European Journal of Applied Mathematics (2025), 1–44
doi:10.1017/S0956792525000178

PAPER

On the long-time asymptotics of the modified Camassa–Holm equation with step-like initial data

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Received: 16 October 2024; **Revised:** 17 March 2025; **Accepted:** 24 March 2025

Keywords: modified Camassa–Holm equation; step-like initial data; Riemann–Hilbert problem; long-time asymptotics; airy function

2020 Mathematics Subject Classification: 35Q15, 41A60, 37K15, 35C20

Abstract

We study the long-time asymptotics for the solution of the modified Camassa–Holm (mCH) equation with step-like initial data.

$$m_t + \left(m\left(u^2 - u_x^2\right)\right)_x = 0, \quad m = u - u_{xx},$$
$$u(x,0) = u_0(x) \to \begin{cases} 1/c_+, & x \to +\infty, \\ 1/c_-, & x \to -\infty, \end{cases}$$

where c_+ and c_- are two positive constants. It is shown that the solution of the step-like initial problem can be characterised via the solution of a matrix Riemann–Hilbert (RH) problem in the new scale (y, t). A double coordinate (ξ, c) with $c = c_+/c_-$ is adopted to divide the half-plane $\{(\xi, c) : \xi \in \mathbb{R},\ c > 0,\ \xi = y/t\}$ into four asymptotic regions. Further applying the Deift–Zhou steepest descent method, we derive the long-time asymptotic expansions of the solution $u(y, t)$ in different space-time regions with appropriate g-functions. The corresponding leading asymptotic approximations are given with the slow/fast decay step-like background wave in genus-0 regions and elliptic waves in genus-2 regions. The second term of the asymptotics is characterised by the Airy function or parabolic cylinder model. Their residual error order is $\mathcal{O}(t^{-2})$ or $\mathcal{O}(t^{-1})$, respectively.

1. Introduction

The present paper is concerned with the long-time asymptotic behaviour for the solution of the modified Camassa–Holm (mCH) equation [26, 42]

$$m_t + \left(m\left(u^2 - u_x^2\right)\right)_x = 0, \quad m = u - u_{xx}, \tag{1.1}$$

with step-like initial data

$$u(x,0) = u_0(x) \to \begin{cases} 1/c_+, & x \to +\infty, \\ 1/c_-, & x \to -\infty. \end{cases} \tag{1.2}$$

The mCH (1.1) appeared in [26] as a integrable equation proposed by Fuchssteiner and Fokas and first introduced by Camassa and Holm as a model for the unidirectional propagation of shallow-water waves [8] (see also [15] for a rigorous justification in shallow-water approximation).

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The mCH (1.1) bears some similarity to the celebrated Camassa–Holm (CH) equation

$$m_t + (um)_x + u_x m = 0, \quad m = u - u_{xx}, \tag{1.3}$$

due to the presence of the relation $m = u - u_{xx}$. Different from the CH (1.3), the mCH (1.1) contains the cubic nonlinearity. In view of Fokas and Fuchssteiner [27], Olver and Rosenau [42], the CH (1.3) is obtained from the general method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the Korteweg–de Vries equation, while this method applied to the modified Korteweg–de Vries equation yields the (1.1). Henceforth, the (1.1) was referred to the modified CH equation (see also [43]). The CH (1.3) appeared in [27] as a integrable equation proposed by Fuchssteiner and Fokas and first introduced by Camassa and Holm as a model for the unidirectional propagation of shallow-water waves [8] (see also [15] for a rigorous justification in shallow-water approximation). Therefore, the CH (1.3) has attracted considerable interest and studied extensively due to its rich mathematical structures and remarkable properties, such as peakon and multi-peakon solutions, bi-Hamiltonian structure, algebro-geometric solutions, wave-breaking phenomena [8, 13, 14, 16, 21, 38].

Applying the scaling transformation and taking parameter limit $\epsilon \to 0$,

$$x \mapsto \epsilon x, t \mapsto \epsilon^{-1} t, u \mapsto \epsilon^2 u,$$

the mCH (1.1) can be reduced a short pulse equation [41]

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}.$$

More recently, the mCH (1.1) was considered as a model for the unidirectional propagation for shallow-water waves of mild amplitude over a flat bottom [12], where the solution u is related to the horizontal velocity at a specific water level. It is noted that the global smooth one-soliton solution of the mCH (1.1) with nonzero background data were obtained by using the RH method [2]. On the other hand, the soliton of the mCH (1.1) with zero background data is a weak solution in the form of peaked wave. In addition, the quasi-periodic solutions with periodic background data were constructed by using algebro-geometric method [31]. The wave-breaking and those peakons for the mCH (1.1) with zero background data were also investigated in [30]. The existence of the global peakon solutions and the large time asymptotic behaviour of these kind of non-smooth solitons were investigated in [10]. It is known that the Cauchy problem associated with the mCH (1.1) is the locally well-posed in the Sobolev space $H^s(\mathbb{R})$, $s > 5/2$ [30]. Recently, the long-time asymptotic behaviour of the mCH (1.1) with linear dispersion term was established by using $\bar{\partial}$-steepest descent analysis in [46]. Based on the RH problem established in [2], Boutet de Monvel et al. studied the long-time asymptotic behaviour of the mCH (1.1) under nonzero boundary conditions via nonlinear steepest descent method.

