_{1 \leq \nu_1, \dots, \nu_k \leq j-1} \delta_{(\ell-k)x_j + x_{\nu_1} + \cdots + x_{\nu_k}}. \tag{3.4}$$

(ii) *Assume that* $\gcd(\ell!, N) = 1$. *Let $m \geq 2$ and let Q be a cube of sidelength $\geq N^{-1}$. Then for $j_1 < \cdots < j_K$,*

$$\mathbb{P}\big\{ \Delta_{j_1,\ell}(Q) \neq 0, \dots, \Delta_{j_K,\ell}(Q) \neq 0 \big\} \leq (2^{d+1}|Q| m^{\ell-1})^K.$$

In particular, for each $u \in \Gamma_N^d$,

$$\mathbb{P}\big\{ \Delta_{j_1,\ell}(\{u\}) \neq 0, \dots, \Delta_{j_K,\ell}(\{u\}) \neq 0 \big\} \leq (2N^{-d} m^{\ell-1})^K.$$

(iii) *Assume that* $\gcd(\ell!, N) = 1$. *For $j = 0, \dots, m-1$ let $\mathcal{E}_j$ be a given event in $\mathcal{F}_j$. Let*

$$Y_j = \begin{cases} \sigma_j^{*\ell} - \sigma_{j-1}^{*\ell} - N^{-d}(j^\ell - (j-1)^\ell) & \text{on } \mathcal{E}_{j-1}, \\ 0 & \text{on } \mathcal{E}_{j-1}^{\complement}. \end{cases}$$

Then $\mathbb{E}[Y_j | \mathcal{F}_{j-1}] = 0$. Let $W_0 = 0$ and $W_j = \sum_{\nu=1}^{j} Y_\nu$, for $j = 1, \dots, m$. Then $\{W_j\}_{j=0}^m$ is a martingale adapted to the filtration $\{\mathcal{F}_j\}_{j=0}^m$.

Proof Part (i) follows immediately from the binomial formula. For part (ii), note that by the assumption $\gcd(\ell!, N) = 1$, the random variables $(\ell - k)x_j$, $1 \leq k \leq \ell$, are uniformly distributed. Observe that for any fixed a, the probability of the event $\{(\ell-k)x_d - a \in Q\}$ is at most $2^d|Q|$. Thus the probability of the event that

$$(\ell-k)x_d - a \in Q$$

for some choice of $a = x_{\nu_1} + \cdots + x_{\nu_k}$, $1 \le \nu_1, \dots, \nu_k \le j-1$, does not exceed

$$2^d|Q|(j-1)^{k-1}.$$

Hence $\mathbb{P}\{\Delta_{j,\ell}(Q) \ne 0\} \le 2^d|Q|\sum_{\kappa=0}^{\ell-1} m^\kappa \le 2^{d+1}|Q|m^{\ell-1}$. Now the assertion in part (ii) follows. The second assertion in (ii) is proved similarly.

For (iii), clearly $\{W_j\}_{j=0}^m$ is adapted to the filtration $\{\mathcal{F}_j\}_{j=0}^m$. By assumption, the random variable qx_j is uniformly distributed on Γ_N^d, for $1 \le q \le \ell$. Given fixed $x_1, \dots, x_{j-1}$, by (3.4),

$$\mathbb{E}[\sigma_j^{*\ell}(\{u\}) - \sigma_{j-1}^{*\ell}(\{u\})|x_1, \dots, x_{j-1}] \\ = N^{-d}\sum_{q=0}^{\ell-1}\binom{\ell}{q}(j-1)^q = N^{-d}(j^\ell - (j-1)^\ell).$$

Since $\mathcal{E}_{j-1} \in \mathcal{F}_{j-1}$, we get $\mathbb{E}[Y_j \mathbb{1}_{\mathcal{E}_{j-1}}|\mathcal{F}_{j-1}] = 0$ in this case. On $\mathcal{E}_{j-1}^{\complement}$ we have $Y_j = 0$, which also implies $\mathbb{E}[Y_j \mathbb{1}_{E_{j-1}^{\complement}}|\mathcal{F}_{j-1}] = 0$. Hence $\mathbb{E}[Y_j|\mathcal{F}_{j-1}] = 0$ and this shows $\{W_j\}_{j=0}^m$ is a martingale. ∎

We shall use (a small variant of) an elementary inequality from Körner's paper [18, Lemma 11] which is useful for the estimation of sums of independent Bernoulli variables.

Lemma 3.3 ([18]) *Let $0 < p < 1$, $m \ge 2$ and $2mp \le M \le m$. Then*

$$\sum_{k=M}^m \binom{m}{k} p^k \le \frac{2(mp)^M}{M!}.$$

In particular, if $mp \le 1$ and if $Y_1, \dots, Y_m$ are independent random variables with

$$\mathbb{P}\{Y_j = 1\} = p \quad \text{and} \quad \mathbb{P}\{Y_j = 0\} = 1 - p,$$

then $\mathbb{P}\{\sum_{j=1}^m Y_j \ge M\} \le \frac{2(mp)^M}{M!}$.

Proof Set $u_k = \binom{m}{k}p^k$. Then $u_{k+1}/u_k = \frac{m-k}{k+1}p \le \frac{mp}{k+1} \le \frac{1}{2}$ for $k \ge M$ and thus the sum is estimated by

$$\sum_{k\ge M} u_k \le 2u_M \le \frac{2(mp)^M}{M!}.$$

The second assertion follows since $\mathbb{P}\{\sum_{j=1}^m Y_j \ge M\} = \sum_{k=M}^m \mathbb{P}\{\sum_{j=1}^m Y_j = k\} \le \sum_{k=M}^m u_k$. ∎

For $\ell = 0, 1, 2, \dots$, $0 < \varepsilon < d$ and $h \in \mathbb{N}$, define recursively positive numbers $M(\ell, \varepsilon, h)$ by

$$M(0,\varepsilon,h) = 1 \quad \text{and} \quad M(\ell,\varepsilon,h) = U(\varepsilon,h)\kappa(\ell,h), \quad \ell \ge 1, \tag{3.5}$$

https://doi.org/10.4153/CJM-2016-019-6 Published online by Cambridge University Press

where

$$U(\varepsilon,h) := \max\{\lfloor e^{d+2}\rfloor, \lceil \varepsilon^{-1}(2d+h+1)\rceil\}, \tag{3.6}$$

$$\kappa(\ell,h) := \sum_{q=0}^{\ell-1}\binom{\ell}{q} M(q, d(1-q/\ell), h+1)\,. \tag{3.7}$$

The growth of these constants as functions of ℓ and h is irrelevant for our purposes. For the sake of completeness we give an upper bound.

Lemma 3.4 *Let $\ell\in\mathbb{N}\cup\{0\}$, $0<\varepsilon<d$, and $h\in\mathbb{N}$. The numbers defined in (3.5)–(3.7) satisfy $M(\ell,\varepsilon,h)\le \varepsilon^{-1}(e^{d+3}\ell^2(h+\ell))^{\ell}$.*

Proof We argue by induction with the case $\ell=0$ being trivial. For the induction step we use $\binom{\ell}{q}=\frac{\ell}{\ell-q}\binom{\ell-1}{q}$ and estimate

$$\begin{aligned}\kappa(\ell,h) &\le 1+\sum_{q=1}^{\ell-1}\frac{\ell}{\ell-q}\binom{\ell-1}{q}\frac{(e^{d+3}q^2(h+1+q))^q}{d(1-\frac{q}{\ell})}\\ &\le \ell^2\sum_{q=0}^{\ell-1}\binom{\ell-1}{q}(e^{d+3}(\ell-1)^2(h+\ell))^q\\ &\le \ell^2\big(e^{d+3}(\ell-1)^2(h+\ell)+1\big)^{\ell-1},\end{aligned} \tag{3.8}$$

where in the last line we have used $(1+x)^{1/x}\le e^x$ for $0<x<1$. Thus

$$\kappa(\ell,h)\le e^{1/2}\ell^2(e^{d+3}\ell^2(h+\ell))^{\ell-1}. \tag{3.9}$$

Now one checks that $U(\varepsilon,h)\le e^{d+2}h\varepsilon^{-1}$ and (3.9) yields for $\ell\ge1$

$$M(\ell,\varepsilon,h)\le e^{d+2}h\varepsilon^{-1}\kappa(\ell,h)\le \varepsilon^{-1}(e^{d+3}\ell^2(h+\ell))^{\ell}. \qquad\blacksquare$$

Lemma 3.5 *Let $\ell\in\mathbb{N}\cup\{0\}$, $0<\varepsilon<d$, and $h\in\mathbb{N}$. Let $M(\ell,\varepsilon,h)$ be as in (3.5). Let N be an integer such that $N>2\ell$ and $\gcd(N,\ell!)=1$. Let $E_m(\ell,\varepsilon,h)$ be the event that $\sigma_m^{*\ell}(Q)\le M(\ell,\varepsilon,h)$ holds for all cubes of measure at most $m^{-\ell}N^{-\varepsilon}$, and let $E(\ell,\varepsilon,h)$ be the intersection of the $E_m(\ell,\varepsilon,h)$ where $m\le N^{\frac{d-\varepsilon}{\ell}}$. Then $E(\ell,\varepsilon,h)$ has probability at least $1-N^{-h}$.*

Proof We argue again by induction on ℓ. If $\ell=0$, then $\sigma^{*0}=\delta_0$, and the statement clearly holds with $M(0,\varepsilon,h)=1$, for $\varepsilon\ge0$ and $h\in\mathbb{N}$. Assume that the statements hold for $0,1,\dots,\ell-1$; we prove that it also holds for ℓ. Let

$$F\equiv F(\ell-1,h)=\bigcap_{q=1}^{\ell-1}E(q,\varepsilon_{q,\ell},h+1),\quad \text{with } \varepsilon_{q,\ell}=d\Big(1-\frac{q}{\ell}\Big).$$

By the induction hypothesis, the event $F^{\complement}$ has probability at most $\ell N^{-h-1}\le\frac12N^{-h}$ since we assume $N>2\ell$. We now proceed to estimate the probability of $E(\ell,\varepsilon,h)^{\complement}\cap F$.

