J. Austral. Math. Soc. (*Series A*) **31** (1981), 26–45

THE PROXIMALLY CONTINUOUS INTEGRALS

D. N. SARKHEL and A. K. DE

(Received 4 May 1977; revised 24 June 1980)

Communicated by E. Strzelecki

Abstract

This paper introduces certain generalizations of the notions of approximate limit, continuity and derivative and of absolute continuity, of real functions, leading to generalized integrals of Perron and Denjoy types comprising the AP-integral of Burkill (1931) and Sonouchi and Utagawa (1949) and the AD-integral of Kubota (1963), respectively. The generalizations are all substantiated by appropriate examples.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 26 A 03, 26 A 24, 26 A 39.
Key words and phrases. AP-integral, AD-integral, approximately continuous, density, sparse set, proximally continuous, Darboux continuity, proximally absolutely continuous, Lusin's condition (N), PP_r-integral, PP_t-integral, PP-integral, PD-integral.

1. Introduction

In this paper we define three integrals of Perron type and one of Denjoy type, each of the first three comprising the AP-integral of Burkill (1931) as generalized by Sonouchi and Utagawa (1949) and the last one comprising the AD-integral of Kubota (1963).

If F is an indefinite AP- or AD-integral of a function f on $[a, b]$, then F is approximately continuous on $[a, b]$ and $(ap)F' = f$ a.e. on $[a, b]$. We observe that if this differential property of an integral is of primary interest, then approximate continuity *everywhere* does not seem to be quite natural to expect, although some meaningful continuity property everywhere is desirable. Each of the integrals we define possesses the said differential property and is continuous everywhere not necessarily in the approximate sense but in a certain proximal sense, which implies continuity in Darboux sense.

We first introduce in Section 3 the notion of *sparse sets* which generalizes the notion of sets of density zero. Using this notion we then introduce in Section 4 the notions of *proximal* limit, continuity and derivative of a function, which generalize the corresponding notions in the approximate sense. Besides, in Section 5 we introduce the notion of *proximal absolute continuity*, PAC, of a function, which generalizes the notion of AC. Using these concepts we define in Section 6 the three integrals of Perron type, the PP_r-, PP_l- and PP-integrals, in terms of major and minor functions defined by unifying and generalizing the ideas of Bauer (1915), Hake (1921) and Saks (1937). Finally in Section 7 we define the integral of Denjoy type, the PD-integral, using the descriptive method of Denjoy (1916) as adapted by Saks (1937).

2. Preliminaries

Conventions. To save space, left-hand analogs of our notations, definitions, theorems and lemmas will remain understood. By a *set* we shall mean a subset of the real line.

Notations. Throughout, we shall use the following notations: R = the real line, R_e = the extended real line, $|E|$ = the outer Lebesgue measure of a set E, $d^+(E, x)$ $[d_+(E, x)]$ = the upper [lower] right density of a set E at a point x, E' = the complement of a set E, and J = the set of positive integers.

With each point $x \in R$ there are associated four extreme densities of a set E, and when these are all equal to one another, their common value is the density of E at x. As a typical definition,

$$d^+(E, x) = \limsup_{y \to x+} \frac{|E \cap (x, y)|}{|(x, y)|}.$$

In the sequel we will use the following facts without further reference. If E is measurable then $d_+(E, x) + d^+(E', x) = 1$, since $|E \cap (x, y)| + |E' \cap (x, y)| = |(x, y)|$ for all $y > x$. For any set E there is a measurable set $A \supset E$ such that $|E \cap M| = |A \cap M|$ for every measurable set M (Saks (1937); (6.7), p. 70). We call such a set A a measurable cover of E. If A, B are measurable covers of E, F respectively, then $A \cup B$ is a measurable cover of $E \cup F$. For, $A \cup B$ is a measurable superset of $E \cup F$, and, for any measurable set M, we have

$$\begin{aligned} |(A \cup B) \cap M| &= |A \cap M| + |B \cap A' \cap M| \\ &= |E \cap M| + |F \cap A' \cap M| \\ &= |(E \cap M) \cup (F \cap A' \cap M)| \quad \because A \text{ is measurable} \\ &\leqslant |(E \cup F) \cap M|, \end{aligned}$$

https://doi.org/10.1017/S1446788700018462 Published online by Cambridge University Press

whence $|(A \cup B) \cap M| = |(E \cup F) \cap M|$ since $E \cup F \subset A \cup B$. Finally, since $|E \cap I| = |A \cap I|$ for any interval I, we note that at any given point the extreme densities of a set and those of a measurable cover of the set are equal in corresponding pairs.

We now prove three lemmas, of which the last one is the most important single result of this paper.

LEMMA 2.1. (i) *An arbitrary set E has density* 0 *or* 1 *a.e. on R.* (ii) *If a subset E of a measurable set M has density* 0 *a.e. on $M \cap E'$, then E is measurable.*

PROOF. Let A be a measurable cover of E. By density theorem (Saks (1937); (10.2), p. 129) A has density 1 at almost all points of A and has density 0 at almost all points of A'. Consequently E has density 1 a.e. on A and has density 0 a.e. on A', which proves (i). To prove (ii), we note that by hypothesis E has density 0 a.e. on $A \cap (M \cap E')$. So A has density 0 a.e. on $A \cap (M \cap E')$, which implies, by density theorem, that $|A \cap M \cap E'| = 0$. Thus $A \cap M \cap E'$ is measurable. Consequently, since $A \cap M$ is a measurable superset of E, it follows that E is measurable.

LEMMA 2.2. *Let $d^+(E, x) = 1$. Then there exists, for every $k > 0$, an open interval $(a, b) \subset (x, x + k)$ with $|(x, a)| < k|(x, b)|$ such that $|E \cap (x, y)| > (1 - k)|(x, y)|$ for all $y \in (a, b)$.*

PROOF. Since $d^+(E, x) = 1$, there is $b \in (x, x + k)$ such that

$$|E \cap (x, b)| > (1 - k^2/(1 + k))|(x, b)|.$$

Taking $a = (x + kb)/(1 + k)$, we see that $|(x, a)| = (k/(1 + k))|(x, b)| < k|(x, b)|$ and, for all $y \in (a, b)$, noting that (x, y), $\{y\}$ and (y, b) are pairwise disjoint measurable sets with $|\{y\}| = 0$, we have

$$\begin{aligned} |E \cap (x, y)| &= |E \cap (x, b)| - |E \cap (y, b)| \\ &> (1 - k^2/(1 + k))|(x, b)| - |(y, b)| \\ &= |(x, y)| - k|(x, a)| \\ &> (1 - k)|(x, y)|, \end{aligned}$$

which proves the lemma.

LEMMA 2.3. *Let $A \subset [a, b]$ be such that* (i) $a \in A$, (ii) $d^-(A, x) < 1$ *for every $x \in B = [a, b] \setminus A$ and* (iii) $d^+(B, x) < 1$ *for every $x \in A$. Then $B = \varnothing$.*

Proof. Suppose, for a contradiction, that $B \neq \varnothing$. Put

$$A_0 = \{x \in A \mid d^+(B, x) = 0\} \quad \text{and} \quad B_0 = \{x \in B \mid d^-(A, x) = 0\}.$$

By Lemma 2.1, the condition (ii) implies that $|B \setminus B_0| = 0$ and that A is measurable, and the condition (iii) implies that $|A \setminus A_0| = 0$ and that B is measurable. Now, since $A \cup B = [a, b]$ and since $|A \setminus A_0| = 0$, the condition (iii) implies that each point of $A \cap [a, b)$ is a limit point of A_0 on the right. Similarly, each point of $B \cap (a, b]$ is a limit point of B_0 on the left.