Initial value problems for nonlinear evolution equations with step-like initial data have attracted much attention since the early 1970s [34]. The implementation of the rigorous asymptotic analysis to step-like initial value problems for integrable equations started in the paper [7], which extended the methods from Deift, Venakides, and Zhou [17]. Since then, problems with step-like initial data have also been considered for a variety of integrable systems such as the KdV equation [22], the focusing and defocusing NLS equations [1, 4–6, 25, 32], the modified KdV equation [29, 35] and Camassa–Holm equation [39]. A wide range of important physical phenomena manifest themselves in the behaviour of solutions of such step-like initial value problems for large times, e.g., rarefaction waves [32], modulated waves [45], elliptic waves [4] and so on. The main feature in the long-time behaviour that distinguishes step-like initial conditions from decaying initial conditions is the formation of an oscillatory region that connects the different behaviour at $x \to \pm\infty$ of the solution. These oscillatory regions are typically described by elliptic or hyperelliptic modulated waves. Very recently, Karpenko, Shepelsky, and Teschl develop the RH formalism to the mCH (1.1) with step-like initial data (1.2) and give a representation for the solution of this problem in terms of the solution of an associated RH problem [33].

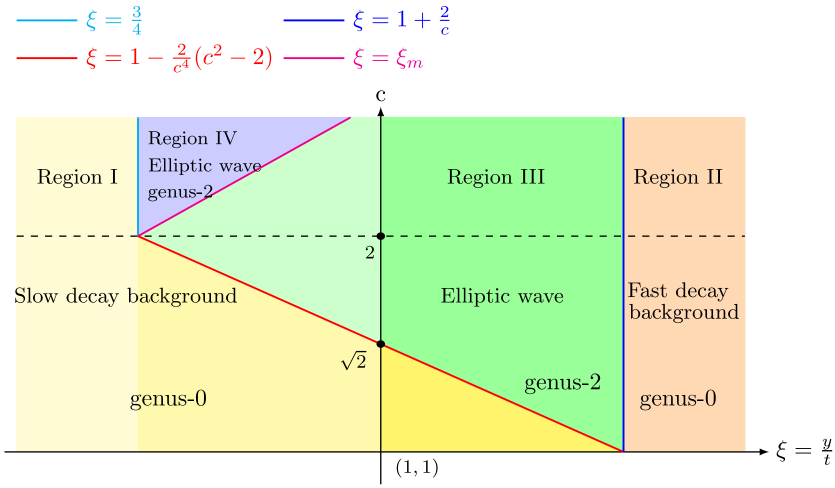

Figure 1. *Asymptotic approximations of the mCH equation in different space-time-(ξ, c) regions, where the Regions I (yellow) and II (orange) corresponding to genus-0, they are slow-decay and fast-decay background regions, respectively; The Regions III (green) and IV (purple) corresponding to genus-2 region, they are the first-type and second-type elliptic wave regions. Here, ξ_m is the critical condition that under the case of Region III, the stationary point of g-function merges to c. The Region I is a unit of three subregions. We use three shades of yellow to distinguish these three subregions. The Region III is a unit of two subregions, where we use two shades of green to distinguish it.*

1.1. Statement of results

The purpose of our present work is to investigate the long-time asymptotic behaviour of the mCH (1.1) with step-like initial data (1.2). Notice that under the transformation

$$u(x, t) \mapsto c_-u(x, c_-^2 t), \tag{1.4}$$

$c_-u(x, c_-^2 t)$ is also a solution of the mCH (1.1), and the condition (1.2) becomes

$$u(x, 0) = u_0(x) \to \begin{cases} 1/c, & x \to +\infty, \\ 1, & x \to -\infty, \end{cases} \tag{1.5}$$

where $c = c_+/c_-$. Therefore, without loss of generality, let

$$c_- = 1, \qquad c_+ = c \geq 1, \tag{1.6}$$

in (1.2). For brevity, we will continue to adopt the notations c_- and c_+, but their exact values are given by (1.6).

We find that the types of asymptotic expansions for the mCH (1.1) are closely related to the scope of two parameter $\xi = y/t$ and c, where y is a new space variable defined by

$$y(x, t) = x + \int_{-\infty}^{x} \left(m(s, t) - \frac{1}{c_-} \right) ds. \tag{1.7}$$

So in our paper we adopt double coordinates (ξ, c) to divide the upper half plane $\{(\xi, c) : \xi \in \mathbb{R}, c > 1\}$ into four different space-time regions (see Figure 1), in which we will present different leading order asymptotic approximations for the mCH (1.1) with step-like initial value (1.5). Our results are subject to the following assumption:

Assumption 1. *The reflection coefficients defined by (2.16), associated to the initial data u_0, are analytic on $\mathbb{C} \setminus [-c, c]$.*

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This assumption is set similar to [5, 6]. On one hand, the initial data are smooth and approaches to the backgrounds quickly enough such that the reflecting coefficients are meromorphic on $\mathbb{C}\setminus[-c,c]$. It is only made to simplify the proof and only affect the error order of the final asymptotic formulas in our main result. It allows us to avoid the technical work associated with the introduction of $\bar\partial$-extensions of the jump matrices to perform the steepest descent analysis like in [46]. On the other hand, the assumption ensures that the initial data are "no soliton" and generic, namely, the corresponding spectral problem has no eigenvalue and spectral singularity. Then the reflection coefficients defined by (2.16) have no poles. The combination of these two aspects results in the analyticity of the reflecting coefficients on $\mathbb{C}\setminus[-c,c]$.