Fix $m\le N^{\frac{d-\varepsilon}{\ell}}$, and fix a cube Q, with $N^{-d}\le|Q|\le N^{-\varepsilon}m^{-\ell}$. Notice that

$$\frac{d}{\ell}=\frac{d-\varepsilon_{q,\ell}}{q}.$$

Therefore, if $\kappa(\ell,h)$ is as in (3.12), we see, using (3.3), that $\Delta_{j,\ell}(Q) \le \kappa(\ell,h)$ holds on F, for $j = 1,\dots,m$. Now let $U \ge 2^{d+2}$ be an integer and let $\mathcal{A}^Q_{U,m}$ be the event that

$$\sum_{j=1}^{m} \Delta_{j,\ell}(Q) \ge U\kappa(\ell,h). \tag{3.10}$$

Now by (3.2) and (3.5) the event $E_m(\ell,\varepsilon,h)^{\complement}$ is contained in the union over the $\mathcal{A}^Q_{U(\varepsilon,h),m}$ when Q ranges over the cubes with measure at most $N^{-\varepsilon}m^{-\ell}$. Let $\mathfrak{Q}$ be the collection of all cubes of measure $N^{-\varepsilon}m^{-\ell}$, that have corners in Γ_N^d. Then $\#(\mathfrak{Q}) \le (2N)^d$. Notice that every cube of measure less than $N^{-\varepsilon}m^{-\ell}$ is contained in at most 3^d cubes in $\mathfrak{Q}$. Hence

$$\mathbb{P}(E_m(\ell,\varepsilon,h)^{\complement} \cap F) \le (6N)^d \max_{Q\in\mathfrak{Q}} \mathbb{P}(\mathcal{A}^Q_{U(\varepsilon,h),m} \cap F). \tag{3.11}$$

Now in order to estimate $\mathbb{P}(\mathcal{A}^Q_{U,m} \cap F)$, we observe that if (3.10) holds on F, then there are at least U indices j with $\Delta_{j,\ell}(Q) \ne 0$. Thus we may assume $m \ge U$. Now we see from Lemma 3.2 (ii), that for $U \le k \le m$ and for any choice of indices $1 \le j_1 < \cdots < j_k \le m$, $\mathbb{P}\{\Delta_{j_\nu,\ell}(Q) \ne 0,\ \nu = 1,\dots,k\} \le (2^{d+1}|Q|m^{\ell-1})^k$. Thus

$$\mathbb{P}(\mathcal{A}^Q_{U,m} \cap F) \le \sum_{k=U}^{m} \binom{m}{k} (2^{d+1}|Q|m^{\ell-1})^k.$$

Now let $p := 2^{d+1}|Q|m^{\ell-1}$. Since $|Q| < m^{-\ell}$, we have $mp \le 2^{d+1}$. Since we assume $U \ge 2^{d+2}$, we get from Lemma 3.3,

$$\sum_{k=U}^{m} \binom{m}{k} \big(2^{d+1}|Q|m^{\ell-1}\big)^k \le \frac{2(mp)^U}{U!} \le \frac{2(2^{d+1}N^{-\varepsilon})^U}{U!}.$$

Thus we get from (3.11)

$$\mathbb{P}(E_m(\ell,\varepsilon,h)^{\complement} \cap F) \le (6N)^d \frac{2(2^{d+1}N^{-\varepsilon})^{U(\varepsilon,h)}}{U(\varepsilon,h)!}.$$

It is not difficult to check that $\frac{6^d\cdot 2(2^{d+1})^U}{U!} \le 1$ for $U > e^{d+2} - 1$; this can be verified by taking logarithms and replacing $\log U$ with the smaller constant $\int_1^{U-1} \ln(t)\,dt$. Since in addition $U = U(\varepsilon,h) \ge \lceil \frac{2d+h+1}{\varepsilon} \rceil$, we then get $N^{d-\varepsilon U} \le \frac{1}{2}N^{-d-h}$ and thus

$$\mathbb{P}(E_m(\ell,\varepsilon,h)^{\complement} \cap F) \le \frac{1}{2}N^{-h-d}.$$

We have already remarked that $\mathbb{P}(F^{\complement}) < \frac{1}{2}N^{-h}$. Thus,

$$\mathbb{P}\big(E(\ell,\varepsilon,h)^{\complement}\big) \le \mathbb{P}(F^{\complement}) + \sum_{m\le N^{(d-\varepsilon)/\ell}} \mathbb{P}\big(E_m(\ell,\varepsilon,h)^{\complement} \cap F\big) \le N^{-h}.$$

This completes the proof. ∎

Lemma 3.6 *Let $\ell \in \mathbb{N}$, $0 < \beta \le d/\ell$, and $h \in \mathbb{N}$. Let N be an integer such that $N > \max\{2\ell, e^{e^e}\}$ and $\gcd(N,\ell!) = 1$. Let $\mathcal{E}_m(\ell,\beta,h)$ denote the event that*

$$\sigma_m^{*\ell}(Q) \le (\beta\ell)^{-1}\big(10^{d+1}\ell^2(\ell+h)\big)^{\ell} \frac{\log N}{\log\log N}$$

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holds for all cubes of measure at most $N^{-\beta\ell}$, *and let* $\mathcal{E}(\ell,\beta,h) = \bigcap_{m\le N^\beta}\mathcal{E}_m(\ell,\beta,h)$. *Then* $\mathcal{E}(\ell,\beta,h)$ *has probability at least* $1-N^{-h}$.

Proof Let

$$\widetilde{\kappa}(\ell,\beta,h) := \sum_{q=0}^{\ell-1}\binom{\ell}{q}M(q,\beta(\ell-q),h+1) \tag{3.12}$$

and let $V \ge e^{2d+8}h$ be a positive integer. Let $\widetilde{E}_m(\ell,h,V)$ denote the event that

$$\sigma_m^{*\ell}(Q) \le \widetilde{\kappa}(\ell,\beta,h)V\frac{\log N}{\log\log N}$$

holds true for all cubes with measure at most $N^{-\ell\beta}$. We shall show that for sufficiently large V, the complement of this event has small probability.

We condition on the event

$$\begin{aligned}\widetilde{F} &= \bigcap_{q=1}^{\ell-1}\mathcal{E}(q,\beta(\ell-q),h+1)\\ &= \Big\{\sigma_m^{*q}(Q) \le M(q,\beta(\ell-q),h+1)\\ &\qquad\qquad \forall Q \text{ with } |Q| \le m^{-q}N^{-\beta(\ell-q)}, 1\le m\le N^{\frac{d-\beta(\ell-q)}{q}}\Big\}.\end{aligned}$$

By Lemma 3.5, $\mathbb{P}(\widetilde{F}^{\complement}) \le \ell N^{-h-1} \le \frac{1}{2}N^{-h}$.

We shall now estimate $\mathbb{P}(\widetilde{E}_m(\ell,h,V)^{\complement}\cap\widetilde{F})$. The assumptions $m \le N^\beta$, $|Q| \le N^{-\ell\beta}$ with $\beta \le d/\ell$ imply for $q \le \ell-1$ that $m \le N^{\frac{d-\beta(\ell-q)}{q}}$ (since $d-\beta\ell \ge 0$) and $|Q| \le m^{-q}N^{-(\ell-q)\beta}$. Thus we can use (3.3) to see that $\Delta_{j,\ell}(Q) \le \widetilde{\kappa}(\ell,\beta,h)$ on $\widetilde{F}$, for $j = 1,\dots,m$.