We assert that if $x_0 \in A_0$, $y_0 \in B_0$ and $x_0 < y_0$, then for any $\varepsilon > 0$ there are points $x_1 \in A_0 \cap (x_0, y_0)$ and $y_1 \in B_0 \cap (x_0, y_0)$ such that

$$(1) \qquad |A \cap (t, y_0)| < \varepsilon|(t, y_0)| \quad \text{for all } t \in (x_1, y_0),$$

$$(2) \qquad |B \cap (x_0, t)| < \varepsilon|(x_0, t)| \quad \text{for all } t \in (x_0, y_1).$$

To see these, first observe that if y_0 is a limit point of A on the left, then by the preceding paragraph y_0 is necessarily a limit point of A_0 on the left, and hence the existence of x_1 follows at once from the fact that $d^-(A, y_0) = 0$. If y_0 is not a limit point of A on the left, noting that x_0 is necessarily a limit point of A on the right, we have $x_0 < s = \sup A \cap (x_0, y_0) < y_0$. Then, since $A \cap (s, y_0) = \varnothing$, we have $d^+(B, s) = 1$, which by (iii) implies that $s \notin A$. Consequently s must be a limit point of A, and hence also of A_0, on the left. We simply choose a point $x_1 \in A_0 \cap (x_0, s)$ with $|(x_1, s)| < \varepsilon|(s, y_0)|$. Then, recalling that $A \cap (s, y_0) = \varnothing$, we readily verify (1) for this x_1.

Existence of y_1 is shown similarly by considering the point $\inf B \cap (x_0, y_0)$.

Now $a \in A$ by (i). Therefore, since $B \subset [a, b]$ and since $B \neq \varnothing$, we have $a < b'$ for some $b' \in B$. Consequently, since a is a limit point of A_0 on the right and since b' is a limit point of B_0 on the left, there exist points $a_0 \in A_0$ and $b_0 \in B_0$ such that $a_0 < b_0$. Starting with b_0 and applying (1) and (2) alternately, we select successively points $a_1, b_1, a_2, b_2, \ldots$ such that, for all $n \in J$, $a_n \in A_0$, $b_n \in B_0$, $a_{n-1} < a_n < b_n < b_{n-1}$,

$$(3) \qquad |A \cap (t, b_{n-1})| < (1/n)|(t, b_{n-1})| \quad \text{for all } t \in (a_n, b_{n-1})$$

and

$$(4) \qquad |B \cap (a_n, t)| < (1/n)|(a_n, t)| \quad \text{for all } t \in (a_n, b_n).$$

Then, for all $n \in J$, we have (using (3) and (4))

$$\begin{aligned} |(a_{n+1}, b_{n+1})| &= |A \cap (a_{n+1}, b_{n+1})| + |B \cap (a_{n+1}, b_{n+1})| \\ &\leq |A \cap (a_{n+1}, b_{n-1})| + |B \cap (a_n, b_{n+1})| \\ &< (1/n)|(a_{n+1}, b_{n-1})| + (1/n)|(a_n, b_{n+1})| \\ &< (2/n)|(a, b)|, \end{aligned}$$

whence $\lim a_n = \lim b_n$. Denoting this common limit by c, we have $a_n < c < b_n$ for all $n \in J$, so that both (3) and (4) hold with $t = c$. Consequently $d_+(A, c) = 0 = d_-(B, c)$. These give $d^+(B, c) = 1 = d^-(A, c)$, which by (ii) and (iii) contradicts the fact that $c \in [a, b] = A \cup B$, and the proof ends.

3. Sparse sets

DEFINITION 3.1. A set E is said to be sparse at a point $x \in R$ on the right if there exists, for every $\varepsilon > 0$, a $k > 0$ such that every interval $(a, b) \subset (x, x + k)$, with $|(x, a)| < k|(x, b)|$, contains at least one point y such that $|E \cap (x, y)| < \varepsilon|(x, y)|$.

The family of sets sparse at x on the right is denoted by $S(x+)$, and E is said to be sparse at x if $E \in S(x)$ where $S(x) = S(x+) \cap S(x-)$. ($S(x-)$ denotes, by convention, the family of sets sparse at x on the left.)

We set $S_0(x+) = \{E | E \subset R \text{ and } d^+(E, x) = 0\}$, $S_0(x-) = \{E | E \subset R \text{ and } d^-(E, x) = 0\}$ and $S_0(x) = S_0(x+) \cap S_0(x-)$.

From definitions, it follows at once that $S_0(x+) \subset S(x+)$, $S_0(x-) \subset S(x-)$ and $S_0(x) \subset S(x)$, for every $x \in R$. The following example together with its left-hand analog shows that these inclusions are all proper.

EXAMPLE 3.1. Given any $x \in R$ and any $r \in (0, 1)$, we will construct a set $E \in S(x+)$ for which $d^+(E, x) > 1 - r$. Since the outer Lebesgue measure is translation invariant, it is enough to consider $x = 0$.

Let $c = (1 + r)/r$, $a_n = c^{-n^2-1}$ and $b_n = c^{-n^2}$. Put

$$E = \bigcup_{n \in J} (a_n, b_n).$$

Given $\varepsilon > 0$, we fix $m \in J$ with $1/m < \varepsilon$. Since $b_{n+1}/a_n = c^{-2n} \to 0$, there is $p \in J$ such that $mb_{n+1} < a_n$ for all $n \geqslant p$. Therefore, if $n \geqslant p$, for $y \in [mb_{n+1}, a_n]$ we have $|E \cap (0, y)| = |E \cap (0, b_{n+1})| \leqslant (1/m)mb_{n+1} < \varepsilon|(0, y)|$. Hence, if $k = \min\{1/mc, a_p\}$, then

$$(*) \qquad \{y \in (0, k) \,|\, |E \cap (0, y)| \geqslant \varepsilon|(0, y)|\} \subset \bigcup_{n > p} (a_n, mb_n).$$

Now, consider any interval $(a, b) \subset (0, k)$ with $|(0, a)| < k|(0, b)|$. If we suppose that $(a, b) \subset (a_n, mb_n)$ for some n, then we have $|(0, a)| \geqslant |(0, a_n)| = (1/mc)|(0, mb_n)| \geqslant k|(0, b)|$, which contradicts the choice of (a, b). Consequently, since the intervals $\{(a_n, mb_n)\}_{n > p}$ are pairwise disjoint, we cannot have $(a, b) \subset \bigcup_{n > p} (a_n, mb_n)$. Hence, noting that $(a, b) \subset (0, k)$, it follows from $(*)$

that $|E \cap (0, y)| < \varepsilon|(0, y)|$ for at least one $y \in (a, b)$. Thus $E \in S(0+)$. Further, for all $n \in J$, we have $|E \cap (0, b_n)| > |(a_n, b_n)| = (1 - 1/c)b_n$, whence $d^+(E, 0) \geq 1 - 1/c = 1/(1 + r) > 1 - r$.

The following theorem gives some interesting and illuminating characterizations of sparseness.

THEOREM 3.1. *Given $x \in R$, $E \subset R$ and a measurable cover A of E, the following conditions are equivalent.*

(i) $A \in S(x+)$.

(ii) $E \in S(x+)$.

(iii) $F \subset R, d^+(F, x) < 1 \Rightarrow d^+(E \cup F, x) < 1$.

(iv) $F \subset R$, $d_+(F, x) = 0$ *and* $d^+(F, x) < 1 \Rightarrow d_+(E \cup F, x) = 0$ *and* $d^+(E \cup F, x) < 1$.

(v) $F \subset R, d_+(F, x) = 0 \Rightarrow d_+(E \cup F, x) = 0$.