Theorem 1.1. *Let $u(x,t)$ be the solution for the initial-value problem (1.1) and (1.5). Denote $\xi=y/t$ with y defined in (1.7). As $t\to\infty$, the long-time asymptotics of the mCH (1.1) are given as follows.*

Region I: *(i) $\{(\xi,c): c\geq 1,\ \xi<3/4\}$; (ii) $\{(\xi,c): 1\leq c\leq\lambda_1,\ 3/4<\xi<1\}$; (iii) $\{(\xi,c): 1\leq c\leq \lambda_1,\ 1\leq\xi<3\}$, with*

$$\lambda_1 := \lambda_1(\xi)=\left(\frac{1-\sqrt{4\xi-3}}{1-\xi}\right)^{1/2}, \tag{1.8}$$

whose branch is selected by $\lambda_1(1)=\sqrt{2}$. It is a slow decay step-like background constant region with genus-0 and admits asymptotic expansion

$$u(x,t)=u(x(y,t),t)=1+u^{(1)}(\xi)t^{-\frac12}+\mathcal{O}(t^{-1}), \tag{1.9}$$

$$x=y-2I_\delta^1-2y^{(1)}(\xi)t^{-\frac12}+\mathcal{O}(t^{-1}), \tag{1.10}$$

where $u^{(1)}$ defined in (7.2) comes from the parabolic cylinder function, and I_δ^1, $u^{(1)}$ and $y^{(1)}$ are given in (3.3) and (7.3), respectively.

Region II: $\{(\xi,c):\xi>1+2/c,\ c\geq 1\}$. *It is a fast decay step-like background constant region with genus-0. We have asymptotic expansion*

$$u(x,t)=u(x(y,t),t)=c^{-1}+\mathcal{O}(e^{-Ct}), \tag{1.11}$$

$$x(y,t)=y-2\ln\left(\delta(\infty)e^{I_\delta^1+ia(y,t)}\right)+\mathcal{O}(e^{-Ct}), \tag{1.12}$$

where $a(y,t)=-\frac{i}{2}(c+1)y+\frac{it}{2}\left(c^{-2}+c\right)$, I_δ^1 and $\delta(\infty)$ are given in Proposition 5 and C is a positive constant.

Region III: *Genus-2 elliptic wave region.*

(i) $\{(\xi,c): c>\sqrt{2},\ 1\leq\xi<1+2/c\}\cup\{(\xi,c): 2<c,\ 1+\frac{2}{c^4}(c^2-2)<\xi<1+\frac{2}{c}\}$, we have asymptotic expansion

$$u(x,t)=u(x(y,t),t)=u^{(3)}(y,t;\xi)+t^{-1}\mathcal{E}(\xi)+\mathcal{O}(t^{-2}),$$

$$x(y,t)=y-2\ln\left(-ie^{-itg(\infty)+it(p_+^{(-)}-g_+)(0)}\delta_\infty(0)\delta_+(0)M_{12,+}^{mod}(0)\right)$$

$$+2i\frac{H_{11}^{(0)}M_{12,+}^{mod}(0)+H_{12}^{(0)}M_{22,+}^{mod}(0)}{M_{12,+}^{mod}(0)}t^{-1}+\mathcal{O}(t^{-2}),$$

where $u^{(3)}(y,t;\xi)$ is constructed by the Riemann theta function associated with the genus 2 Riemann surface shown in (7.5), and $\mathcal{E}(\xi)$ given in (7.6) comes from the combined effect of the Riemann theta function and Airy Model. $g(\infty)$, M^{mod}, $g(z)$, $\delta_\infty(0)$, $\delta_+(0)$ and $H^{(0)}$ are shown in (5.5), (5.13), Proposition 6, 7, and 8, respectively.

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(ii) $\{(\xi, c) : \sqrt{2} < c < 2,\ 1 - \frac{2}{c^4}(2 - c^2) < \xi < 1\} \cup \{(\xi, c) : \xi_m < \xi < 1,\ c > 2\}$, *we have asymptotic expansion*

$$u(x, t) = u(x(y, t), t) = u^{(4)}(y, t; \xi) + t^{-1/2}\mathcal{E}(\xi) + \mathcal{O}(t^{-1}),$$
$$x(y, t) = y - 2\ln\left(-ie^{-itg(\infty)+it(p_+^{(-)}-g_+)(0)}\delta_\infty(0)\delta_+(0)M^{mod}_{12,+}(0)\right)$$
$$+ 2i\frac{H^{(0)}_{11}M^{mod}_{12,+}(0) + H^{(0)}_{12}M^{mod}_{22,+}(0)}{M^{mod}_{12,+}(0)}t^{-1/2} + \mathcal{O}(t^{-1}),$$

where $u^{(4)}(y, t; \xi)$, $\mathcal{E}(\xi)$, *has same expansion as (7.5), (7.6) but the function* $\delta_\infty(0)$, $\delta_+(0)$, $H^{(0)}$ *and* $H^{(1)}$ *are shown in Proposition 9 and 11, respectively.* ξ_m *is the critical velocity that the stationary points of the g-function given in Proposition 6 merges to c.* $\mathcal{E}(\xi)$, $H^{(0)}$ *and* $H^{(1)}$ *represent the contribution of the pairs of stationary points out of cut via parabolic cylinder model.*

Region IV: *Genus-2 elliptic wave region.* $\{\frac{3}{4} < \xi < \xi_m, 2 < c\}$, *we have asymptotic expansion*

$$u(x, t) = u(x(y, t), t) = u^{(5)}(y, t; \xi) + t^{-1}\mathcal{E}(\xi) + \mathcal{O}(t^{-2}),$$
$$x(y, t) = y - 2\ln\left(-ie^{-itg(\infty)+it(p_+^{(-)}-g_+)(0)}\delta_\infty(0)\delta_+(0)M^{mod}_{12,+}(0)\right)$$
$$+ 2i\frac{H^{(0)}_{11}M^{mod}_{12,+}(0) + H^{(0)}_{12}M^{mod}_{22,+}(0)}{M^{mod}_{12,+}(0)}t^{-1} + \mathcal{O}(t^{-2}).$$

where $u^{(5)}(y, t; \xi)$ *and* $\mathcal{E}(\xi)$ *has same expansion as (7.5), (7.6) but the functions* $g(\infty)$, $g(z)$, $\delta_\infty(0)$, $\delta_+(0)$, $H^{(0)}$, $H^{(1)}$ *and* M^{mod} *are shown in Proposition 12, 13, and 14, formula (6.19), respectively.* $\mathcal{E}(\xi)$, $H^{(0)}$ *and* $H^{(1)}$ *represent the common contribution of two local Airy Model of two pairs of stationary points.*

Remark 1.2. We divide the (ξ, c) plane in four parts as shown in above theorem accounting to the g-function appeared in the analysis. Although both Region III and Region IV are genus-2 regions, their g-functions have different expressions.