Let $\widetilde{\mathcal{A}}^Q_{V,m}$ be the event that

$$\sum_{j=1}^{m}\Delta_{j,\ell}(Q) \ge V_N\widetilde{\kappa}(\ell,\beta,h),\ \text{where } V_N = \Big\lfloor V\frac{\log N}{\log\log N}\Big\rfloor. \tag{3.13}$$

Let $\widetilde{\mathcal{A}}_{V,m}$ be the event that (3.13) holds for some cube with measure at most $N^{-\ell\beta}$. Arguing as in the proof of Lemma 3.5 we find that

$$\mathbb{P}(\widetilde{\mathcal{A}}_{V,m}\cap\widetilde{F}) \le 2\cdot(6N)^d\frac{2^{(d+1)V_N}}{V_N!}.$$

We need to verify that

$$2\cdot(6N)^d\frac{2^{(d+1)V_N}}{V_N!} \le N^{-d-1-h} \tag{3.14}$$

for $V \ge e^{2d+8}h$ and $N > e^{e^e}$. We take logarithms and replace $\log V_N!$ with the lower bound $\int_1^{V_N-1}\log t\,dt = (V_N-1)\log(V_N-1) - V_N + 2$. Then (3.14) follows from

$$\begin{aligned}\log 2 + d\log 6 + V_N(1+(d+1)\log 2) - 2 - (V_N-1)\log(V_N-1)\\ < -(h+1+d)\log N.\end{aligned} \tag{3.15}$$

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Since, by assumption, $V \geq e^{2d+10}$ and $N > e^{e^e}$, crude estimates show that (3.15) is implied by

$$\frac{V_N}{2}\log(V_N - 1) \geq (d+h+1)\log N. \tag{3.16}$$

For $N \geq e^{e^e}$ we have $\log\log\log N \leq \frac{1}{2}\log\log N$ and therefore $\log(V_N - 1) \geq \frac{1}{2}\log\log N$. Thus (3.16) is implied by $V \geq 4(h+2+d)$, which holds since we assume $V \geq e^{2d+8}h$ and $N \geq e^{e^e}$. Thus (3.14) holds. We thus get $\mathbb{P}(\widetilde{E}_m(\ell,h,V)^{\mathsf{C}} \cap \widetilde{F}) \lesssim N^{-d-h-1}$ and hence

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{m\leq N^\beta} \widetilde{E}_m(\ell,h,V)^{\mathsf{C}}\Big) &\lesssim \mathbb{P}(\widetilde{F}^{\mathsf{C}}) + \sum_{m\leq N^\beta} \mathbb{P}(\widetilde{E}_m(\ell,h,V)^{\mathsf{C}} \cap \widetilde{F}) \\ &\lesssim \tfrac{1}{2}N^{-h} + N^{\beta-d-h-1} \leq N^{-h}.\end{aligned}$$

It remains to show that

$$V\widetilde{\kappa}(\ell,\beta,h) \leq \frac{1}{\beta\ell}\big(10^{d+1}\ell^2(\ell+h)\big)^{\ell} \tag{3.17}$$

for $V = e^{2d+10}$. For $\widetilde{\kappa}(\ell,\beta,h)$ we have, by Lemma 3.4,

$$\widetilde{\kappa}(\ell,\beta,h) \leq 1 + \sum_{q=1}^{\ell-1} \frac{\ell}{\ell-q}\binom{\ell-1}{q}\frac{(e^{d+3}q^2(h+1+q))^q}{\beta(\ell-q)}$$

and the right-hand side is estimated by $(\beta\ell)^{-1}\kappa_*(\ell,h)$, where $\kappa_*(\ell,h)$ is the expression in line (3.8). The estimation that follows in the proof of Lemma 3.4 yields

$$\widetilde{\kappa}(\ell,\beta,h) \leq e^{1/2}\frac{\ell}{\beta}(e^{d+3}\ell^2(h+\ell))^{\ell-1}$$

and thus clearly (3.17) follows. ∎

Lemma 3.7 *Let $\ell \in \mathbb{N}$, $h \in \mathbb{N}$, and $B \geq 1$. There exist positive constants $N_0(B,\ell)$ and $M_0(B,\ell,h,d)$ so that for $N \geq N_0(\ell,B)$, the event*

$$\max_{m\leq (BN^d\log N)^{1/\ell}} \max_{u\in\Gamma_N^d} \sigma_m^{*\ell}(\{u\}) \leq M_0(\ell,B,h,d)\log N$$

has probability at least $1 - N^{-h}$.

Proof If $\ell \geq 2$, we may assume that

$$B\log N \leq N^{\frac{d}{2(\ell-1)}} \text{ for } N \geq N_0(\ell,B). \tag{3.18}$$

Let

$$\widehat{\kappa}(\ell,h) := \sum_{q=0}^{\ell-1}\binom{\ell}{q} M\Big(q, \frac{d}{2}\Big(1-\frac{q}{\ell}\Big), h+1\Big).$$

and let

$$\mathcal{V} \geq 2d + h + 1 + 20B. \tag{3.19}$$

Let $\widehat{E}_m(\ell,h,\mathcal{V})$ denote the event that $\sigma_m^{*\ell}(\{u\}) \leq \widehat{\kappa}(\ell,h)\mathcal{V}\log N$ holds true for all $u \in \Gamma_N^d$. We condition on the event $\widehat{F} = \bigcap_{q=1}^{\ell-1} E(q, \frac{d}{2}(1-\frac{q}{\ell}), h+1)$, again with the sets

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on the right-hand side defined as in the statement of Lemma 3.5. Then the event $\widehat{F}^{\complement}$ has probability at most $\ell N^{-h-1} \le \frac{1}{2} N^{-h}$.

It remains to estimate $\sum_{m \le (BN^d \log N)^{1/\ell}} \mathbb{P}(\widehat{E}_m(\ell, h, V)^{\complement} \cap \widehat{F})$. If we apply the condition $E(q, \frac{d}{2}(1-\frac{q}{\ell}), h+1)$ only for cubes of measure N^{-d}, then we see that

$$\sigma_m^{*q}(\{u\}) \le M\Big(q, \frac{d}{2}\Big(1-\frac{q}{\ell}\Big), h+1\Big), \quad m \le N^{\frac{d}{2q}+\frac{d}{2\ell}}, 1 \le q \le \ell - 1.$$

In order to apply it for all $m \le (BN^d \log N)^{1/\ell}$, we must have $(BN^d \log N)^{1/\ell} \le N^{\frac{d}{2q}+\frac{d}{2\ell}}$ which is implied by (3.18).

By (3.3) we have $\Delta_{j,\ell}(\{u\}) \le \widehat{\kappa}(\ell, h)$ on $\widehat{F}$, for $j = 1, \dots, m$. Let $\widehat{\mathcal{A}}^u_{\mathcal{V},m}$ be the event that

$$\sigma_m^{*\ell}(\{u\}) \equiv \sum_{j=1}^m \Delta_{j,\ell}(\{u\}) \ge \mathcal{V}_N \widehat{\kappa}(\ell, h), \quad \text{where } \mathcal{V}_N = \lfloor \mathcal{V} \log N \rfloor, \tag{3.20}$$

and let $\widehat{\mathcal{A}}_{\mathcal{V},m}$ be the event that (3.20) holds for all $u \in \Gamma_N^d$.

Now we estimate $\widehat{\mathcal{A}}^u_{\mathcal{V},m}$ on $\widehat{F}$. Notice that if (3.20) holds on $\widehat{F}$, there are at least $\mathcal{V}_N$ indices j so that $\Delta_{j,\ell}(\{u\}) \ne 0$ (and we may assume $m \ge \mathcal{V}_N$). We argue as in the proof of Lemma 3.5 using Lemma 3.2 (ii) to see that

$$\mathbb{P}(\widehat{\mathcal{A}}^u_{\mathcal{V},m} \cap \widehat{F}) \le \sum_{k=\mathcal{V}}^m \binom{m}{k} (2N^{-d} m^{\ell-1})^k .$$

In order to apply Lemma 3.3 we must have $\mathcal{V}_N \ge 2mp$ with $p = 2N^{-d} m^{\ell-1}$, and this is certainly satisfied if $\mathcal{V} \ge 8B$. Under this condition we thus get

$$\mathbb{P}(\widehat{\mathcal{A}}^u_{\mathcal{V},m} \cap \widehat{F}) \le \frac{2(2N^{-d} m^\ell)^{\mathcal{V}_N}}{\mathcal{V}_N!} \le \frac{2(2B \log N)^{\mathcal{V}_N}}{\mathcal{V}_N!}.$$

We use the inequality

$$\frac{T^n}{n!} \le e^{-n}, \text{ for } T \ge 1 \text{ and } n \ge e^2 T. \tag{3.21}$$

To verify this, one takes logarithms and uses $\log(n!) \ge n \log n - n + 1$. Thus the inequality follows from $n(\log T - \log n) \le -2n$ which is true for $n \ge e^2 T$.

We apply (3.21) with $T = 2B \log N$ and $n = \mathcal{V}_N$. Note that by the assumption (3.19) we get $\mathcal{V}_N \ge e^2 T$. Therefore

$$\frac{2(2B \log N)^{\mathcal{V}_N}}{\mathcal{V}_N!} \le 2e^{-\mathcal{V}_N} \le 2e^{1-\mathcal{V} \log N} \le N^{-(2d+h+1+10B)}.$$

Thus

$$\begin{aligned}\mathbb{P}\Big(\bigcup_{m \le (BN^d \log N)^{1/\ell}} \widehat{E}_m(\ell, h, \mathcal{V})^{\complement}\Big) &\le \mathbb{P}(\widehat{F}^{\complement}) + \sum_{m \le (BN^d \log N)^{1/\ell}} \sum_{u \in \Gamma_N^d} \mathbb{P}(\widehat{\mathcal{A}}^u_{\mathcal{V},m} \cap \widehat{F}) \\ &\le \frac{1}{2} N^{-h} + (BN^d \log N)^{1/\ell} N^d N^{-10B} N^{-2d-h-1} \\ &\le N^{-h}\end{aligned}$$

and we get the assertion of the lemma. ∎

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Remark It is also possible to give a proof of Lemma 3.7 based on the second version of Hoeffding's inequality (A.2) in the appendix (*cf.* [7]).

The following proposition can be seen as a discrete analog to Theorem C (iv).