PROOF. Since $|E \cap I| = |A \cap I|$ for any interval I, so (i) $\Leftrightarrow$ (ii). We will show further that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) Assume (iii) false. Then there is a set F such that $d^+(F, x) < 1$ but $d^+(E \cup F, x) = 1$. Choose $\varepsilon > 0$ so that $d^+(F, x) + 3\varepsilon < 1$. There is $h > 0$ such that, for all $y \in (x, x + h)$,

$$|F \cap (x, y)| < (d^+(F, x) + \varepsilon)|(x, y)|. \tag{5}$$

Now, since $d^+(E \cup F, x) = 1$, for any positive k ($< \min\{h, \varepsilon\}$) there exists, by Lemma 2.2, an interval $(a, b) \subset (x, x + h)$ with $|(x, a)| < k|(x, b)|$ such that, for all $y \in (a, b)$,

$$|(E \cup F) \cap (x, y)| > (1 - \varepsilon)|(x, y)|. \tag{6}$$

Since for all $y \in (a, b)$ we have, by (5) and (6),

$$\begin{aligned}|E \cap (x, y)| &\geq |(E \cup F) \cap (x, y)| - |F \cap (x, y)| \\ &> (1 - 2\varepsilon - d^+(F, x))|(x, y)| > \varepsilon|(x, y)|,\end{aligned}$$

clearly $E \notin S(x+)$. Hence we conclude that (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv) Let $d_+(F, x) = 0$ and let B be a measurable cover of F. Then $d_+(B, x) = 0$, so that $d^+(B', x) = 1$, and hence

$$\begin{aligned}d^+(E \cup (A' \cap B'), x) &= d^+(A \cup (A' \cap B'), x) \\ &= d^+(A \cup B', x) = 1.\end{aligned}$$

Therefore, assuming (iii), $d^+(A' \cap B', x) = 1$. This gives $d_+(A \cup B, x) = 0$, whence $d_+(E \cup F, x) = 0$. Hence, plainly, (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (v) Let $d_+(F, x) = 0$ and $d^+(F, x) = 1$, and let B be a measurable cover of F. Then $d_+(B, x) = 0$ and $d^+(B, x) = 1$, so that $d^+(B', x) = 1$ and

$d_+(B', x) = 0$. The last condition gives $d_+(A' \cap B', x) = 0$. Assuming (iv), we get as in the preceding proof $d^+(A' \cap B', x) = 1$, so that $d_+(A \cup B, x) = 0$, whence $d_+(E \cup F, x) = 0$. Hence, plainly, (iv) $\Rightarrow$ (v).

(v) $\Rightarrow$ (ii) Assume (ii) false. Then there exist an $\varepsilon > 0$ and a sequence $\{(a_n, b_n)\}$ of open intervals such that, for all $n \in J$,

$$x < b_{n+1} < a_n, \tag{7}$$

$$|(x, a_n)| < 2^{-n}|(x, b_n)| \tag{8}$$

and

$$|E \cap (x, y)| \geqslant \varepsilon|(x, y)| \quad \text{for all } y \in (a_n, b_n). \tag{9}$$

By (7) and (8), $a_{n+1} < \frac{1}{2}(x + b_{n+1}) < a_n$ for all $n \in J$. So, if

$$F = \bigcup_{n \in J} \left(\tfrac{1}{2}(x + b_{n+1}), a_n\right),$$

using (8) we have, for all $n \in J$,

$$\begin{aligned}\left|F \cap \left(x, \tfrac{1}{2}(x + b_{n+1})\right)\right| &= |F \cap (x, a_{n+1})| \leqslant |(x, a_{n+1})| \\ &< 2^{-n-1}|(x, b_{n+1})| = 2^{-n}\left|\left(x, \tfrac{1}{2}(x + b_{n+1})\right)\right|.\end{aligned}$$

Hence $d_+(F, x) = 0$. However, for $y \in [b_{n+1}, a_n]$,

$$|F \cap (x, y)| > \left|\left(\tfrac{1}{2}(x + b_{n+1}), y\right)\right| \geqslant \tfrac{1}{2}|(x, y)|. \tag{10}$$

By (9) and (10), $d_+(E \cup F, x) > 0$. Hence we conclude that (v) $\Rightarrow$ (ii), and this completes the proof of the theorem.

COROLLARY 3.1.1. (i) *If* $E \in S(x+)$, *then* $d_+(E, x) = 0$ *and* $d^+(E, x) < 1$. (ii) *If* $A, B \in S(x+)$, *then* $A \cup B \in S(x+)$ *and, for any* $E \subset A$, $E \in S(x+)$.

PROOF. (i) Since $d_+(\varnothing, x) = 0$ and $d^+(\varnothing, x) = 0 < 1$, by Theorem 3.1((ii) $\Rightarrow$ (iv)) we have $d_+(E, x) = d_+(E \cup \varnothing, x) = 0$ and $d^+(E, x) = d^+(E \cup \varnothing, x) < 1$, for every $E \in S(x+)$.

(ii) Consider any set F for which $d_+(F, x) = 0$. By Theorem 3.1((ii) $\Rightarrow$ (v)) we have $d_+(B \cup F, x) = 0$ and, hence, again $d_+(A \cup (B \cup F), x) = 0$, that is, $d_+((A \cup B) \cup F, x) = 0$. Hence, by Theorem 3.1((v) $\Rightarrow$ (ii)), we conclude that $A \cup B \in S(x+)$. Again, we have $d_+(A \cup F, x) = 0$. Hence, for any $E \subset A$, we have $d_+(E \cup F, x) = 0$, which implies that $E \in S(x+)$.

https://doi.org/10.1017/S1446788700018462 Published online by Cambridge University Press

4. Proximal continuity and derivative

Let the functions $f, g\colon X \to R_e$ be given, where $X \subset R$.

DEFINITION 4.1. For any $x \in R$, define

$$P^+f(x) = \inf\{r \in R_e | \{y \in X | f(y) > r\} \in S(x+)\},$$
$$P_+ f(x) = \sup\{r \in R_e | \{y \in X | f(y) < r\} \in S(x+)\},$$
$$\bar{P}f(x) = \inf\{r \in R_e | \{y \in X | f(y) > r\} \in S(x)\},$$
$$\underline{P}f(x) = \sup\{r \in R_e | \{y \in X | f(y) < r\} \in S(x)\}.$$

Then $P^+f(x)$ is called the upper right proximal limit, $P_+ f(x)$ the lower right proximal limit, $\bar{P}f(x)$ the upper proximal limit and $\underline{P}f(x)$ the lower proximal limit, of f at x. When $x \in X$, f is said to be proximally continuous at x if $\bar{P}f(x) = \underline{P}f(x) = f(x)$. If f is proximally continuous at each point of a subset $E \subset X$, we say that f is proximally continuous on E.

These definitions are analogous to those in the approximate sense; for example, denoting by 'A' the approximate limit process, $A^+f(x)$ is the result of replacing $S(x+)$ by $S_0(x+)$ in the definition of $P^+f(x)$.

If $X \notin S(x+)$, we have $P_+ f(x) \leqslant P^+f(x)$. For, if not, choose any r_0 with $P^+f(x) < r_0 < P_+ f(x)$. Then, since by Corollary 3.1.1 subsets of members of $S(x+)$ are also members of $S(x+)$, it is clear that $A, B \in S(x+)$ where $A = \{y \in X | f(y) \geqslant r_0\}$ and $B = \{y \in X | f(y) \leqslant r_0\}$. Hence by Corollary 3.1.1 $X = A \cup B \in S(x+)$, which is contrary to hypothesis. Similarly, $\underline{P}f(x) \leqslant \bar{P}f(x)$ if $X \notin S(x)$.

Recalling that $S_0(x+) \subset S(x+)$ and $S_0(x) \subset S(x)$, we evidently have $P^+f(x) \leqslant A^+f(x)$, $A_+ f(x) \leqslant P_+ f(x)$ and $\bar{P}f(x) \leqslant \bar{A}f(x)$, $\underline{A}f(x) \leqslant \underline{P}f(x)$ for all x.