Remark 1.3. Region I and Region III are comprised of the union of two and three subregions, respectively. In Regions I and III, the subleading term of the asymptotic behaviour in these subregions are different, because these subregions have different number of stationary points. When $\xi \to 1_-$, a pair of stationary points approaches to infinity while a pair of stationary points approaches to $\pm\sqrt{2}$. We find that there is no transition region on the shared boundary $\xi = 1$ in Region I. So does it on the shared boundary $\xi = 1$ in Region III. But as $\xi \to 3/4$, the stationary points will merge, which implies that the asymptotic behaviour may be expressed in terms of solutions of the second Painlevé equation. Our results also hold for $c = 1$.

Remark 1.4. Our result also implies that $x/t = y/t + \mathcal{O}(t^{-1})$. So the division of regions in the (y, t) plane approximates to it on (x, t) plane as $t \to \infty$.

Compared with the works [2, 3, 33], our work has the following different features:

- Consider the mCH (1.1) with a nonzero boundary condition, Boutet de Monvel et al. in [2] constructed its RH problem and exact solutions. Further they obtained long-time asymptotics of the solution by using Deift–Zhou steepest descent method [3]. In our present work, we consider the mCH (1.1) with the step-like initial data condition (1.2), which can reduce the nonzero boundary condition as a special case of (1.2) by taking $c = 1$. Moreover, our long-time asymptotics with the step-like initial data condition becomes more challenging than that [3] which is only described by parabolic cylinder model. Our result requires a elliptic wave model in genus-2, the Airy function model and also the parabolic cylinder model.

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- In Ref. [33], though Karpenko et al. considered the mCH (1.1) with step-like initial data which is the same as ours, they only established its RH problem without consideration of long-time asymptotics. While we focus on its long-time asymptotic behaviours for different space-time regions on the whole (x, t)-plane.

1.2. Out line of the paper

Our paper is arranged as follows. In Section 2, we study the eigenfunctions and scattering data associated with step-like initial value (1.5). Further we analyse their analyticity, symmetries and asymptotic to construct the RH problem for $M(z)$ of step-like initial value problem, which will be used to analyse long-time asymptotics of the mCH equation in our paper. In Section 3 and Section 4, we construct the RH problem associated with the Regions I and II, further transform it into a model RH problem. In Sections 5 and 6, to analyse the RH problem in the regions III and IV, we introduce a g-function in genus two Riemann surface and transform the original RH problem to a RH problem $M^{(2)}(z)$, which is further decomposed into a $M^{mod}(z)$ model problem and an inner local problems. The $M^{mod}(z)$ contributes to the leading term of the asymptotics and is given by Riemann theta functions attached to a hyperelliptic Riemann surface in subsection 5.2.1 and subsection 6.3 in different region. Finally, in Section 7, we give the proof of Theorem 1.1.

2. Direct scattering and the RH problem

2.1. Spectral analysis on the lax pair

The mCH (1.1) admits the Lax pair [2]

$$\Phi_x = X\Phi, \qquad \Phi_t = T\Phi, \tag{2.1}$$

where

$$X = \frac{1}{2}(izm\sigma_2 - \sigma_3), \tag{2.2}$$

$$T = \left(z^{-2} + \frac{u^2 - u_x^2}{2}\right)\sigma_3 - i\left(z^{-1}(u - u_x) + \frac{z}{2}\left(u^2 - u_x^2\right)m\right)\sigma_2 \tag{2.3}$$

and $z \in \mathbb{C}$ is spectrum parameter. Here, we introduce the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since the Lax pair (2.1) admit spectral singularity at $z = \infty$ and $z = 0$, the asymptotic behaviour of the eigenfunction Φ as $z \to \infty$ and $z \to 0$ need to be controlled.

Case I. $z = \infty$. For any real constant $C \neq 0$, we denote a matrix function relying on C

$$D_C(z) = \frac{1}{2}\begin{pmatrix} \phi_C(z) + \phi_C(z)^{-1} & \phi_C(z)^{-1} - \phi_C(z) \\ \phi_C(z)^{-1} - \phi_C(z) & \phi_C(z) + \phi_C(z)^{-1} \end{pmatrix}, \ \phi_C(z) = \left(\frac{C+z}{C-z}\right)^{1/4}, \tag{2.4}$$

where $\phi_C(z)$ is analytic on $\mathbb{C} \setminus [-C, C]$ and the branch is chosen such that as $z \to \infty$, $\phi_C(z) \sim e^{-\frac{i\pi}{4}} + \mathcal{O}(z^{-1})$. Denote

$$\lim_{z\to\infty} D_C(z) = D_C(\infty) = \frac{\sqrt{2}}{2}(I + i\sigma_1), \tag{2.5}$$

which is independent of C. For convenience, we use the notation $f_\pm(z)$ of some function f to denote the boundary values of f from the $\pm$ sides of the oriented jump contours. We set the orientation of all curve