Proposition 3.8 *Given integers $\kappa \geq 1$, $\ell \geq \kappa + 1$ and $h \geq 1$, there exists $N_\kappa(\ell, h) \geq 1$ and $M_\kappa(\ell, h, d) > 0$ such that for all $N \geq N_\kappa(\ell, h)$ with $\gcd(\ell!, N) = 1$, the event*

$$\max_{m \leq (N^d \log N)^{\frac{1}{\ell - \kappa}}} \max_{u \in \Gamma_N^d} \frac{\left|\sigma_m^{*\ell}(\{u\}) - m^\ell N^{-d}\right|}{(m^\ell N^{-d})^{1/2}} \leq M_\kappa(\ell, h, d)(\log N)^{1+\frac{\kappa}{2}} \tag{3.22}$$

has probability at least $1 - N^{-h}$.

Proof We prove this by induction on κ.

The case $\kappa = 1$. Let $B_0 \geq d + h + 1$, sufficiently large. We first remark that for $m^\ell \leq B_0 N^d \log N$, inequality (3.22) is implied by Lemma 3.7, provided that N is sufficiently large. We thus may assume that

$$m \geq (B_0 N^d \log N)^{1/\ell}. \tag{3.23}$$

Following [18], we will treat the telescopic sums

$$\sigma_m^\ell(\{u\}) - m^\ell N^{-d} = \sum_{j=1}^m \sigma_j^{*\ell} - \sigma_{j-1}^{*\ell} - N^{-d}(j^\ell - (j-1)^\ell)$$

as a sum of martingale differences with respect to the filtration of σ-algebras $\mathcal{F}_j$, with $\mathcal{F}_j$ generated by the random variables $x_1, \dots, x_j$; see Lemma 3.2 (iii).

By Lemma 3.7, there is a constant $M_0 = M_0(\ell, B_0, h, d)$ so that

$$\mathbb{P}\Big(\max_{1 \leq q \leq \ell - 1} \max_{1 \leq j \leq (B_0 N^d \log N)^{1/q}} \max_{u \in \Gamma_N^d} \sigma_j^{*q}(\{u\}) \leq M_0 \log N\Big) \geq 1 - N^{-2(d+h+1)},$$

provided that N is large enough. Note that

$$(B_0 N^d \log N)^{1/\ell} \leq \min_{1 \leq q \leq \ell - 1} (B_0 N^d \log N)^{1/q},$$

provided that N is large enough. Let $\mathcal{E}_{j-1}$ denote the event

$$\mathcal{E}_{j-1} = \left\{\sigma_{j-1}^{*q}(\{u\}) \leq M_0 \log N \text{ for } 1 \leq q \leq \ell - 1 \text{ and all } u \in \Gamma_N^d\right\}. \tag{3.24}$$

Then $\mathbb{P}\big(\bigcup_{1 \leq j \leq (N^d \log N)^{\frac{1}{\ell-1}}} \mathcal{E}_j^{\complement}\big) \leq N^{-2(d+h+1)}$. Define for fixed $u \in \Gamma_N^d$

$$Y_j \equiv Y_{j,u} := \begin{cases} \sigma_j^{*\ell} - \sigma_{j-1}^{*\ell} - N^{-d}(j^\ell - (j-1)^\ell) & \text{on } \mathcal{E}_{j-1}, \\ 0 & \text{on } \mathcal{E}_{j-1}^{\complement}. \end{cases}$$

We shall apply Lemma 3.2 (iii) to the martingale $\{W_j\}_{j=0}^m$ with $W_0 = 0$ and $W_j = \sum_{\nu=1}^j Y_\nu$ for $j \geq 1$. We prepare for an application of Hoeffding's inequality (Lemma A.1) and estimate the conditional expectation of $e^{\lambda Y_j}$ given fixed $x_1, \dots, x_{j-1}$.

Claim: For $|\lambda| \leq (2^\ell M_0 \log N)^{-1}$,

$$\mathbb{E}[e^{\lambda Y_j} | x_1, \dots, x_{j-1}] \leq \exp\left(3 m^{\ell-1} N^{-d} (2^\ell M_0)^2 (\log N)^2 \lambda^2\right). \tag{3.25}$$

Proof of (3.25). Given $(x_1,\dots,x_{j-1})$, if inequality (3.24) does not hold, then we have $Y_j = 0$ and thus $\mathbb{E}[e^{\lambda Y_j}|x_1,\dots,x_{j-1}] = 1$. Thus in this case (3.25) holds trivially. We thus need to bound (3.25) on $\mathcal{E}_{j-1}$. First observe

$$N^{-d}(j^\ell - (j-1)^\ell) \le \ell j^{\ell-1}N^{-d} \le \ell m^{\ell-1}N^{-d} \le \ell \log N$$

by assumption. By (3.3) and (3.24),

$$\sigma_j^{*\ell}(\{u\}) - \sigma_{j-1}^{*\ell}(\{u\}) \le \sum_{k=0}^{\ell-1}\binom{\ell}{k} M_0 \log N \le 2^\ell M_0 \log N.$$

Hence we get $|Y_j| \le 2^\ell M_0 \log N$. On the other hand, writing $Z_j = \Delta_{j,\ell}(\{u\}) = \sigma_j^{*\ell}(\{u\}) - \sigma_{j-1}^{*\ell}(\{u\})$, we have, by (3.4),

$$\mathbb{P}(Z_j \ne 0|x_1,\dots,x_{j-1}) \le N^{-d}\sum_{k=0}^{\ell-1}(j-1)^k \le 2m^{\ell-1}N^{-d}.$$

We use these observations to estimate, for $0 < |\lambda| \le (2^\ell M_0 \log N)^{-1}$, the term $\mathbb{E}[e^{\lambda Y_j}]$, which in the following calculation is an abbreviation for the expectation conditional on $x_1,\dots,x_{j-1}$. Since the expectation of Y_j with respect to x_j is zero, we obtain

$$\begin{aligned}\mathbb{E}[e^{\lambda Y_j}] &= \sum_{k=0}^{\infty}\frac{\lambda^k\mathbb{E}[Y_j^k]}{k!} = 1 + \sum_{k=2}^{\infty}\frac{\lambda^k\mathbb{E}[Y_j^k]}{k!}\\ &= 1 + \mathbb{P}(Z_j = 0)\sum_{k=2}^{\infty}\frac{|\lambda|^k\mathbb{E}\big[|Y_j|^k \mid Z_j = 0\big]}{k!}\\ &\quad + \mathbb{P}(Z_j \ne 0)\sum_{k=2}^{\infty}\frac{|\lambda|^k\mathbb{E}\big[|Y_j|^k \mid Z_j \ne 0\big]}{k!}.\end{aligned}$$

We have $m^{\ell-1}N^{-d} \le \log N$ and thus

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{|\lambda|^k\mathbb{E}\big[|Y_j|^k \mid Z_j = 0\big]}{k!} &\le \sum_{k=2}^{\infty}\frac{(|\lambda|\ell m^{\ell-1}N^{-d})^k}{k!}\\ &\le (\lambda\ell m^{\ell-1}N^{-d})^2\sum_{k=2}^{\infty}\frac{|\lambda\ell\log N|^k}{k!} \le (\lambda\ell m^{\ell-1}N^{-d})^2.\end{aligned}$$

Also

$$\begin{aligned}\mathbb{P}(Z_j \ne 0)\sum_{k=2}^{\infty}\frac{|\lambda|^k\mathbb{E}\big[|Y_j|^k \mid Z_j \ne 0\big]}{k!} &\le \mathbb{P}(Z_j \ne 0)\sum_{k=2}^{\infty}\frac{|\lambda 2^\ell M_0\log N|^k}{k!}\\ &\le 2m^{\ell-1}N^{-d}(\lambda 2^\ell M_0 \log N)^2.\end{aligned}$$

Combining the two estimates we get

$$\begin{aligned}\mathbb{E}[e^{\lambda Y_j} \mid x_1,\dots,x_{j-1}] &\le 1 + 3m^{\ell-1}N^{-d}(2^\ell M_0\log N)^2\\ &\le \exp(3m^{\ell-1}N^{-d}(2^\ell M_0 \log N)^2),\end{aligned}$$

thus proving (3.25).

We now apply Hoeffding's inequality (*cf.* (A.1)) with the parameters

$$a_j^2 = 6m^{\ell-1}N^{-d}(2^\ell M_0 \log N)^2,$$
$$A = \sum_{j=1}^m a_j^2 = 6m^\ell N^{-d}(2^\ell M_0 \log N)^2,$$
$$\delta = (2^\ell M_0 \log N)^{-1},$$
$$t = 2\sqrt{A(d+h+1)\log N} = M_1(m^\ell N^{-d})^{1/2}(\log N)^{3/2},$$

where $M_1 = M_0 2^\ell \sqrt{24(d+h+1)}$. For (A.1) to hold, we must have $t \le A\delta$ which one checks to be equivalent with $(d+h+1)\log N \le \frac{3}{2} m^\ell N^{-d}$, and thus valid by (3.23). Thus, by (A.1),

$$\begin{aligned}\mathbb{P}\Big(\Big|\sum_{j=1}^m Y_{j,u}\Big| \ge M_1(m^\ell N^{-d})^{1/2}(\log N)^{3/2}\Big) &\le 2\exp(-t^2/2A)\\ &= 2\exp(-2(d+h+1)\log N)\\ &= 2N^{-2(d+h+1)}.\end{aligned}$$

Allowing $u \in \Gamma_N^d$ and $m \le (N^d \log N)^{\frac{1}{\ell-1}}$ to vary, we see that

$$\mathbb{P}\Big(\max_{u\in\Gamma_N^d} \max_{m\le(N^d\log N)^{\frac{1}{\ell-1}}} \frac{\big|\sum_{j=1}^m Y_{j,u}\big|}{(m^\ell N^{-d})^{1/2}} \ge M_1(\log N)^{3/2}\Big)$$
$$\le 2N^{-d-2h-2}N^d(N^d\log N)^{\frac{1}{\ell-1}} \le N^{-2h-1}$$

if N is large enough. Now $\sigma_m^{*\ell}(\{u\}) - m^\ell N^{-d} - \sum_{j-1}^m Y_{j,u} = 0$ on $\bigcap_{1\le j\le m} \mathcal{E}_{j-1}$ and thus

$$\mathbb{P}\Big(\max_{u\in\Gamma_N^d} \max_{m\le(N^d\log N)^{\frac{1}{\ell-1}}} \Big|\sigma_m^{*\ell}(\{u\}) - m^\ell N^{-d} - \sum_{j=1}^m Y_{j,u}\Big| \ne 0\Big)$$
$$\le \sum_{1\le j-1\le(N^d\log N)^{\frac{1}{\ell-1}}} \mathbb{P}(\mathcal{E}_{j-1}^{\complement}) \le (N^d\log N)^{\frac{1}{\ell-1}}N^{-2(d+h+1)} \le N^{-2h-1}$$

if N is large enough. This establishes the assertion for $\kappa = 1$.