Thus, if $X \notin S(x+)$, we have $A_+ f(x) \leqslant P_+ f(x) \leqslant P^+f(x) \leqslant A^+f(x)$. If $X \notin S(x)$, we have $\underline{A}f(x) \leqslant \underline{P}f(x) \leqslant \bar{P}f(x) \leqslant \bar{A}f(x)$, so that approximate continuity of f at the point x implies proximal continuity. The converse of this result is false (see Example 6.1).

We note finally that if $X \in S(x+)$, then $P^+f(x) = -\infty$ and $P_+ f(x) = \infty$. In particular, if X is a closed interval $[a, b]$ then $P^+f(b) = -\infty$ and $P_+ f(b) = \infty$. Also $P^-f(a) = -\infty$ and $P_- f(a) = \infty$.

Using Corollary 3.1.1(ii), the reader can verify that:

(i) $P_+(-f)(x) = -P^+f(x)$,

(ii) $\underline{P}(-f)(x) = -\bar{P}f(x)$,

(iii) $P_+(cf)(x) = c \cdot P_+ f(x)$ if c is a positive constant,

https://doi.org/10.1017/S1446788700018462 Published online by Cambridge University Press

(iv) $\underline{P}f(x) = \min\{P_+ f(x), P_- f(x)\}$,

(v) $P_+(f + g)(x) \geqslant P_+ f(x) + P_+ g(x)$,

equality holding in (v) if $P_+ f(x) = P^+f(x)$, it being assumed that the 'sums' in (v) are well defined in R_e.

Similar other results can be deduced from these. In particular, real valued proximally continuous functions on a given set form a linear space.

THEOREM 4.1. *Let X be connected, and let f be proximally continuous on X. Then $f(X)$ is connected.*

PROOF. Suppose, for a contradiction, that $f(X)$ is not connected. Then there are points $a, b \in X$ and $k \in R \setminus f(X)$ such that $f(a) < k < f(b)$. Let I denote the closed interval with end-points a and b. Then $I \subset X$ since X is connected. Put

$$A_1 = \{x \in I \mid f(x) < k\}, \quad \text{and} \quad A_2 = \{x \in I \mid f(x) > k\}.$$

For any point $x_1 \in A_1$ we have, since f is proximally continuous at x_1, $\overline{P}f(x_1) = f(x_1) < k$, so that $A_2 \in S(x_1)$ and, hence, by Corollary 3.1.1(i) and its left-hand analog we have $d^+(A_2, x_1) < 1$ and $d^-(A_2, x_1) < 1$. Also, for any point $x_2 \in A_2$ we have, since f is proximally continuous at x_2, $\underline{P}f(x_2) = f(x_2) > k$, so that $A_1 \in S(x_2)$ and, hence, as above $d^-(A_1, x_2) < 1$ and $d^+(A_1, x_2) < 1$. Since, moreover, $a \in A_1$, $b \in A_2$, $A_2 = I \setminus A_1$ and $A_1 = I \setminus A_2$, we have a contradiction to Lemma 2.3, whether $a < b$ or $b < a$, and this completes the proof.

NOTE 4.1. Theorem 4.1 plainly implies that proximal continuity of f on an interval $I_0 \subset X$ implies Darboux continuity (intermediate value property) of f on I_0.

Henceforth we suppose that $f(X) \subset R$.

DEFINITION 4.2. Given $a \in X$, define $F\colon X \to R$ by $F(a) = 0$ and $F(x) = (f(x) - f(a))/(x - a)$ otherwise. Then the extended real number $P^+F(a)$ is called the upper right proximal derivate of f at a, and it is denoted by $PD^+f(a)$. The numbers $P_+F(a)$, $P^-F(a)$, $P_-F(a)$, $\overline{P}F(a)$ and $\underline{P}F(a)$ are named and denoted analogously. If $\overline{PD}f(a) = \underline{PD}f(a)$, this common value is called the proximal derivative of f at a, and it is denoted by $PDf(a)$.

The approximate derivates $AD^+f(a)$, etc., of f at a are defined similarly in terms of the approximate limits of F at a. The approximate derivative $ADf(a)$, when exists, is also denoted by $(ap)f'(a)$. The observations following Definition 4.1 and the relations between the extreme proximal limits mentioned there are evidently true also for proximal and approximate derivates.

THEOREM 4.2. *If $PD_+ f > -\infty$, or, more generally, if $P_+ f \geqslant f$, on a measurable subset $M \subset X$, then f is measurable on M.*

PROOF. Given $a \in R$, let $E = \{x \in M | f(x) \leqslant a\}$. Consider any point $c \in M \cap E'$. The condition $P_+ f(c) \geqslant f(c) > a$ implies that $E \in S(c+)$, so that by Corollary 3.1.1 $d_+(E, c) = 0$. Consequently, by Lemma 2.1(i), E has density 0 a.e. on $M \cap E'$ and, hence, by Lemma 2.1(ii) E is measurable, because M is a measurable superset of E. Hence we conclude that f is measurable on M.

THEOREM 4.3. *Suppose that X is connected, $P^- f \leqslant f \leqslant P_+ f$ on X and $f(E)$ has void interior, where $E = \{x \in X | \max\{PD_+ f(x), PD_- f(x)\} \leqslant 0\}$. Then f is nondecreasing.*

PROOF. Suppose, for a contradiction, that $f(b) < f(a)$ for some points $a, b \in X$ where $a < b$. Then $[a, b] \subset X$ since X is connected. Now, since $f(E)$ has void interior, we can choose a point $k \notin f(E)$ such that $f(b) < k < f(a)$. We put

$$A = \{x \in [a, b] | f(x) > k \text{ or, else, } f(x) = k \text{ and } PD_+ f(x) > 0\}$$

and $B = [a, b] \setminus A$. Then, since $k \notin f(E)$, $PD_- f(x) > 0$ for every $x \in B$ at which $f(x) = k$. If $x \in A$ and $f(x) > k$, then by hypothesis $P_+ f(x) \geqslant f(x) > k$, so that $B \in S(x+)$ since $f(y) \leqslant k$ for all $y \in B$. If $x \in A$ and $f(x) = k$, then $PD_+ f(x) > 0$ and, hence, we must have $B \in S(x+)$ because $f(y) - f(x) = f(y) - k \leqslant 0$ for all $y \in B$. Hence, recalling Corollary 3.1.1, we see that $d^+(B, x) < 1$ for all $x \in A$. Similarly, $d^-(A, x) < 1$ for all $x \in B$. Since, moreover, $a \in A$ and $b \in B$, we arrive at a contradiction to Lemma 2.3, and the proof ends.

REMARK 4.1. The preceding theorem is a positive improvement over Theorem 1.1 of Sonouchi and Utagawa (1949), which states that if f: $[a, b] \to R$ is measurable and $\underline{ADf} \geqslant 0$ on $[a, b]$, then f is nondecreasing. For, in this theorem we may suppose that $\underline{ADf} > 0$, since, otherwise, we could consider the function $f(x) + \varepsilon x$ for arbitrary $\varepsilon > 0$; also, the condition $\underline{ADf} > 0$ trivially implies that $A^- f \leqslant f \leqslant A_+ f$.

5. Proximal absolute continuity

By a *subdivision* of a set E we shall mean a finite family (possibly empty) of pairwise disjoint open intervals whose endpoints belong to E. A sequence $\{E_n\}$ of sets whose union is E will be called an *E-form* with *parts* E_n. An expanding

https://doi.org/10.1017/S1446788700018462 Published online by Cambridge University Press

E-form will be called an *E-chain*. If $E_0 \subset E$ and $E \setminus E_0$ is countable (finite or denumerable), an E_0-chain will be called a co-countable E-chain or *c.c. E-chain*.