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on the real axis to be directed from the left to the right in this paper. From $\phi_{C,+}(z) = -i\phi_{C,-}(z)$, it follows that

$$D_{C,+}(z) = i\sigma_1 D_{C,-}(z), \quad z \in \Sigma_+,$$

Under the initial value (1.5), we define two gauge transformations

$$\Psi^{\pm}(z;x,t) = D_{c_\pm}(z)\Phi(z;x,t), \tag{2.6}$$

which satisfy the following Lax pair

$$\Psi^{\pm}_x = \left(-\frac{i}{2}m\sqrt{z^2-c_\pm^2}\sigma_3 + P^{\pm}\right)\Psi^{\pm}, \tag{2.7}$$

$$\Psi^{\pm}_t = \left(i\sqrt{z^2-c_\pm^2}\left(\frac{m(u^2-u_x^2)}{2} + \frac{1}{c_\pm z}\right)\sigma_3 + L^{\pm}\right)\Psi^{\pm}, \tag{2.8}$$

with

$$P^{\pm} := i\frac{c_\pm m - 1}{2\sqrt{z^2-c_\pm^2}}\left(c_\pm\sigma_3 + iz\sigma_2\right),$$

$$L^{\pm} := i\left(\frac{c_\pm(u^2-u_x^2)(1-c_\pm m)}{2\sqrt{z^2-c_\pm^2}} - \frac{u-1/c_\pm}{\sqrt{z^2-c_\pm^2}}\right)\sigma_3 + \frac{u_x}{c_\pm}\sigma_1$$
$$-\left(\frac{z(u^2-u_x^2)(1-c_\pm m)}{2\sqrt{z^2-c_\pm^2}} - \frac{c_\pm u - 1}{z\sqrt{z^2-c_\pm^2}}\right)\sigma_2,$$

and the branch of the square root is chosen such that $\sqrt{z^2-c_\pm^2} \sim ic_\pm$, $z \to 0$ in $\mathbb{C}^+$, where $\mathbb{C}^\pm$ denote the upper/lower half complex plane and $c_\pm$ are exactly given in (1.6). For convenience, we denote $\Sigma_\pm = [-c_\pm, c_\pm]$ as the branch cut of $\phi_{c_\pm}(z)$.

Furthermore, we introduce

$$\mu^{\pm}(z;x,t) = \Psi^{\pm}(z;x,t)e^{itp^{(\pm)}(z)\sigma_3}, \tag{2.9}$$

where $p^{(\pm)}(z)$ are defined by

$$tp^{(\pm)}(z) := tp^{(\pm)}(z;x,t) = \frac{\sqrt{z^2-c_\pm^2}}{2}\left(\int_{\pm\infty}^{x}(m(s)-1/c_\pm)ds + \frac{x}{c_\pm} - \frac{2t}{c_\pm z^2} - \frac{t}{c_\pm^3}\right). \tag{2.10}$$

In this paper, whenever convenient, we use $f(z)$ to denote $f(z;x,t)$ to emphasise the dependence on z. Then $\mu^{\pm}(z;x,t)$ solve the two Volterra-type integral equations

$$\mu^{\pm}(z;x,t) = I + \int_{\pm\infty}^{x} e^{\frac{i}{2}\hat{\sigma}_3\sqrt{z^2-c_\pm^2}\int_x^s m(v,t)dv}\left[P^{\pm}(z;s,t)\mu^{\pm}(z;s,t)\right]ds. \tag{2.11}$$

It follows from (2.11) that the Jost functions $\mu^{\pm}(z) := \mu^{\pm}(z;x,t)$ admit two kinds of symmetries

$$\mu^{\pm}(z) = \sigma_1\overline{\mu^{\pm}(\bar{z})}\sigma_1 = \sigma_2\overline{\mu^{\pm}(-z)}\sigma_2^{-1}.$$

Again applying (2.11), it is accomplished that $\det\mu^{\pm}(z) = 1$, and

$$\mu^{\pm}(z) \to I, \quad z \to \infty.$$

Thus it appears that $\mu^{\pm}(z)$ are analytical in $\mathbb{C}\setminus\Sigma_\pm$, respectively. Let

$$\tilde{\mu}^{\pm}(z;x,t) = D^{-1}_{c_\pm}(z)\mu^{\pm}(z;x,t), \tag{2.12}$$

then the Volterra-type integrals (2.11) about $\tilde{\mu}^{\pm}(z) := \tilde{\mu}^{\pm}(z;x,t)$ are changed into

$$\tilde{\mu}^{\pm}(z;x,t) = D^{-1}_{c_\pm}(z) + \int_{\pm\infty}^{x} D^{-1}_{c_\pm}(z)e^{\frac{i}{2}\sqrt{z^2-c_\pm^2}\int_x^s m(l,t)dl\hat{\sigma}_3}D_{c_\pm}(z)$$
$$\cdot\left(X(z;s,t) + \frac{i}{2}m(s,t)\sqrt{z^2-c_\pm^2}D_{c_\pm}(z)^{-1}\sigma_3 D_{c_\pm}(z)\right)\tilde{\mu}^{\pm}(z;s,t)ds, \tag{2.13}$$

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where X is defined in (2.2). It follows from (2.13) that the Jost functions $\tilde{\mu}^{\pm}(z)$ have no more than $-\frac{1}{4}$-weak singularity at $z=\pm 1$ and $z=\pm c$ as

$$\tilde{\mu}^{\pm}(z)=\mathcal{O}\big((z\mp 1)^{-1/4}\big), \quad \tilde{\mu}^{\pm}(z)=\mathcal{O}\big((z\mp c)^{-1/4}\big).$$

Since $D_{c_\pm}(z)^{-1}\Psi^{\pm}(z;x,t)$ are two fundamental matrix solutions of the Lax pair (2.1), they are related by a scattering matrix function $S(z)$ independent of x and t

$$D_{c_+}(z)^{-1}\Psi^{+}(z;x,t)=D_{c_-}(z)^{-1}\Psi^{-}(z;x,t)S(z), \tag{2.14}$$

$$S(z)=\begin{pmatrix} s_{11}(z) & s_{12}(z) \\ s_{21}(z) & s_{22}(z)\end{pmatrix}, \qquad \det S(z)=1.$$