The induction step. We now assume $\kappa \ge 2$, $\ell \ge \kappa + 1$, and that the assertion holds for $1 \le \kappa' < \kappa$. Let $h \ge 1$ and fix j with $1 \le j \le (N^d \log N)^{\frac{1}{\ell-\kappa}}$.

Define the event $E_{j-1} = E_{j-1}(\ell, \kappa - 1, N)$ by the following three conditions:

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$$\max_{u \in \Gamma_N^d} \sigma_{j-1}^{*q}(\{u\}) \le \mathcal{C} \log N \text{ for } 1 \le q \le \ell - \kappa. \tag{3.26}$$

$$\max_{u \in \Gamma_N^d} \sigma_{j-1}^{*q}(\{u\}) \le \mathcal{C} \log N \tag{3.27}$$

for those q with $\ell - \kappa + 1 \le q \le \ell - 1$, and $j - 1 \le (N^d \log N)^{1/q}$.

$$\max_{u \in \Gamma_N^d} \Big| \sigma_{j-1}^{*q}(\{u\}) - \frac{(j-1)^q}{N^d} \Big| \le \mathcal{C} \Big(\frac{(j-1)^q}{N^d} \Big)^{1/2} (\log N)^{1+\frac{\kappa'}{2}} \tag{3.28}$$

for those q, κ' with $\kappa' < \kappa$, $q \le \ell$,

$(N^d \log N)^{\frac{1}{q-\kappa'+1}} \le j - 1 \le (N^d \log N)^{\frac{1}{q-\kappa'}}$.

Then by Lemma 3.7 and by the induction hypothesis, there exist $N_{\kappa-1} = N_{\kappa-1}(\ell)$ and $\mathcal{C} = \mathcal{C}_{\kappa-1}(\ell, h, d) \ge 1$ so that for all $N \ge N_{\kappa-1}$, the event E_{j-1} has probability at least $1 - N^{-2(h+d+1)}$. We define

$$\Upsilon_j \equiv \Upsilon_{j,u} := \begin{cases} \sigma_j^{*\ell}(\{u\}) - \sigma_{j-1}^{*\ell}(\{u\}) - N^{-d}(j^\ell - (j-1)^\ell) & \text{on } E_{j-1}, \\ 0 & \text{on } E_{j-1}^\complement \end{cases}$$

and claim that

$$|\Upsilon_{j,u}| \le \mathcal{C} 2^{\ell+2} \Big(\frac{m^{\ell-1}}{N^d} \Big)^{1/2} (\log N)^{\frac{\kappa+1}{2}}. \tag{3.29}$$

To see (3.29) we decompose using (3.3)

$$\sigma_j^{*\ell} - \sigma_{j-1}^{*\ell} - \frac{j^\ell - (j-1)^\ell}{N^d} = \sum_{q=0}^{\ell-\kappa} \binom{\ell}{q} \delta_{(\ell-q)x_j} * \sigma_{j-1}^{*q} - \sum_{q=0}^{\ell-\kappa} \binom{\ell}{q} \frac{(j-1)^q}{N^d} + \sum_{q=\ell-\kappa+1}^{\ell-1} \binom{\ell}{q} \delta_{(\ell-q)x_j} * \Big(\sigma_{j-1}^{*q} - \frac{(j-1)^q}{N^d} \Big).$$

Now we have $m \le (N^d \log N)^{\frac{1}{\ell-\kappa}}$ and thus $\sum_{q=0}^{\ell-\kappa} \binom{\ell}{q} \frac{(j-1)^q}{N^d} \le 2^\ell m^\ell N^{-d} \le 2^\ell \log N$. On $E_{j-1}^\complement$ we have by (3.26) $\sum_{q=0}^{\ell-\kappa} \binom{\ell}{q} \delta_{(\ell-q)x_j} * \sigma_{j-1}^{*q}(\{u\}) \le 2^\ell \mathcal{C} \log N$. If $\ell - \kappa + 1 \le q \le \ell - 1$, each j with $j - 1 \le (N^d \log N)^{\frac{1}{\ell-\kappa}}$ satisfies either $(j-1) \le (N^d \log N)^{1/q}$ or $(N^d \log N)^{\frac{1}{q-\kappa'+1}} < j - 1 \le (N^d \log N)^{\frac{1}{q-\kappa'}}$ for some κ' with $1 \le \kappa' < \kappa$. If

$$(j-1) \le (N^d \log N)^{1/q},$$

we use (3.27) to bound $|\sigma_{j-1}^{*q}\{u\} - \frac{(j-1)^q}{N^d}|$ by $(\mathcal{C}+1)\log N$. If

$$(N^d \log N)^{\frac{1}{q-\kappa'+1}} < j - 1 \le (N^d \log N)^{\frac{1}{q-\kappa'}},$$

we use (3.28) to bound $|\sigma_{j-1}^{*q}\{u\} - \frac{(j-1)^q}{N^d}|$ by $\mathcal{C}((j-1)^q N^{-d})^{1/2} (\log N)^{1+\kappa'/2}$ and hence by $\mathcal{C}(m^{\ell-1} N^d)^{1/2} (\log N)^{\frac{\kappa+1}{2}}$. Now sum and combine everything to get (3.29).

Now given (3.29) we can apply the Azuma–Hoeffding inequality (Corollary A.3) with

$$a_j = 2^{\ell+2}\mathcal{C}\Big(\frac{m^{\ell-1}}{N^d}\Big)^{1/2}(\log N)^{\frac{\kappa+1}{2}},$$
$$A = \sum_{j=1}^{m} a_j^2 = (2^{\ell+2}\mathcal{C})^2 m^\ell N^{-d}(\log N)^{\kappa+1},$$
$$t = \sqrt{2A(2d+2h+2)\log N} = M_\kappa (m^\ell N^{-d})^{1/2}(\log N)^{1+\frac{\kappa}{2}}$$

with $M_\kappa(\ell,h,d) = (2d+2h+2)^{1/2}2^{\ell+2}\mathcal{C}_{\kappa-1}(\ell,h,d)$. We get

$$\mathbb{P}\Big(\Big|\sum_{j=1}^{m}\Upsilon_{j,u}\Big| \geq M_\kappa(m^\ell N^{-d})^{1/2}(\log N)^{1+\frac{\kappa}{2}}\Big)$$
$$\leq 2\exp(-t^2/2A) = 2\exp(-2(d+h+1)\log N) = 2N^{-2(d+h+1)}.$$

To conclude we argue as in the beginning of the induction. Allowing $u \in \Gamma_N^d$ and $m \leq (N^d \log N)^{\frac{1}{\ell-\kappa}}$ to vary, we see that

$$\mathbb{P}\Big(\max_{u\in\Gamma_N^d}\ \max_{m\leq(N^d\log N)^{\frac{1}{\ell-\kappa}}} \frac{\big|\sum_{j=1}^{m} Y_{j,u}\big|}{(m^\ell N^{-d})^{1/2}} \geq M_\kappa(\log N)^{1+\frac{\kappa}{2}}\Big)$$
$$\leq 2N^{-2d-2h-2}N^d(N^d\log N)^{\frac{1}{\ell-\kappa}} \leq N^{-2h-1}$$

if N is large enough. Moreover

$$\mathbb{P}\Big(\max_{u\in\Gamma_N^d}\ \max_{m\leq(N^d\log N)^{\frac{1}{\ell-\kappa}}} \Big|\sigma^{*\ell}(\{u\}) - \frac{m^\ell}{N^d}\Big| \geq M_\kappa(\log N)^{1+\frac{\kappa}{2}}\Big)$$
$$\leq \ell N^{-2h-1} + \sum_{1\leq j\leq(N^d\log N)^{\frac{1}{\ell-\kappa}}} \mathbb{P}(E_{j-1}^{\complement}) \leq N^{-h}$$

if $N \geq N_\kappa(\ell)$ is large enough. ∎

Proof of Proposition 2.5 Let $P = m = \lfloor N^\beta \rfloor$, with N large. Then the inequalities for σ_P and $P^{-1}\sigma_P$ in Lemma 3.1, Lemma 3.6, and Proposition 3.8 hold with positive (and high) probability. Proposition 2.5 is an immediate consequence. ∎

4 Fourier Restriction and Multiplier Estimates

Proof of Theorem A The restriction estimate is equivalent with the bound

$$\|\widehat{g\mu}\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|g\|_{L^2(\mu)}. \tag{4.1}$$

If $\mu^{*n} \in L^\infty(\mathbb{R}^d)$, then (4.1) for $p = \frac{2n}{2n-1}$ follows from a special case of an inequality in [6], namely $\|\widehat{g\mu}\|_{2n}^{2n} \lesssim \|\mu^{*n}\|_\infty \|g\|_{L^2(\mu)}^{2n}$. In conjunction with Theorem C this proves Theorem A. ∎

4.1 Multipliers of Bochner–Riesz Type

For $p \le q \le 2$ we formulate $L^p \to L^q$ versions of the multiplier Theorem B stated in the introduction. The main result is the following.