Given $E \subset X \subset R$ and f: $X \to R$, let $\underline{V}(f, E; r)$ [$V(f, E; r)$] denote, for every $r > 0$, the supremum of the sums $\Sigma(f(b_k) - f(a_k))$ [$\Sigma|f(b_k) - f(a_k)|$] for all subdivisions $\{(a_k, b_k)\}$ of E with $\Sigma(b_k - a_k) < r$. Then $0 \leqslant \underline{V}(f, E; r) \leqslant V(f, E; r) \leqslant \infty$. Also, if $E_1 \subset E$ and $r_1 \leqslant r$, then $\underline{V}(f, E_1; r_1) \leqslant \underline{V}(f, E; r)$ and $V(f, E_1; r_1) \leqslant V(f, E; r)$. We define $\underline{V}(f, E; 0) = \inf_{r>0} \underline{V}(f, E; r)$, $V(f, E; 0) = \inf_{r>0} V(f, E; r)$ and $V(f, E) = \sup_{r>0} V(f, E; r)$.

We note that f is absolutely continuous, AC, [of bounded variation, VB,] on E if and only if $V(f, E; 0) = 0$ [$V(f, E) < \infty$]. The function f is said to be AC above [AC below] (Kennedy (1930–31)) on E if $\underline{V}(f, E; 0) = 0$ [$\underline{V}(-f, E; 0) = 0$]. It is ACG above [ACG below] on E, if it is AC above [AC below] on each part of an E-form. It is ACG on E, if it is both ACG above and ACG below on E (Kubota (1963); Definition 2.2). (This definition of ACG includes the one given by Saks (1937; p. 223); it does not imply any kind of continuity of $f|E$.) We propose further generalizations of these notions.

DEFINITION 5.1. If for an $\varepsilon > 0$ there is a c.c. E-chain $\{E_n\}$ such that $\underline{V}(f, E_n; 0) < \varepsilon$ for every n, then f is said to be proximally ε-AC above, ε-PAC above, on E. If f is ε-PAC above on E for every $\varepsilon > 0$, then f is said to be PAC above on E.

If $-f$ is ε-PAC above [PAC above] on E, we say that f is ε-PAC below [PAC below] on E. If f is both PAC above and PAC below on E, we say that f is *proximally absolutely continuous*, PAC, on E. If f is PAC above [PAC below] on each part of an E-form, then f is said to be PACG above [PACG below] on E. If f is both PACG above and PACG below on E, we say that f is PACG on E.

Clearly, these properties are all hereditary.

LEMMA 5.1. *Let $\{A_n\}$, $\{B_n\}$ denote any two c.c. E-chains. Then $\{A_n \cap B_n\}$ is a c.c. E-chain.*

PROOF. Let $A = \bigcup_n A_n$, $B = \bigcup_n B_n$, $E_n = A_n \cap B_n$ and $E_0 = \bigcup_n E_n$. Since both $\{A_n\}$ and $\{B_n\}$ are expanding, so is $\{E_n\}$. Clearly $E_0 \subset A \cap B$. But, if $x \in A \cap B$, then $x \in A_m \cap B_n$ for some m, n, so that $x \in A_k \cap B_k = E_k$ where $k = \max\{m, n\}$. Consequently $E_0 = A \cap B$. Since both $E \setminus A$ and $E \setminus B$ are countable, it follows that $E \setminus E_0 = E \setminus (A \cap B) = (E \setminus A) \cup (E \setminus B)$ is countable. Hence $\{E_n\}$ is a c.c. E-chain, which proves the lemma.

NOTE 5.1. From any two E-forms $\{A_n\}$, $\{B_n\}$ we get an E-form with parts $A_m \cap B_n$, considering all $m, n \in J$.

LEMMA 5.2. *For all nonnegative real numbers a, b and r, we have* (i) $\underline{V}(f, E; r) \leq V(f, E; r) = V(-f, E; r) \leq \underline{V}(f, E; r) + \underline{V}(-f, E; r)$ *and* (ii) $\underline{V}(af + bg, E; r) \leq a \cdot \underline{V}(f, E; r) + b \cdot \underline{V}(g, E; r)$.

Here g denotes another real valued function whose domain includes E. The proofs are straightforward. The only point that deserves mention is that $\Sigma_k |f(b_k) - f(a_k)| = \Sigma_i (f(b_i) - f(a_i)) + \Sigma_j ((-f)(b_j) - (-f)(a_j))$, where i ranges over those suffixes k for which $f(b_k) - f(a_k) > 0$ and j ranges over the remaining ones.

Using Lemmas 5.1, 5.2 and Note 5.1, we easily obtain:

THEOREM 5.1. (i) *f is AC on E if and only if $\underline{V}(f, E; 0) = 0 = \underline{V}(-f, E; 0)$.*

(ii) *f is PAC on E if and only if there exists, for any $\varepsilon > 0$, a c.c. E-chain $\{E_n\}$ such that $V(f, E_n; 0) < \varepsilon$ for every n.*

(iii) *If f is ε_1-PAC above on E and g is ε_2-PAC above on E, then $af + bg$ is $(a\varepsilon_1 + b\varepsilon_2)$-PAC above on E for any two positive real numbers a, b.*

(iv) *If f and g are both PAC [PACG] on E, then $af + bg$ is PAC [PACG] on E for any two real numbers a, b.*

It is now clear that each of the conditions AC above, AC below, ACG above, ACG below, AC and ACG implies the corresponding condition in the proximal sense. On the other hand:

ANNOUNCEMENT 5.1. For any closed interval I, there exist functions which are approximately continuous and PAC on I, but which are neither ACG above nor ACG below on I.

Our construction of such a function (which should be approximately continuous) requires much space and will be given elsewhere. However, barring approximate continuity, we have the following example which reveals certain other interesting features.

EXAMPLE 5.1. Let $\{c_{2n}\}_{n \in J}$ denote an enumeration (with distinct terms) of the set of rational numbers, and let $c_{2n-1} = c_{2n} + 2^{1/2}$. Then $\{c_{2n}\}$ and $\{c_{2n-1}\}$ are disjoint dense subsets of R. Define

$$u(x) = \sum_{c_n < x} (-1)^n \cdot 2^{-n}, \qquad x \in R.$$

We note that the series $\Sigma(-1)^n 2^{-n}$ is absolutely convergent. Then u is clearly VB on R. Also, $\{c_n\}$ is precisely the set of the points of discontinuity of u. Now,

https://doi.org/10.1017/S1446788700018462 Published online by Cambridge University Press

given $\varepsilon > 0$, we can find $p \in J$ such that

$$\text{(A)} \qquad \sum_{n>p} 2^{-n} < \varepsilon.$$

We choose $h > 0$ sufficiently small so that no two distinct members of $\{(c_k - h, c_k + h)\}_{k \leqslant p}$ intersect. For each $n \in J$, let

$$R_n = R \setminus \bigcup_{k \leqslant p} ((c_k - h/n, c_k) \cup (c_k, c_k + h/n)).$$

Then R_n is clearly an R-chain (more than a c.c. R-chain!). Fix $n \in J$. If $\{(a_i, b_i)\}$ is any subdivision of R_n with $\Sigma(b_i - a_i) < h/n$, then evidently none of the intervals $[a_i, b_i]$ contains any point c_k with $k \leqslant p$, and, hence, simple computations show that

$$\text{(B)} \qquad \sum |u(b_i) - u(a_i)| \leqslant \sum_{n>p} 2^{-n}.$$

From (B) and (A) we get $V(u, R_n; h/n) < \varepsilon$, whence $V(u, R_n; 0) < \varepsilon$. Hence by Theorem 5.1(ii) u is PAC on R.

We assert that u is neither ACG above nor ACG below on any interval I. In fact, we will show that u is neither ACG above nor ACG below on $E = I \setminus \{c_n\}$. The interesting part of this is that $u|E$ is continuous and PAC on E.