Combining the transformations (2.9), (2.12) with the (2.14), it deduced that

$$S(z)=e^{itp^{(-)}(z;x,t)\sigma_3}(\tilde{\mu}^{-}(z;x,t))^{-1}\tilde{\mu}^{+}(z;x,t)e^{-itp^{(+)}(z;x,t)\sigma_3}, \tag{2.15}$$

which is analytic on $\mathbb{C}\setminus\Sigma_+$. It also implies that the scattering matrix $S(z)$ has no more than $-\frac{1}{4}$-weak singularity at $z=\pm 1$ and $z=\pm c$. On the other hand, it is also deduced that

$$S(z)\sim e^{\frac{1}{2}Hz\sigma_3}, \quad z\to\infty,$$

where H is a constant given by

$$H=\left(1-\frac{1}{c}\right)x+\left(\frac{1}{c^3}-1\right)t+\int_{-\infty}^{x}(m(s,t)-1)\,ds+\int_{x}^{+\infty}\left(m(s,t)-\frac{1}{c}\right)ds.$$

Define two reflecting coefficients by

$$r_1(z)=\frac{s_{21}(z)}{s_{11}(z)}, \qquad r_2(z)=\frac{s_{12}(z)}{s_{22}(z)}. \tag{2.16}$$

As $z\to\pm c$, they then admit asymptotic behaviour $1-r_1(z)r_2(z)=\mathcal{O}((z\mp c)^{1/2})$.

To construct the RH problem, the jump of the Jost functions $\tilde{\mu}^{\pm}(z)$ on the cut $\Sigma_\pm$ need to be analysed in the following proposition under standard proof.

Proposition 1. *The functions $\mu^{\pm}$, $\tilde{\mu}^{\pm}$, S and the reflecting coefficients r_1, r_2 admit the jump relations*

(i) For $z\in\Sigma_\pm$,

$$\mu_+^{\pm}(z)=\sigma_1\mu_-^{\pm}(z)\sigma_1.$$

(ii) For $z\in\Sigma_-$,

$$\tilde{\mu}_{11,+}^{\pm}(z)=-i\tilde{\mu}_{12,-}^{\pm}(z), \quad \tilde{\mu}_{21,+}^{\pm}(z)=-i\tilde{\mu}_{22,-}^{\pm}(z),$$
$$s_{11,\pm}(z)=s_{22,\mp}(z), \quad s_{12,\pm}(z)=s_{21,\mp}(z), \quad r_{1,\pm}(z)=r_{2,\mp}(z).$$

(iii) For $z\in\Sigma_+\setminus\Sigma_-$, $\tilde{\mu}^{+}(z)$ has same jump as above equation while $\tilde{\mu}^{-}(z)$ has no jump. And

$$s_{11,+}(z)=-is_{12,-}(z), \quad s_{21,+}(z)=-is_{22,-}(z), \quad r_{1,\pm}(z)r_{2,\mp}(z)=1.$$

(iv) For $z\in\mathbb{R}\setminus\Sigma_+$,

$$r_1(z)=\overline{r_2(z)}.$$

Under the Assumption 1 $r_1(z)$ and $r_2(z)$ are analytic in $\mathbb{C}\setminus\Sigma_+$.

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Case II: $z=0$.

The Lax pair (2.7)–(2.8) is rewritten in the form

$$\Psi_x^{\pm}=-\frac{i\sqrt{z^2-c_{\pm}^2}}{2c_{\pm}}\sigma_3\Psi^{\pm}+P_0^{\pm}\Psi^{\pm},$$
$$\Psi_t^{\pm}=i\sqrt{z^2-c_{\pm}^2}\left(\frac{1}{2c_{\pm}^3}+\frac{1}{c_{\pm}z^2}\right)\sigma_3\Psi^{\pm}+L_0^{\pm}\Psi^{\pm},$$

where $c_{\pm}$ are exactly given in (1.6) and

$$P_0^{\pm}=iz\frac{c_{\pm}m-1}{2c_{\pm}\sqrt{z^2-c_{\pm}^2}}\left(z\sigma_3+ic_{\pm}\sigma_2\right),$$
$$L_0^{\pm}=L_{\pm}+i\sqrt{z^2-c_{\pm}^2}\left(\frac{m(u^2-u_x^2)}{2}+\frac{1}{c_{\pm}z}-\frac{1}{2c_{\pm}^3}+\frac{1}{c_{\pm}z^2}\right)\sigma_3.$$

By making transformation

$$\mu_0^{\pm}(z;x,t)=\Psi_{\pm}(z;x,t)e^{iq^{(\pm)}(z;x,t)\sigma_3},$$
$$q^{(\pm)}(z;x,t)=\frac{i\sqrt{z^2-c_{\pm}^2}}{2c_{\pm}}\left[x-\left(\frac{1}{c_{\pm}^2}+\frac{2}{z^2}\right)t\right],$$

$\mu_0^{\pm}(z):=\mu_0^{\pm}(z;x,t)$ admit a new Lax pair

$$\mu_{0,x}^{\pm}=-\frac{i\sqrt{z^2-c_{\pm}^2}}{2c_{\pm}}[\sigma_3,\mu_0^{\pm}]+P_0^{\pm}\mu_0^{\pm}, \tag{2.17}$$
$$\mu_{0,t}^{\pm}=i\sqrt{z^2-c_{\pm}^2}\left(\frac{1}{2c_{\pm}^3}+\frac{1}{c_{\pm}z^2}\right)[\sigma_3,\mu_0^{\pm}(z)]+L_0^{\pm}\mu_0^{\pm}, \tag{2.18}$$

which can be written into two Volterra type integrals

$$\mu_0^{\pm}(z)=I+\int_{\pm\infty}^{x}e^{\frac{i}{2c_{\pm}}\hat{\sigma}_3\sqrt{z^2-c_{\pm}^2}(s-x)}\left[P_0^{\pm}(z;s,t)\mu_0^{\pm}(z;s,t)\right]ds.$$