Theorem 4.1 *Let $1 \le p \le q \le 2$, and let $N > d(1/q - 1/2)$ be an integer. Let μ be a Borel probability measure on $\mathbb{R}^d$, and assume that the Fourier restriction theorem holds:*

$$\sup_{\|f\|_p \le 1} \Big(\int |\widehat{f}|^2 \, d\mu \Big)^{1/2} \le A_p < \infty. \tag{4.2}$$

For $r \le 1$, let

$$\varpi(r) = \sup_{x \in \mathbb{R}^d} \mu(B(x,r)), \tag{4.3}$$

and let $\eta_r \in C^\infty$ be supported in $\{\xi : r/4 \le |\xi| \le r\}$ and satisfy the differential inequalities $r^{|\beta|}\|\partial^\beta \eta_r\|_\infty \le 1$ for all multiindices β with $|\beta| \le N$. Let $h = \eta_r * \mu$. Then for all $f \in L^p(\mathbb{R}^d)$, $\|\mathcal{F}^{-1}[h\widehat{f}]\|_q \lesssim r^{d-\frac{d}{q}} A_p (\varpi(r))^{1/2} \|f\|_p$, where the implicit constant is independent of r and η.

Proof The proof is an adaptation of the argument by Fefferman and Stein [10]. Let $\Phi \in C^\infty(\mathbb{R}^d)$ supported in $\{x, |x| \le 1\}$ so that $\Phi(x) = 1$ for $|x| \le 1/2$. Let

$$\Phi_{0,r}(x) = \Phi(rx) \quad \text{and} \quad \Phi_{n,r}(x) = \Phi(2^{-n}rx) - \Phi(2^{-n+1}rx), \qquad n \ge 1.$$

Then we decompose $h = \sum_{n \ge 0} h_n$, where $\mathcal{F}^{-1}[h_n](x) = \mathcal{F}^{-1}[h](x)\Phi_{n,r}(x)$.

We first examine the L^∞ norm of $h_n = h * \widehat{\Phi}_{n,r}$. Observe by the support property of η_r and $\|\eta_r\|_\infty \le 1$ that $|h(\xi)| \le \mu(B(\xi, r)) \le \varpi(r)$. Moreover,

$$|h_n(\xi)| \le \varpi(r) \int |\widehat{\Phi}_{n,r}(y)| \, dy \lesssim \varpi(r)$$

since the L^1 norm of $\widehat{\Phi}_{n,r}$ is uniformly bounded in n and r. For $n \ge 1$, the last estimate can be improved, since then $\Phi_{n,r}$ vanishes near 0 and therefore all moments of $\widehat{\Phi}_{n,r}$ vanish. This allows us to write

$$\begin{aligned} h_n(\xi) &= \int \widehat{\Phi}_{n,r}(y) \int \Big[\eta_r(\xi - w - y) - \sum_{j=0}^{N-1} \frac{1}{j!} (\langle y, \nabla \rangle)^j \eta(\xi - w) \Big] \, d\mu(w) \, dy \\ &= \int_0^1 \frac{(1-s)^{N-1}}{(N-1)!} \int \widehat{\Phi}_{n,r}(y) \int (\langle y, \nabla \rangle)^N \eta_r(\xi - sy - w) \, d\mu(w) \, dy \, ds. \end{aligned}$$

Assuming $N_1 > N + d$, this gives

$$|h_n(\xi)| \le C(N_1) \varpi(r) \int \Big(\frac{|y|}{r} \Big)^N \frac{(2^n/r)^d}{(1 + 2^n|y|/r)^{N_1}} \, dy$$

and then

$$\|h_n\|_\infty \le C_N 2^{-nN} \varpi(r). \tag{4.4}$$

Since $\mathcal{F}^{-1}[h_n]$ is supported on a ball of radius $2^n r^{-1}$, we get the estimate

$$\|\mathcal{F}^{-1}[h_n] * f\|_q \lesssim (2^n r^{-1})^{d(\frac{1}{q} - \frac{1}{2})} \|\mathcal{F}^{-1}[h_n] * f\|_2 . \tag{4.5}$$

To see this, one decomposes $f = \sum_Q f_{Q,n}$, where the cubes Q form a grid of cubes of sidelength $2^n/r$ with f_Q supported in Q, and $\mathcal{F}^{-1}[h_n] * f$ supported in the corresponding double cube. In view of this support property, $\|\sum_Q \mathcal{F}^{-1}[h_n] * f\|_q \le C_d(\sum_Q \|\mathcal{F}^{-1}[h_n] * f\|_q^q)^{1/q}$ and (4.5) follows by Hölder's inequality.

Next, by Plancherel's theorem,

$$\|\mathcal{F}^{-1}[h_n] * f\|_2^2 = \|h_n \widehat{f}\|_2^2 \le \|h_n\|_\infty \int |\widehat{f}(\xi)|^2 |h_n(\xi)| d\xi$$

and

$$\begin{aligned}\int |\widehat{f}(\xi)|^2 |h_n(\xi)|\, d\xi &\le \int |\widehat{f}(\xi)|^2 \int |\eta_r * \widehat{\Phi}_{n,r}(\xi - w)|\, d\mu(w) d\xi \\ &= \int |\eta_r * \widehat{\Phi}_{n,r}(\xi)| \int |\widehat{f}(\xi + w)|^2 d\mu(w)\, d\xi \\ &\le A_p^2 \|\eta_r * \widehat{\Phi}_{n,r}\|_1 \|f\|_p^2,\end{aligned}$$

where for the last inequality we have applied the assumed Fourier restriction inequality to the function $f e^{-i\langle w,\cdot\rangle}$.

Now $\|\eta_r * \widehat{\Phi}_{n,r}\|_1 \lesssim \|\eta_r\|_1 \lesssim r^d$ and for $n \ge 1$, we also get (using Taylor's theorem as above) $\|\eta_r * \widehat{\Phi}_{n,r}\|_1 \le \int \|\langle y, \nabla\rangle^N \eta\|_1 |\widehat{\Phi}_{n,r}(y)| dy \lesssim 2^{-Nn} r^d$. The above estimates yield

$$\|\mathcal{F}^{-1}[h_n] * f\|_2 \lesssim \|m_n\|_\infty^{1/2} 2^{-nN/2} r^{d/2} A_p \|f\|_p \lesssim 2^{-nN} r^{d/2} \sqrt{\varpi(r)} A_p \|f\|_p\,,$$

by (4.4). We combine this with (4.5) to get

$$\|\mathcal{F}^{-1}[h_n] * f\|_q \lesssim 2^{-n(N - d(\frac{1}{q} - \frac{1}{2}))} r^{d - d/q} \sqrt{\varpi(r)} A_p \|f\|_p\,,$$

and finish by summing in n. ∎

As a corollary we get one direction of the statement in Theorem B for the multiplier m_λ as in (1.3)

Corollary 4.2 *Let μ be a Borel probability measure on $\mathbb{R}^d$, ϖ as in (4.3), and assume that $\varpi(r) \le C_\varepsilon r^{\alpha - \varepsilon}$ for all $\varepsilon > 0$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ and define, for $\lambda > 0$,*

$$m_\lambda(\xi) = \int_{\mathbb{R}^d} \chi(\xi - \eta)|\xi - \eta|^{\lambda - \alpha} d\mu(\eta).$$

Assume that $1 \le p \le q \le 2$ and that (4.2) holds. Then the inequality

$$\|\mathcal{F}^{-1}[m_\lambda \widehat{f}]\|_q \lesssim \|f\|_p \tag{4.6}$$

holds for $\lambda > d(\frac{1}{q} - \frac{1}{2}) - \frac{d-\alpha}{2}$. If, in addition, $\int_0^1 [t^{-\alpha}\varpi(t)]^{1/2} \frac{dt}{t} < \infty$, then (4.6) holds for $\lambda \ge d(\frac{1}{q} - \frac{1}{2}) - \frac{d-\alpha}{2}$.

Proof Decompose $\chi(\xi)|\xi|^{\lambda-\alpha} = \sum_{j=0}^\infty 2^{-j(\lambda-\alpha)} \eta_j(\xi)$ where (for a suitable constant C_N) the function $C_N^{-1}\eta_j$ satisfies the assumption of Theorem 4.1 with $r = 2^{-j}$. Thus

$$\|2^{-j(\lambda-\alpha)} \eta_j * \mu\|_{M_p^q} \lesssim 2^{-j(\lambda - d(\frac{1}{q} - \frac{1}{2}) + \frac{d-\alpha}{2})} \sqrt{2^{-j\alpha}\varpi(2^{-j})}.$$

The corollary follows. ∎

https://doi.org/10.4153/CJM-2016-019-6 Published online by Cambridge University Press

We now discuss the necessity of the condition on λ. One may test the convolution operator on a Schwartz function whose Fourier transform equals 1 on the (compact) support of m_λ. Therefore, the condition $m_\lambda \in M_p^q$ implies $\mathcal{F}^{-1}[m_\lambda] \in L^q$.