Consider any E-form $\{E_n\}$. We note that the set E is a G_δ, because I is a G_δ and $\{c_n\}$ is an F_σ. Then, by Baire's theorem (Saks (1937); (9.2), p. 54), there exist an index n and an open interval $I_0 \subset I$ such that $E \cap I_0 \neq \varnothing$ and E_n is dense in $E \cap I_0$. Since E is evidently dense in I_0, it follows that E_n is dense in I_0. By our choice of the sequence $\{c_n\}$, we can find a point $c_m \in I_0$ with an even or odd index as we like. Since E_n is dense in I_0, given any $r > 0$ we can find points $a, b \in E_n$ with $a < c_m < b$ such that $b - a < r$, and such that

$$\text{(C)} \qquad \sum_{a<c_n<c_m} 2^{-n} + \sum_{c_m<c_n<b} 2^{-n} < 2^{-m-1}.$$

Suppose m is even by choice. Then we have

$$u(b) - u(a) = \sum_{a<c_n<c_m} (-1)^n 2^{-n} + 2^{-m} + \sum_{c_m<c_n<b} (-1)^n 2^{-n},$$

whence by (C) we get $u(b) - u(a) > 2^{-m} - 2^{-m-1} = 2^{-m-1}$. Since $r > 0$ is arbitrary, it follows that $\underline{V}(u, E_n; 0) \geqslant 2^{-m-1}$. Thus u is not AC above on the set E_n. Similarly, supposing m odd by choice, we get $\underline{V}(-u, E_n; 0) > 2^{-m-1}$, so that u is not AC below on the set E_n. Thus every E-form has at least one part on which u is neither AC above nor AC below. In particular, therefore, u is neither ACG above nor ACG below on E.

Definition 5.2 (Saks (1937); p. 224). The function f is said to satisfy Lusin's condition (N) on E if $|f(H)| = 0$ for every $H \subset E$ with $|H| = 0$.

THEOREM 5.2 (see Saks (1937); (6.1), p. 225). *Let f be PACG on E. Then f satisfies Lusin's condition* (N) *on E.*

PROOF. Consider any subset $H \subset E$ with $|H| = 0$. Suppose first that f is PAC on H. Then for any $\varepsilon > 0$ there exists, by Theorem 5.1(ii), a c.c. H-chain $\{H_n\}$ such that $V(f, H_n; 0) < \varepsilon$ for every n. So there is a sequence $\{r_n\}$ of positive numbers such that $V(f, H_n; r_n) < \varepsilon$ for every n. Now, $|H| = 0$ implies $|H_n| = 0$ for all n. Fix an index n. Since $|H_n| = 0$, H_n can be covered by a family $\{I_k\}$ of pairwise disjoint open intervals with $\Sigma|I_k| < r_n$. Since $|f(H_n \cap I_k)|$ cannot exceed the oscillation of f on $H_n \cap I_k$, simple computations show that $|f(H_n)| \leq \Sigma_k|f(H_n \cap I_k)| \leq V(f, H_n; r_n) < \varepsilon$. Consequently, since the sequence $\{f(H_n)\}$ is expanding and since the outer Lebesgue measure is regular, we have (cf. Saks (1937); (6.1), p. 51) $|f(\bigcup_n H_n)| = |\bigcup_n f(H_n)| = \lim|f(H_n)| \leq \varepsilon$. Hence $|f(H)| \leq \varepsilon$, because $H \setminus \bigcup_n H_n$ is countable. Since $\varepsilon > 0$ is arbitrary, we get $|f(H)| = 0$.

Now, in the general case, f being PACG on E there is an E-form $\{E_n\}$ on each part of which f is PAC. Then by above $|f(H \cap E_n)| = 0$ for all n. Hence, since $\bigcup_n E_n = E \supset H$, it follows that $|f(H)| = |\bigcup_n f(H \cap E_n)| = 0$, which completes the proof.

THEOREM 5.3. *Suppose that f is PACG and measurable on E. Then* $(ap)f'$ *exists finitely a.e. on E.*

PROOF. Since f is PACG on E, it is PAC on each part of an E-form $\{E_n\}$. Fix an index k. By Theorem 5.1(ii) there is a c.c. E_k-chain $\{A_n\}$ such that $V(f, A_n; 0) < 1$ for all n. So there is a sequence $\{r_n\}$ of positive numbers such that $V(f, A_n; r_n) < 1$ for all n. For any integer j (positive, negative or zero), let $A_{nj} = A_n \cap [jr_n/2, (j+1)r_n/2]$. Then A_{nj} has diameter less than r_n and, hence, $V(f, A_{nj}) = V(f, A_{nj}; r_n) \leq V(f, A_n; r_n) < 1$. Thus f is VB on A_{nj}. Since, further, f is measurable on $E \supset A_{nj}$, it follows that (Saks (1937); (4.2), p. 222) $(ap)f'$ exists finitely a.e. on A_{nj}. Since $\bigcup_{j=-\infty}^{\infty} A_{nj} = A_n$ and $E_k \setminus \bigcup_n A_n$ is countable and $\bigcup_{k=1}^{\infty} E_k = E$, it follows that $(ap)f'$ exists finitely a.e. on E, completing the proof.

THEOREM 5.4. *Let X be connected and let there exist, for some $\varepsilon > 0$, a function $u: X \to R$ such that u is ε-PAC above on E and such that either* (i) $D_+(f+u) > -\infty$ *n.e. on E or* (ii) $\underline{AD}\,(f+u) > -\infty$ *n.e. on E. Then f is ε-PAC below on E.*

PROOF. Since u is ε-PAC above on E, there is a c.c. E-chain $\{A_n\}$ such that

$$\underline{V}(u, A_n; 0) < \varepsilon \quad \text{for all } n \in J.$$

Now, put $g = f + u$. In case (i), let E_n denote for each $n \in J$ the set of points $x \in A_n$ such that $g(y) - g(x) > -n\,(y - x)$ whenever $0 < y - x < 1/n$, $y \in X$. Observe that if $D_+\, g(x) > -\infty$ at a point $x \in \bigcup_n A_n$, then there are positive integers m_x, n_x and p_x such that $x \in A_{m_x}$, $D_+\, g(x) > -n_x$, and $g(y) - g(x) > -n_x(y - x)$ whenever $0 < y - x < 1/p_x$, $y \in X$; consequently, since the sequence $\{A_n\}$ is expanding, for every positive integer $n > \max\{m_x, n_x, p_x\}$ we have $x \in A_n$, and $g(y) - g(x) > -n(y - x)$ whenever $0 < y - x < 1/n$, $y \in X$, so that $x \in E_n$. Hence, since $E \setminus \bigcup_n A_n$ is countable and since by (i) $D_+\, g > -\infty$ n.e. on $\bigcup_n A_n$, it follows that $E \setminus \bigcup_n E_n$ is countable. Also, the sequence $\{E_n\}$ is evidently expanding. Thus $\{E_n\}$ is a c.c. E-chain. Since $g(x_2) - g(x_1) > -n(x_2 - x_1)$ for any two points $x_1, x_2 \in E_n$ with $0 < x_2 - x_1 < 1/n$, we have further $\underline{V}(-g, E_n; 0) = 0$ for all $n \in J$. Therefore, using Lemma 5.2(ii) and noting that $E_n \subset A_n$, we have

$$\underline{V}(-f, E_n; 0) = \underline{V}(-g + u, E_n; 0) \leqslant \underline{V}(-g, E_n; 0) + \underline{V}(u, E_n; 0) \leqslant \underline{V}(u, A_n; 0) < \varepsilon$$

for all n. Hence f is ε-PAC below on E.

