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THE GENERALIZED SYLVESTER MATRIX EQUATION, RANK MINIMIZATION AND ROTH'S EQUIVALENCE THEOREM

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Abstract

Roth's theorem on the consistency of the generalized Sylvester equation $AX - YB = C$ is a special case of a rank minimization theorem.

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Let $A \in K^{m\times m}$, $B \in K^{n\times n}$ and $C \in K^{m\times n}$ be matrices over a field K. Set $\mathrm{Gl}(n) = \{M \in K^{n\times n} \mid \det M \neq 0\}$. The following theorem is due to Roth [1]. It gives a necessary and sufficient condition for the consistency of the generalized Sylvester equation (1) in terms of an equivalence of two associated matrices.

THEOREM 1. *The matrix equation*

$$AX - YB = C \tag{1}$$

is solvable with $X, Y \in K^{m\times n}$ *if and only if there exist matrices* $P, Q \in \mathrm{Gl}(m+n)$ *such that*

$$P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} Q.$$

We shall see that Roth's theorem is a special case of a result on rank minimization. It is the purpose of this note to prove the following.

THEOREM 2. *We have*

$$\begin{aligned} &\min\{\mathrm{rank}(AX - YB - C) \mid X, Y \in K^{m\times n}\} \\ &\quad = \min\left\{\mathrm{rank}\left[P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} Q\right] \,\middle|\, P, Q \in \mathrm{Gl}(m+n)\right\}. \end{aligned}$$

PROOF. Let $X, Y \in K^{m\times n}$ and $P, Q \in \mathrm{Gl}(m+n)$. Set

$$\phi(X, Y) = AX - YB - C \quad \text{and} \quad \Phi(P, Q) = P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} Q,$$

and

$$\gamma = \min \operatorname{rank}\{\phi(X, Y) \mid X, Y \in K^{m\times n}\}$$

and

$$\Gamma = \min \operatorname{rank}\{\Phi(P, Q) \mid P, Q \in \mathrm{Gl}(m+n)\}.$$

From [2] we know that

$$\gamma = \operatorname{rank}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \operatorname{rank}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

For $P, Q \in \mathrm{Gl}(m+n)$, we obtain

$$\begin{aligned} \operatorname{rank} \Phi(P, Q) &\geq \operatorname{rank} P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \operatorname{rank}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} Q \\ &= \operatorname{rank}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} - \operatorname{rank}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \gamma. \end{aligned}$$

Hence $\Gamma \geq \gamma$. With $X, Y \in K^{m\times n}$, we associate the matrices

$$P_Y = \begin{pmatrix} I & Y \\ 0 & I \end{pmatrix} \quad \text{and} \quad Q_X = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$

Then

$$\Phi(P_Y, Q_X) = \begin{pmatrix} 0 & -\phi(X, Y) \\ 0 & 0 \end{pmatrix}.$$

Hence

$$\Gamma \leq \min\{\operatorname{rank} \Phi(P_Y, Q_X) \mid X, Y \in K^{m\times n}\} = \min\{\operatorname{rank} \phi(X, Y)\} = \gamma.$$

This completes the proof. □

The following theorem deals with the Sylvester equation (2). It is known as Roth's similarity theorem.

THEOREM 3 [1]. *The matrix equation*

$$AX - XB = C \tag{2}$$

is solvable with $X \in K^{m\times n}$ *if and only if there exists a matrix* $P \in \mathrm{Gl}(m+n)$ *such that*

$$P\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} P.$$

There is evidence that Theorem 3 is also a special case of a result on rank minimization. We conjecture that the identity

$$\begin{aligned}&\min\{\operatorname{rank}(AX - XB - C) \mid X \in K^{m\times n}\}\\ &\quad = \min\left\{\operatorname{rank}\left[P\begin{pmatrix} A & C \\ 0 & B\end{pmatrix} - \begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}P\right] \,\middle|\, P \in \mathrm{Gl}(m+n)\right\}\end{aligned}$$

holds, which extends Theorem 3.

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