Lemma 4.3 *Let μ be a Borel measure supported on a set of Hausdorff dimension α and assume that $|\widehat{\mu}(x)| \le C_\gamma(1+|x|)^{-\gamma/2}$ for every $\gamma < \alpha$. Let $\lambda > \alpha - d$, m_λ be as in (1.3), and $\chi \in C_c^\infty$ with $\widehat{\chi}$ nonnegative and $\widehat{\chi}(0) > 0$. Let $K_\lambda = \mathcal{F}^{-1}[m_\lambda]$, $1 \le q \le 2$, and assume $K_\lambda \in L^q$. Then $\lambda \ge d(\frac{1}{q} - \frac{1}{2}) - \frac{d-\alpha}{2}$.*

Proof We argue as in Mockenhaupt [25]. The positivity conditions on χ and formulas for fractional integrals imply that for $\gamma < \alpha$ there exist $c > 0$, $c_\gamma > 0$, such that for $|x| \ge 1$, $|K_\lambda(x)| \ge c|x|^{\alpha-\lambda-d}|\widehat{\mu}(x)| \ge c_\gamma|\widehat{\mu}(x)|^{1+\frac{2(\lambda+d-\alpha)}{\gamma}}$. The second inequality follows by the assumption on $\widehat{\mu}$ and $\lambda > \alpha - d$. The displayed inequality and the condition $K_\lambda \in L^q$ implies $\widehat{\mu} \in L^r$, for $r > q(1 + 2(\lambda + d - \alpha)\alpha^{-1})$. It was shown in [28] that $\widehat{\mu} \in L^r$ implies $r \ge 2d/\alpha$; indeed this follows from the fact that $\dim_{\mathrm{H}}(\operatorname{supp}\mu) = \alpha$ implies that the energy integral $I_\beta(\mu)$ is infinite for $\beta > \alpha$, and Hölder's inequality. We now have the condition $\frac{2d}{\alpha} \le (1 + \frac{2(\lambda+d-\alpha)}{\gamma})q$, which is equivalent with $\lambda \ge d(\frac{1}{q} - \frac{1}{2}) - \frac{d-\alpha}{2} - (\alpha-\gamma)(\frac{d}{\alpha q} - \frac{1}{2})$. This holds for all $\gamma < \alpha$ and the assertion follows. ∎

4.2 Failure of Ahlfors–David Regularity

Before closing this section, we note that the measures for which the endpoint $L^{\frac{2d}{2d-\alpha}} \to L^2(\mu)$ restriction estimate hold cannot be Ahlfors–David regular. This can be seen as a consequence of a result of Strichartz [32]. For the convenience of the reader we give a short direct proof. We remark that some related results also appear in the recent thesis by Senthil Raani [29].

Proposition 4.4 *Let μ be a Borel probability measure supported on a compact set $E \subset \mathbb{R}^d$ and for $\rho \ge 1$, let*

$$\mathcal{B}_\rho(\mu) = \Big(\int_{\rho \le |\xi| \le 2\rho} |\widehat{\mu}(\xi)|^{\frac{2d}{\alpha}} \, d\xi \Big)^{\frac{\alpha}{2d}}.$$

Suppose that there exist $0 < \alpha < d$ and a constant $c > 0$ such that $\mu(B(x,r)) \ge cr^\alpha$ for all $x \in E$ and $0 < r < 1$. Then

(i) $\limsup_{\rho\to\infty} \mathcal{B}_\rho(\mu) > 0$.

(ii) *$\mathcal{F}$ does not extend to a bounded operator from $L^{\frac{2d}{2d-\alpha}}(\mathbb{R}^d)$ to $L^2(\mu)$.*

Proof Let χ be a nonnegative C^∞ function so that $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$. Let $R \gg 1$ and observe that, by assumption,

$$\begin{aligned} cR^{-\alpha} &\le \int \mu(B(x, R^{-1}))\, d\mu(x) \le \iint \chi(R(x-y))\, d\mu(y) d\mu(x) \\ &= \langle \widetilde{\mu} * \mu, \chi(R\cdot) \rangle = \int |\widehat{\mu}(\xi)|^2 R^{-d} \widehat{\chi}(R^{-1}\xi)\, d\xi. \end{aligned}$$

https://doi.org/10.4153/CJM-2016-019-6 Published online by Cambridge University Press

And therefore, $R^{d-\alpha} \le C_N \int |\widehat{\mu}(\xi)|^2 (1 + R^{-1}|\xi|)^{-N} \, d\xi$. Let $\mathcal{A}_0 = B(0,1)$ and $\mathcal{A}_j = B(0,2^j) \setminus B(0,2^{j-1})$ for $j \ge 1$. Then

$$R^{d-\alpha} \le C_N \Big(\int_{\mathcal{A}_0} |\widehat{\mu}(\xi)|^2 \, d\xi + \sum_{j\ge 1} \min\{1, (2^{j-1}R^{-1})^{-N}\} \int_{\mathcal{A}_j} |\widehat{\mu}(\xi)|^2 \, d\xi \Big)$$

$$\text{(4.7)} \qquad \le C_N' \Big(1 + \sum_{j\ge 1} \min\{1, (2^j R^{-1})^{-N}\} 2^{j(d-\alpha)} \mathcal{B}_{2^j}(\mu)^2 \Big),$$

by Hölder's inequality.

Now in order to prove (i), we argue by contradiction and assume that (i) does not hold, *i.e.*, $\lim_{\rho\to\infty} \mathcal{B}_\rho(\mu) = 0$. Since μ is compactly supported, the expressions $\mathcal{B}_\rho(\mu)$ are all finite, and by our assumption it follows that $\sup_\rho \mathcal{B}_\rho(\mu) \le B < \infty$. We use (4.7) for some $N > d - \alpha$ and obtain for $R \ge 1$

$$R^{d-\alpha} \le C_{d,\alpha} \Big(1 + B^2 R^{\frac{d-\alpha}{2}} + R^{d-\alpha} \sup_{\rho \ge \sqrt{R}} \mathcal{B}_\rho(\mu)^2 \Big),$$

and letting $R \to \infty$, this yields a contradiction.

To prove (ii) we observe that by duality (4.1) holds with $p' = 2d/\alpha$. We take $g \in C_c^\infty$ so that $g = 1$ on $\operatorname{supp}(\mu)$, and it follows that $\widehat{\mu} \in L^{2d/\alpha}$. This in turn implies $\lim_{\rho\to\infty} \mathcal{B}_\rho(\mu) = 0$ in contradiction to the result in (i). ■

A Some Standard Probabilistic Inequalities

For the convenience of the reader we include the proof of some standard probabilistic inequalities used in this paper. We will need the following version of Hoeffding's inequality, a slight variant of the one in [18].

Lemma A.1 *Let $\{W_j\}_{j=0}^m$ be a bounded real-valued martingale adapted to the filtration $\{\mathcal{F}_j\}_{j=0}^m$. Suppose that $a_j > 0$ for $1 \le j \le m$ and that*

$$\mathbb{E}[e^{\lambda(W_j - W_{j-1})} | \mathcal{F}_{j-1}] \le e^{a_j^2 \lambda^2/2} \text{ for all } |\lambda| < \delta.$$

Let $A = \sum_{j=1}^m a_j^2$. Then

$$\text{(A.1)} \qquad \mathbb{P}\big(|W_m - W_0| \ge t \big) \le 2e^{-\frac{t^2}{2A}}, \qquad 0 < t < A\delta,$$

$$\text{(A.2)} \qquad \mathbb{P}\big(|W_m - W_0| \ge t \big) \le 2e^{A\delta^2/2} e^{-\delta t}, \qquad t \ge A\delta.$$

Proof Observe that if $0 < \lambda < \delta$,

$$\mathbb{E}e^{\lambda(W_m - W_0)} = \mathbb{E}\big[e^{\lambda(W_{m-1}-W_0)} \mathbb{E}[e^{\lambda(W_m - W_{m-1})} \mid \mathcal{F}_{m-1}] \big]$$
$$\le e^{a_m^2\lambda^2/2} \mathbb{E}[e^{\lambda(W_{m-1}-W_0)}].$$

By iterating this step, we get $\mathbb{E}e^{\lambda(W_m - W_0)} \le e^{A\lambda^2/2}$.

Now $\mathbb{P}\{W_m - W_0 \ge t\} = \mathbb{P}\{e^{\lambda(W_m - W_0)} \ge e^{\lambda t}\}$ and Tshebyshev's inequality gives

$$\mathbb{P}\{W_m - W_0 \ge t\} \le e^{-\lambda t} \mathbb{E}e^{\lambda(W_m - W_0)} \le e^{-\lambda t + A\lambda^2/2}.$$

If $0 < t < A\delta$, we set $\lambda = t/A$, and if $t > A\delta$, we set $\lambda = \delta$. For these choices the displayed inequality gives

$$\mathbb{P}\{W_m - W_0 \geq t\} \leq \begin{cases} e^{-\frac{t^2}{2A}} & \text{for } 0 < t < A\delta, \\ e^{A\delta^2/2} e^{-\delta t} & \text{for } t \geq A\delta. \end{cases} \tag{A.3}$$

Similarly, still for $0 < \lambda < \delta$, $\mathbb{P}\{W_m - W_0 \leq -t\} = \mathbb{P}\{e^{-\lambda(W_m - W_0)} \geq e^{\lambda t}\}$ and argue as above to see that $\mathbb{P}\{W_m - W_0 \leq -t\}$ is also bounded by the right-hand side of (A.3). This implies the asserted inequality. ■

To verify the assumption in Lemma A.1, the following calculus inequality is useful [14, Lemma 1].