In case (ii), for each $n \in J$ and $x \in E$ put

$$B_{nx} = \{y \in X \mid y \neq x, (g(y) - g(x))/(y - x) \leqslant -n\}.$$

Let F_n denote the set of points $x \in A_n$ such that

$$|B_{nx} \cap (x, x + h)| < \tfrac{1}{2}h \quad \text{whenever } 0 < h < 1/n, \tag{11}$$

$$|B_{nx} \cap (x - h, x)| < \tfrac{1}{2}h \quad \text{whenever } 0 < h < 1/n. \tag{12}$$

Using the meaning of $\underline{AD}g(x) > -\infty$ and arguing as above, we show that $\{F_n\}$ is a c.c. E-chain. If $x_1, x_2 \in F_n$ and $0 < x_2 - x_1 < 1/n$, then by (11) and (12) we get $|(B_{nx_1} \cup B_{nx_2}) \cap (x_1, x_2)| < |(x_1, x_2)|$. So there are points $y \in (x_1, x_2) \setminus (B_{nx_1} \cup B_{nx_2})$. For any such y (which must belong to X since X is connected) we have $g(y) - g(x_1) > -n(y - x_1)$ and $g(x_2) - g(y) > -n(x_2 - y)$, whence $g(x_2) - g(x_1) > -n(x_2 - x_1)$, which implies as before $\underline{V}(-g, F_n; 0) = 0$ for all n. Since $F_n \subset A_n$, it follows as above that $\underline{V}(-f, F_n; 0) < \varepsilon$ for all n. Consequently f is ε-PAC below on E, and the proof ends.

6. The proximally continuous Perron integrals

DEFINITION 6.1. Given $f\colon I = [a, b] \to R_e$, a function $u\colon I \to R$ is termed a right upper (major) function of f in the proximal sense if $u(a) = 0$, $P^-u \leqslant u \leqslant P_+u$ on I, $PD_+u > -\infty$ n.e. on I and $PD_+u \geqslant f$ a.e. on I.

A function l is termed a right lower (minor) function of f in the proximal sense if $-l$ is a right upper function of $-f$ in the proximal sense.

Henceforth u and l, with or without suffixes, will denote respectively upper and lower functions of f as defined above.

THEOREM 6.1. *$u - l$ is nondecreasing and nonnegative.*

PROOF. Arguing as in the theory of ordinary Perron integral, we have $PD_+(u - l) > -\infty$ n.e. on I. Also, there is a subset $E \subset I$ of measure zero such that $PD_+(u - l) \geqslant 0$ on $I \setminus E$. By a well-known method (see Saks (1937); p. 203, supra) we can find, for any $\varepsilon > 0$, a nondecreasing function v: $I \to R$ such that $v(b) - v(a) < \varepsilon$ and $v'(x) = \infty$ for every $x \in E$. Let $g(x) = u(x) - l(x) + v(x) + \varepsilon$, $x \in I$. Then evidently g fulfils the conditions of Theorem 4.3 and so g is nondecreasing, whence the theorem follows since $\varepsilon > 0$ is arbitrary and since $u(a) - l(a) = 0$.

DEFINITION 6.2. The function f is said to be right Perron integrable in the proximal sense, PP_r-integrable, if there exist a sequence $\{u_n\}$ and a sequence $\{l_n\}$ such that (i) u_n is $(1/n)$-PAC below on I, (ii) l_n is $(1/n)$-PAC above on I and (iii) $\lim u_n(b) = \lim l_n(b)$. When f is PP_r-integrable, the common limit in (iii) is called the definite PP_r-integral of f, and it is denoted by $PP_r - \int_a^b f$.

Our definition of the value of $PP_r - \int_a^b f$ is to be justified by showing its uniqueness. To this end, let $\{u_n\}$ and $\{\bar{u}_n\}$ be two sequences of upper functions, and let $\{l_n\}$ and $\{\bar{l}_n\}$ be two sequences of lower functions, of f, such that $\lim u_n(b) = \lim l_n(b)$ $(= L$, say) and $\lim \bar{u}_n(b) = \lim \bar{l}_n(b)$ $(= \bar{L}$, say). By Theorem 6.1, we have $u_n(b) \geqslant \bar{l}_n(b)$ and $\bar{u}_n(b) \geqslant l_n(b)$ for all n. Therefore $L \geqslant \bar{L}$ and $\bar{L} \geqslant L$, whence $L = \bar{L}$. We note further that $l_m(b) \leqslant u_n(b)$ for all m, n, which gives, in particular, that L is finite.

As a simple consequence of Theorem 6.1, we now get:

THEOREM 6.2. *The function f is PP_r-integrable if and only if there exist, for any $\varepsilon > 0$, an upper function u and a lower function l such that u is ε-PAC below on I, l is ε-PAC above on I and $u(b) - l(b) < \varepsilon$.*

REMARK 6.1. An ordinary upper function u defined by Bauer (1915) is continuous and fulfils the condition $\underline{D}u > -\infty$ on I, that defined by Hake (1921) is continuous and fulfils the condition $D_+u > -\infty$ on I, and that defined by Saks (1937) is not necessarily continuous but fulfils the condition $\underline{D}u > -\infty$ on I. We observe that the condition $\underline{D}u > -\infty$ trivially implies that $\limsup_{t \to x-} u(t) \leqslant u(x) \leqslant \liminf_{t \to x+} u(t)$ for all $x \in I$. Thus, our method of defining upper and lower functions is more general than the methods of Bauer, Hake and Saks. We note further that the condition $D_+u > -\infty$ on I trivially implies, by Theorem 5.4, that u is PAC below on I.

Burkill (1931) defined the AP-integral using the method of Bauer, and Sonouchi and Utagawa (1949) modified it using the method of Saks. In either case, an upper function u of f fulfils the conditions $u(a) = 0$ and $-\infty \neq \underline{AD}u \geq f$ on I; and l is a lower function of f if $-l$ is an upper function of $-f$. The condition $\underline{AD}u > -\infty$ implies, by Theorem 5.4, that u is PAC below on I, and also, by definition of $\underline{AD}$, that $A^-u \leq u \leq A_+u$ on I. If f possesses AP-upper functions u and AP-lower functions l, and if $\inf u(b) = \sup l(b)$ $(= L$, say), then f is said to be AP-integrable and the common value L is called the definite AP-integral of f on I. From the relations between the approximate and proximal limits and derivates it is now clear that the PP_r-integral comprises the AP-integral. The following example shows that the PP_r-integral is substantially more general than the AP-integral.

EXAMPLE 6.1. Take the set $E = \bigcup_n(a_n, b_n)$ of Example 3.1, and put $c_n = \frac{1}{2}(a_n + b_n)$. Consider the function F on $I = [0, b_1]$ which is 0 on $I \setminus E$, 1 on $\{c_n\}_{n\in J}$ and is linear on the intervals $[a_n, c_n]$ and $[c_n, b_n]$. Define $f(x) = F'(x)$ if the derivative exists finitely and $f(x) = 0$ otherwise. Then F is both a right upper and a right lower function of f on I in the proximal sense; moreover it is PAC on I. (In fact, F is AC on $[a_n, b_1]$ for all $n \in J$.) Consequently f is PP_r-integrable on I and $F(x) = PP_r - \int_0^x f$ for all $x \in I$. F is proximally continuous, but not approximately continuous, at the origin. Since an indefinite AP-integral is necessarily approximately continuous everywhere (Sonouchi and Utagawa (1949); Theorem 1.4), and since we clearly have $AP - \int_x^{b_1} f = F(b_1) - F(x)$ for all $x \in (0, b_1)$, it follows that f is not AP-integrable on I.

With Theorems 6.1 and 6.2 in hand, it is easy to show that the PP_r-integral possesses properties analogous to those of the AP-integral. We omit the routine statements and proofs, but we use them to prove the following three theorems which are to some extent new either in respect of proof or in content. In these theorems f is assumed to be PP_r-integrable and F denotes the indefinite integral defined by $F(x) = PP_r - \int_a^x f$.

THEOREM 6.3. *F is proximally continuous on* $[a, b]$.

