GROUP ALGEBRAS WITH RADICALS OF SQUARE ZERO

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Over a field of characteristic p the group algebra of a finite group has a non-trivial radical if and only if the order of the group is divisible by the prime p. It would be of interest to determine the powers of the radical in the non-semi-simple case [**2**, p. 61]. In the particular case of p-groups the solution to the problem is known through the work of Jennings [**6**]. We here consider the special case of group algebras whose radicals have square zero and we relate this condition to the structure of the group itself.

We shall prove the following

THEOREM. *Let G be a group of order* $p^a m$ $((p, m) = 1, a \geqq 1)$. *Let K be an algebraically closed field of characteristic p and A the group algebra of G over K and let N be the corresponding radical. Then* $N^2 = \{0\}$ *if and only if* $p^a = 2$.

To prove the theorem we shall first establish a lemma and two corollaries of this lemma.

In the group algebra A, we may choose [**1**, p. 101, (3.3)] mutually orthogonal idempotents $e_1, e_2, \ldots, e_n$ of A such that, if e is the identity of G, then

$$e = e_1 + e_2 + \ldots + e_n$$

and

$$A = Ae_1 + Ae_2 + \ldots + Ae_n,$$

where Ae_i $(i = 1, 2, \ldots, n)$ is an indecomposable left ideal of A and the sum is a direct sum.

LEMMA. $N^2 = \{0\}$ *if and only if* Ae_i $(i = 1, 2, \ldots, n)$ *contains at most two irreducible constituents.*

Proof. Assume that $N^2 = \{0\}$. Then Ae_i/Ne_i is an irreducible constituent of Ae_i [**1**, Cor. 9.2F]. Thus if $Ne_i = \{0\}$, Ae_i has one irreducible constituent. If $Ne_i \neq \{0\}$, then, since $N(Ne_i) = \{0\}$, the upper Loewy series of Ae_i is $Ae_i \supset Ne_i \supset \{0\}$ [**1**, p. 102]. But Ne_i is therefore completely reducible and we know that, as A is a group algebra, Ae_i has a unique minimal left ideal and that this ideal is isomorphic to Ae_i/Ne_i [**4**, pp. 238, 240]. Thus we must have Ne_i irreducible and isomorphic to Ae_i/Ne_i.

Conversely, if Ae_i contains at most two irreducible constituents, either Ae_i is irreducible, in which case $Ne_i = \{0\}$, or, if reducible, Ae_i contains two constituents. Since one constituent is Ae_i/Ne_i, Ne_i must be irreducible and so $N^2 e_i = \{0\}$. Thus we have

$$N^2 = N^2 e \subseteq N^2 e_1 + N^2 e_2 + \ldots + N^2 e_n = \{0\}.$$

COROLLARY 1. $N^2 = \{0\}$ *if and only if* Ae_i $(i = 1, 2, \ldots, n)$ *contains at most two isomorphic irreducible constituents.*

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COROLLARY 2. *In the notation of the fundamental paper of Brauer and Nesbitt* [3, p. 559], $N^2 = \{0\}$ *if and only if either*

$$U_\kappa = F_\kappa \quad or \quad U_\kappa = \begin{pmatrix} F_\kappa & 0 \\ * & F_\kappa \end{pmatrix} \quad (\kappa = 1, 2, \ldots, k).$$

The first corollary is implicit in the proof of the lemma and the second is a restatement of the first in terms of the indecomposable and irreducible representations of G.

Proof of the Theorem. Suppose that $N^2 = \{0\}$. Consider U_1, the indecomposable representation corresponding to the 1-representation of G. Since p^a divides the degree of U_1 [3, (18)], we must have

$$U_1 = \begin{pmatrix} F_1 & 0 \\ * & F_1 \end{pmatrix}.$$

In particular, we see that $p^a = 2$.

Suppose now that $p^a = 2$. Then a 2-Sylow subgroup of G necessarily lies in the centre of its normalizer and so there exists, by Burnside's Theorem, a normal subgroup H of G of index 2 in G [5, Theorem 14. 3.1]. Then U_1 has degree 2 [3, p. 583] and so

$$U_1 = \begin{pmatrix} F_1 & 0 \\ * & F_1 \end{pmatrix}.$$

Now the Kronecker product representation $U_1 \otimes F_\kappa$ contains U_κ as a constituent [3, p. 579; 7, p. 413]. But

$$U_1 \otimes F_\kappa = \begin{pmatrix} F_\kappa & 0 \\ * & F_\kappa \end{pmatrix}.$$

Consequently either $U_\kappa = F_\kappa$ or $U_\kappa = \begin{pmatrix} F_\kappa & 0 \\ * & F_\kappa \end{pmatrix}$ $(\kappa = 1, 2, \ldots, k)$.

The desired conclusion now follows from Corollary 2.

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