Lemma A.2 *Let X be a real-valued random variable with $|X| \leq a < \infty$ and $\mathbb{E}[X|\mathcal{F}] = 0$. Then for any $t \in \mathbb{R}$, $\mathbb{E}[e^{tX} \mid \mathcal{F}] \leq e^{a^2 t^2/2}$.*

Proof Replacing t by at and X by X/a, it suffices to consider the case $a = 1$. By the convexity of the function $x \mapsto e^{tx}$, for $x \in [-1, 1]$ we have

$$e^{tx} \leq \frac{1-x}{2} e^{-t} + \frac{x+1}{2} e^t = \cosh t + x \sinh(t),$$

and thus $\mathbb{E}[e^{tX}|\mathcal{F}] \leq \cosh t + \sinh t\,\mathbb{E}[X|\mathcal{F}]$. The last summand drops by assumption. Finally use that $\cosh t \leq e^{t^2/2}$ for all $t \in \mathbb{R}$, which follows by considering the power series and the inequality $(2k)! \geq 2^k k!$. ■

A combination of Lemma A.1 and Lemma A.2 yields the following corollary.

Corollary A.3 (Azuma–Hoeffding inequality) *Let $\{W_j\}_{j=0}^m$ be a bounded real-valued martingale adapted to filtration $\{\mathcal{F}_j\}_{j=0}^m$. For $1 \leq j \leq m$ let $a_j > 0$ and suppose that $|W_j - W_{j-1}| \leq a_j$. Writing $A = \sum_{j=1}^m a_j^2$, we have $\mathbb{P}(|W_m - W_0| \geq t) \leq 2e^{-t^2/2A}$, for all $t > 0$.*

As a consequence, we obtain a version of Bernstein's inequality.

Corollary A.4 (Bernstein's inequality) *Let $X_1, \dots, X_m$ be complex valued independent random variables with $\mathbb{E}X_j = 0$ and $|X_j| \leq M \in (0, \infty)$ for all $j = 1, \dots, m$. Then, for all $t > 0$*

$$\mathbb{P}\Big(\Big| \frac{1}{m} \sum_{j=1}^m X_j \Big| \geq Mt \Big) \leq 4e^{-mt^2/4}.$$

Acknowledgement X. Chen would like to thank Pablo Shmerkin for helpful discussions regarding convolution squares and Fourier decay properties of Cantor sets, and Malabika Pramanik for clarifying an example by R. Kaufman.

References

[1] J.-G. Bak and A. Seeger, *Extensions of the Stein-Tomas theorem*. Math. Res. Lett. **18**(2011), no. 4, 767–781. http://dx.doi.org/10.4310/MRL.2011.v18.n4.a14

[2] C. Bluhm, *Random recursive construction of Salem sets.* Ark. Mat. **34**(1996), no. 1, 51–63. http://dx.doi.org/10.1007/BF02559506

[3] ———, *On a theorem of Kaufman: Cantor-type construction of linear fractal Salem sets.* Ark. Mat. **36**(1998), no. 2, 307–316. http://dx.doi.org/10.1007/BF02384771

[4] J. Bourgain, *Bounded orthogonal systems and the $\Lambda(p)$-set problem*. Acta Math. **162**(1989), no. 3-4, 227–245. http://dx.doi.org/10.1007/BF02392838

[5] V. Chan, *On convolution squares of singular measures*. Master's Thesis, University of Waterloo, 2010.

[6] X. Chen, *A Fourier restriction theorem based on convolution powers.* Proc. Amer. Math. Soc. **142**(2014), no. 11, 3897–3901. http://dx.doi.org/10.1090/S0002-9939-2014-12148-4

[7] ———, *Restriction of the Fourier transform to Salem sets*. Ph.D. Thesis, University of Wisconsin-Madison 2015.

[8] ———, *Sets of Salem type and sharpness of the L^2-Fourier restriction theorem.* Trans. Amer. Math. Soc. **368**(2016), no.3, 1959–1977. http://dx.doi.org/10.1090/tran/6396

[9] P. Erdös and A. Rényi, *A probabilistic approach to problems of diophantine approximation.* Illinois J. Math. **1**(1957), 303–315.

[10] C. Fefferman, *Inequalities for strongly singular convolution operators.* Acta Math. **124**(1970), 9–36. http://dx.doi.org/10.1007/BF02394567

[11] K. Hambrook, *Explicit Salem sets and applications to metrical Diophantine approximation*. arxiv:1604.00411.

[12] K. Hambrook and I. Łaba, *On the sharpness of Mockenhaupt's restriction theorem*. Geom. Funct. Anal. **23**(2013), no. 4, 1262–1277. http://dx.doi.org/10.1007/s00039-013-0240-9

[13] ———, *Sharpness of the Mockenhaupt-Mitsis-Bak-Seeger restriction theorem in higher dimensions*. arxiv:509.05912.

[14] W. Hoeffding, *Probability inequalities for sums of bounded random variables*. J. Amer. Statist. Assoc. **58**(1963), 13–30. http://dx.doi.org/10.1080/01621459.1963.10500830

[15] J.-P. Kahane, *Some random series of functions.* 2nd edition. Cambridge, Cambridge University Press, 1985.

[16] R. Kaufman, *On the theorem of Jarník and Besicovitch*. Acta Arith. **39**(1981), no. 3, 265–267.

[17] T. W. Körner, *On the theorem of Ivašev-Musatov. III*. Proc. London Math. Soc. (3) **53**(1986), no. 1, 143–192. http://dx.doi.org/10.1112/plms/s3-53.1.143

[18] ———, *On a theorem of Saeki concerning convolution squares of singular measures.* Bull. Soc. Math. France **136**(2008), 439–464.

[19] ———, *Baire category, probabilistic constructions and convolution squares*. Lecture notes, University of Cambridge, 2009.

[20] ———, *Hausdorff and Fourier dimension*. Studia Math. 206 (2011), no. 1, 37–50. http://dx.doi.org/10.4064/sm206-1-3

[21] I. Łaba, *Harmonic analysis and the geometry of fractals, Proceedings of the 2014 International Congress of Mathematicians.* to appear.

[22] I. Łaba and M. Pramanik, *Arithmetic progressions in sets of fractional dimension*. Geom. Funct. Anal. **19**(2009), no. 2, 429–456. http://dx.doi.org/10.1007/s00039-009-0003-9

[23] P. Mattila, *Hausdorff dimension, projections, and the Fourier transform.* Publ. Mat. **48**(2004), no. 1, 3–48. http://dx.doi.org/10.5565/PUBLMAT_48104_01

[24] T. Mitsis, *A Stein-Tomas restriction theorem for general measures*. Publ. Math. Debrecen **60**(2002), no. 1-2, 89–99.

[25] G. Mockenhaupt, *Salem sets and restriction properties of Fourier transform.* Geom. Funct. Anal. **10**(2000), no. 6, 1579–1587. http://dx.doi.org/10.1007/PL00001662

[26] W. Rudin, *Trigonometric series with gaps*. J. Math. Mech. **9**(1960), 203–227.

[27] S. Saeki, *On convolution powers of singular measures*. Illinois J. Math. **24**(1980), 225–232.

[28] R. Salem, *On singular monotonic functions whose spectrum has a given Hausdorff dimension*. Ark. Mat. **1**(1951), no. 4, 353–365. http://dx.doi.org/10.1007/BF02591372

[29] K. S. Senthil Raani, *L^p asymptotics of Fourier transform of fractal measures*. Ph.D. Thesis, Indian Institute of Science, Bangalore, 2015.

[30] P. Shmerkin and V. Suomala, *Spatially independent martingales, intersections, and applications*. Memoirs of the American Mathematical Society, to appear. arxiv:1409.6707.

[31] E. M. Stein and R. Shakarchi, *Functional analysis*. Princeton, Princeton University Press, 2011.

[32] R. Strichartz, *Fourier asymptotics of fractal measures*. J. Funct. Anal. **89**(1990), no. 1, 154–187. http://dx.doi.org/10.1016/0022-1236(90)90009-A

[33] T. Tao, *Some recent progress on the restriction conjecture. Fourier analysis and convexity*. Appl. Numer. Harmon. Anal., Birkhäuser, Boston, MA, 2004. pp. 217–243.

[34] P. A. Tomas, *A restriction theorem for the Fourier transform*. Bull. Amer. Math. Soc. **81**(1975), 477–478. http://dx.doi.org/10.1090/S0002-9904-1975-13790-6

[35] ______, *Restriction theorems for the Fourier transform*. In: Harmonic analysis in Euclidean spaces. Proc. Symp. Pure Math. American Mathematical Society, Providence, RI, 1979,pp. 111–114.

[36] N. Wiener and A. Wintner, *Fourier-Stieltjes transforms and singular infinite convolutions*. Amer. J. Math **60**(1938), 513–522. http://dx.doi.org/10.2307/2371591

[37] T. H. Wolff, *Lectures on harmonic analysis*. I. Łaba and C. Shubin, eds. University Lecture Series 29. American Mathematical Society, Providence, RI, 2003.

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