Taking $z=0$ in above integral equation implies $\mu_0^{\pm}(0)=I$. Moreover, expanding $\mu_0^{\pm}(z)$ at $z=0$ gives that

$$\mu_0^{\pm}(z)=I+\frac{z}{2}\begin{pmatrix}0 & \int_{\pm\infty}^{x}\left(m-\frac{1}{c_{\pm}}\right)e^{x-s}ds\\ -\int_{\pm\infty}^{x}\left(m-\frac{1}{c_{\pm}}\right)e^{s-x}ds & 0\end{pmatrix}+\mathcal{O}(z^2), \tag{2.19}$$

which will be used to reconstruct the potential $u(x,t)$.

Because $\mu_0^{\pm}e^{-iq^{(\pm)}\sigma_3}$ also admit Lax pair (2.7)–(2.8), there exist two matrix functions $C_{\pm}(z)$ independent of x and t such that

$$\mu_0^{\pm}(z;x,t)e^{-iq^{(\pm)}(z;x,t)\sigma_3}C_{\pm}(z)=\mu^{\pm}(z;x,t)e^{-itp^{(\pm)}(z;x,t)\sigma_3}. \tag{2.20}$$

Since $q^{(\pm)}-tp^{(\pm)}=-\frac{1}{2}\sqrt{z^2-c_{\pm}^2}\int_{\pm\infty}^{x}(m-1/c_{\pm})ds$, taking the limits $x\to\pm\infty$, we obtain $C_{\pm}(z)\equiv I$. Invoking (2.12) and $D_{c_{\pm},+}(0)=i\sigma_1$, it follows that

$$\tilde{\mu}_{+}^{\pm}(0)=-i\begin{pmatrix}0 & e^{i(q_{+}^{(\pm)}(0)-tp_{+}^{(\pm)}(0))}\\ e^{-i(q_{+}^{(\pm)}(0)-tp_{+}^{(\pm)}(0))} & 0\end{pmatrix}. \tag{2.21}$$

Consequently, from (2.15) it follows that as $z\to 0\in\mathbb{C}^{+}$,

$$s_{11}(z)=e^{i(q_{+}^{(-)}-q_{+}^{(+)})(0)}+\mathcal{O}(z^2),\ \ s_{22}(z)=e^{-i(q_{+}^{(-)}-q_{+}^{(+)})(0)}+\mathcal{O}(z^2). \tag{2.22}$$

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2.2. Setting up a RH problem with step-like initial data

Define a sectionally analytical matrix

$$M(z) := M(z; x, t) = D_{c_-}(\infty) \times \begin{cases} \left(\tilde{\mu}_1^-(z),\ \frac{\tilde{\mu}_2^+(z)}{s_{22}(z)} e^{it(p^{(+)}-p^{(-)})}\right), & \text{as } z \in \mathbb{C}^+, \\ \left(\frac{\tilde{\mu}_1^+(z)}{s_{11}(z)} e^{-it(p^{(+)}-p^{(-)})},\ \tilde{\mu}_2^-(z)\right), & \text{as } z \in \mathbb{C}^-, \end{cases} \tag{2.23}$$

where $\tilde{\mu}_1^\pm(z)$ and $\tilde{\mu}_2^\pm(z)$ denote the first and second column of $\tilde{\mu}^\pm(z)$, respectively and $D_{c_-}(\infty)$ is defined in (2.5).

In order to construct the RH problem only depending explicitly on the scattering data, via the definition of the new scale $y(x, t)$ in (1.7), we define

$$N(z) := N(z; y, t) = M(z; x(y, t), t). \tag{2.24}$$

Recall the notation $\xi = y/t$ and $c_- = 1$, then $p^{(-)}$ defined in (2.10) can be rewrite as

$$p^{(-)} = \frac{\sqrt{z^2 - 1}}{2}\left(\xi - 1 - 2z^{-2}\right).$$

Then $N(z)$ is a solution of the following RH problem.

RH problem 1.

1. *Analyticity: $N(z)$ is meromorphic in $\mathbb{C} \setminus \mathbb{R}$;*
2. *Symmetry: $N(z) = \sigma_2 N(-z)\sigma_2^{-1} = \sigma_1 \overline{N(\bar{z})}\sigma_1$;*
3. *Jump condition: N has continuous boundary values $N_\pm(z)$ on $\mathbb{R}$ and*

$$N_+(z) = N_-(z)\tilde{V}(z), \qquad z \in \mathbb{R}, \tag{2.25}$$

where

$$\tilde{V}(z) = \begin{cases} \begin{pmatrix} 1 & r_2(z)e^{-2itp^{(-)}} \\ -r_1(z)e^{2itp^{(-)}} & 1 - r_1 r_2 \end{pmatrix}, & \text{as } z \in \mathbb{R} \setminus \Sigma_+, \\ \begin{pmatrix} 1 & r_{2,+}(z)e^{-2itp^{(-)}} \\ -r_{1,-}(z)e^{2itp^{(-)}} & 0 \end{pmatrix}, & \text{as } z \in \Sigma_+ \setminus \Sigma_-, \\ -i\sigma_1, & \text{as } z \in \Sigma_-. \end{cases}$$