PROOF. Fix any $c \in (a, b]$. Consider a sequence $\{u_n\}$ and a sequence $\{l_n\}$ such that $\lim u_n(c) = \lim l_n(c) = F(c)$. Since $l_n(x) \leq F(x) \leq u_n(x)$ for all $x \in I$, we have $P_-l_n(c) \leq P_-F(c) \leq P^-F(c) \leq P^-u_n(c)$. But by definitions of u_n and l_n, $l_n(c) \leq P_-l_n(c)$ and $P^-u_n(c) \leq u_n(c)$. Thus $l_n(c) \leq P_-F(c) \leq P^-F(c) \leq u_n(c)$, whence, letting $n \to \infty$, we get $P_-F(c) = P^-F(c) = F(c)$. Similarly, for any $c \in [a, b)$ we get $P_+F(c) = P^+F(c) = F(c)$. Hence F is proximally continuous on $[a, b]$.

Corollary 6.3.1. *u and l are proximally continuous n.e. on I and they possess finite unique unilateral proximal limits everywhere on I.*

(Hints: $u - F$ and $F - l$ are nondecreasing.)

Theorem 6.4. *F is PAC on I.*

Proof. Fix any $\varepsilon > 0$. By Theorem 6.2 there exist a u and an l such that u is ε-PAC below on I, l is ε-PAC above on I and $u(b) - l(b) < \varepsilon$. Now, $u - F$ is nondecreasing, $u(a) - F(a) = 0$ and $u(b) - F(b) \leqslant u(b) - l(b) < \varepsilon$. Hence $\underline{V}(u - F, I; 0) < \varepsilon$, so that $u - F$ is ε-PAC above on I. Similarly, $F - l$ is ε-PAC above on I. Consequently, from the representations $F = (F - l) + l$ and $-F = (u - F) + (-u)$, it follows by Theorem 5.1(iii) that F is both 2ε-PAC above and 2ε-PAC below on I. Since $\varepsilon > 0$ is arbitrary, the proof ends.

Corollary 6.4.1. *An indefinite AP-integral is PAC.*

Theorem 6.5. $-\infty < (ap)F' = f < \infty$ *a.e. on I.*

Proof. The usual method of proof (see Sonouchi and Utagawa (1949); Theorem 1.5), with minor modifications, gives $-\infty < PD_+F = PD^+F = f < \infty$ a.e. on I. Then, by Theorem 4.2, F is measurable on I and so, by Theorems 6.4 and 5.3, $(ap)F'$ exists finitely a.e. on I (alternatively, recalling Corollary 3.1.1(i), we could use a result of Denjoy-Khintchine (Saks (1937); (10.1), p. 295)), which completes the proof.

By our convention, there is the theory of *left Perron integral* in the proximal sense, PP_l-integral, analogous to that of the PP_r-integral. Lest there arise any confusion, we mention that a left upper function will differ from a right upper function in respect of the use of derivates only. We prove below that these integrals are compatible.

Theorem 6.6. *Let f be both PP_r- and PP_l-integrable. Then the two definite integrals are equal.*

Proof. Let the indefinite PP_r- and PP_l-integrals of f on I be denoted by F_1 and F_2, respectively. Fix $\varepsilon > 0$, and define $F(x) = F_1(x) - F_2(x) + \varepsilon x$, $x \in I$. Then, by Theorem 6.5 and its analog, $PDF = (ap)F' = \varepsilon > 0$ a.e. on I. Therefore, if $E = \{x \in I \mid PD_+F(x) < 0\}$, then $|E| = 0$. Bu, by Theorem 6.4 and its analog, F_1 and F_2 are PAC on I and, hence, by Theorem 5.1(iv) it follows that F

is PAC on I. So by Theorem 5.2 we have $|F(E)| = 0$. Also, by Theorem 6.3 and its analog, F is proximally continuous on I. Hence by Theorem 4.3 F is nondecreasing. Since $\varepsilon > 0$ is arbitrary, it follows that $F_1 - F_2$ is nondecreasing. Similarly, $F_2 - F_1$ is nondecreasing. Therefore $F_1 = F_2$ on I, since $F_1(a) - F_2(a) = 0$, which completes the proof.

We observe that if $f = f_1 + g_1 = f_2 + g_2$ where f_i is PP_r-integrable and g_i is PP_l-integrable $(i = 1, 2)$, then $f_1 - f_2 = g_2 - g_1$ a.e. on I and, hence, by the preceding theorem

$$PP_r - \int_a^b (f_1 - f_2) = PP_l - \int_a^b (g_2 - g_1),$$

whence

$$PP_r - \int_a^b f_1 + PP_l - \int_a^b g_1 = PP_r - \int_a^b f_2 + PP_l - \int_a^b g_2.$$

This justifies the following definition.

DEFINITION 6.3. A function $f\colon I = [a, b] \to R_e$ is said to be Perron integrable in the proximal sense, PP-integrable, if f can be expressed as $f = f_1 + g_1$ where f_1 is PP_r-integrable and g_1 is PP_l-integrable on I, and then the unique number $PP_r - \int_a^b f_1 + PP_l - \int_a^b g_1$ is called the definite PP-integral of f.

Evidently the PP-integral comprises both the PP_r- and PP_l-integrals and shares their properties.

7. The proximally continuous Denjoy integral

If $F_1, F_2\colon I = [a, b] \to R$ are both PACG and both proximally continuous on I, then by Theorems 4.2, 5.3, both $(ap)F_1'$ and $(ap)F_2'$ exist finitely a.e. on I. Hence, if further $(ap)F_1' = (ap)F_2'$ a.e. on I, then $(ap)(F_1 - F_2)' = 0$ a.e. on I, so that, arguing as in the proof of Theorem 6.6, $F_1 - F_2$ is a constant. This observation justifies the following definition.

DEFINITION 7.1. A function $f\colon I = [a, b] \to R_e$ is said to be Denjoy integrable in the proximal sense, PD-integrable, if there exists a function $F\colon I \to R$ such that (i) F is proximally continuous on I, (ii) F is PACG on I and (iii) $(ap)F' = f$ a.e. on I; and the increment $F(b) - F(a)$ is then called the definite PD-integral of f, and F is called an indefinite PD-integral of f on I.

A function F is termed an indefinite AD-integral of the function f if F is approximately continuous and ACG on I and $(ap)F' = f$ a.e. on I (Kubota (1963); Definition 2.3). Hence the PD-integral comprises the AD-integral. The a.e. finite approximate derivatives of the functions in Announcement 5.1 provide examples of functions which are not AD-integrable but are PD-integrable with indeed approximately continuous indefinite integrals. Evidently the PD-integral possesses properties analogous to those of the PP-integral, and it comprises the latter. We note in conclusion that, by Theorem 4.1, indefinite PP- and PD-integrals are necessarily continuous in Darboux sense.

References

H. Bauer (1915), 'Der Perronsche Integralbegriff und seine Beziehung zum Lebesgueschen', *Monatsh. Math. Phys.* **26**, 153–198.

J. C. Burkill (1931), 'The approximately continuous Perron Integral', *Math. Z.* **34**, 270–278.

A. Denjoy (1916), 'Sur la dérivation et son calcul inverse', *C. R. Acad. Sci. Paris Sér. A-B* **162**, 377–380.

H. Hake (1921), 'Über de la Vallée Poussins Ober- und Unterfunktionen einfacher Integrale und die Integral-definition von Perron', *Math. Ann.* **83**, 119–142.

M. D. Kennedy (1930-31), 'Upper and lower Lebesgue Integrals', *Proc. London Math. Soc.* (2) **32**, 21–50.

Yoto Kubota (1963), 'On the approximately continuous Denjoy Integrals', *Tôhoku Math. J.* **15**, 253–264.

S. Saks (1937), *Theory of the integral* (second revised edition, Warsaw, 1937; reprint, Dover Publ., New York, 1964).

G. Sonouchi and M. Utagawa (1949), 'The generalized Perron Integrals', *Tôhoku Math. J.* **1**, 95–99.

Department of Mathematics
University of Kalyani
Kalyani, West Bengal
India