4. *Asymptotic behaviours: $N(z) = I + \mathcal{O}(z^{-1}), \qquad z \to \infty$;*
5. *Singularity: $N(z)$ has singularity at $z = \pm 1$ with*

$$N(z) \sim \left(\mathcal{O}\left((z \mp 1)^{-1/4}\right), \mathcal{O}\left((z \mp 1)^{1/4}\right)\right),\ z \to \pm 1 \text{ in } \mathbb{C}^+, \tag{2.26}$$

$$N(z) \sim \left(\mathcal{O}\left((z \mp 1)^{1/4}\right), \mathcal{O}\left((z \mp 1)^{-1/4}\right)\right),\ z \to \pm 1 \text{ in } \mathbb{C}^-. \tag{2.27}$$

From (2.19), (2.20) and (2.22), it reveals that

$$N(z) = N_+(0) + N_1 z + \mathcal{O}(z^2), \quad z \to 0 \in \mathbb{C}^+, \tag{2.28}$$

where

$$N_+(0) = iD_{c_-}(\infty)\begin{pmatrix} 0 & f_1 \\ f_1^{-1} & 0 \end{pmatrix}, \quad N_1 = iD_{c_-}(\infty)\begin{pmatrix} f_2 & 0 \\ 0 & f_3 \end{pmatrix}$$

with

$$f_1 = \exp\left\{-\frac{1}{2}\int_{-\infty}^{x}(m-1)ds\right\}, \quad f_2 = \frac{e^{\frac{1}{2}\int_{-\infty}^{x}(m-1)ds}}{2}\left(\int_{-\infty}^{x}(m-1)e^{s-x}ds + 1\right),$$

$$f_3 = \frac{e^{-\frac{1}{2}\int_{-\infty}^{x}(m-1)ds}}{2c}\left(1 - \int_{+\infty}^{x}(cm-1)e^{x-s}ds\right).$$

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Thus, via the definition of y in (1.7), it follows that

$$x(y,t)=y-2\ln\left(i\sqrt{2}N_{21,+}(0)\right). \tag{2.29}$$

Direct calculation shows that

$$\begin{aligned}-2\left(N_{12,+}(0)N_{1,11}+N_{21,+}(0)N_{1,22}\right)&=f_1f_2+f_1^{-1}f_3 \\ &=\frac{1}{2}\left(\int_{-\infty}^{x}(m-1)e^{s-x}ds-\int_{+\infty}^{x}(m-1/c)e^{x-s}ds+(1+c)/c\right).\end{aligned} \tag{2.30}$$

Taking $x\to+\infty$ and $x\to-\infty$ in above equation, respectively, and via the fact that $\lim_{x\to-\infty}m=1$, $\lim_{x\to+\infty}m=1/c$, we arrive at that

$$\lim_{x\to+\infty}(f_1f_2+f_1^{-1}f_3)=1/c \qquad \lim_{x\to-\infty}(f_1f_2+f_1^{-1}f_3)=1.$$

Via taking the derivative with respect to x on (2.30), we obtain

$$-2\left(N_{12,+}(0)N_{1,11}+N_{21,+}(0)N_{1,22}\right)+2\partial_x^2(N_{12,+}(0)N_{1,11}+N_{21,+}(0)N_{1,22})=m.$$

Therefore, we arrive at the following reconstruction formula

$$u(x,t)=-2(N_{12}(0)N_{1,11}+N_{21,+}(0)N_{1,22}), \tag{2.31}$$

2.3. An almanac of jump matrix factorisations

The jump matrix $\tilde{V}(z)$ admits the following decomposition from the symmetry of the reflecting coefficients r_1, r_2 in Proposition 1, which will be used in the asymptotic analysis in the next section.

On the interval $\mathbb{R}\setminus\Sigma_+$,

$$\begin{aligned}\tilde{V}(z)&=\begin{pmatrix}1 & 0\\ -r_1e^{2itp^{(-)}} & 1\end{pmatrix}\begin{pmatrix}1 & r_2e^{-2itp^{(-)}}\\ 0 & 1\end{pmatrix}\\ &=\begin{pmatrix}1 & \frac{r_2e^{-2itp^{(-)}}}{1-r_1r_2}\\ 0 & 1\end{pmatrix}(1-r_1r_2)^{-\sigma_3}\begin{pmatrix}1 & 0\\ \frac{-r_1e^{2itp^{(-)}}}{1-r_1r_2} & 1\end{pmatrix}.\end{aligned} \tag{2.32}$$

On the interval $\Sigma_+\setminus\Sigma_-$,

$$\begin{aligned}\tilde{V}(z)&=\begin{pmatrix}1 & 0\\ -r_{1,-}(z)e^{2itp^{(-)}} & 1\end{pmatrix}\begin{pmatrix}1 & r_{2,+}(z)e^{-2itp^{(-)}}\\ 0 & 1\end{pmatrix}\\ &=\begin{pmatrix}1 & \frac{r_{2,-}(z)e^{-2itp^{(-)}}}{1-r_{1,-}(z)r_{2,-}(z)}\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & \frac{r_{2,-}(z)}{e^{2itp^{(-)}}}\\ \frac{-e^{2itp^{(-)}}}{r_{2,-}(z)} & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-r_{1,+}(z)e^{2itp^{(-)}}}{1-r_{1,+}(z)r_{2,+}(z)} & 1\end{pmatrix}.\end{aligned}$$

On the interval Σ_-,

$$\tilde{V}(z)=-i\begin{pmatrix}1 & 0\\ -r_{1,-}(z)e^{2itp^{(-)}} & 1\end{pmatrix}\sigma_1\begin{pmatrix}1 & r_{2,+}(z)e^{-2itp^{(-)}}\\ 0 & 1\end{pmatrix}.$$

The long-time asymptotic of RH problem 1 is affected by the growth or decay of the exponential function $e^{\pm 2itp^{(-)}}$ with

$$p^{(-)}=\frac{\sqrt{z^2-1}}{2}\left(\xi-1-2z^{-2}\right),\ \partial_zp^{(-)}=\frac{1}{2\sqrt{z^2-1}z^3}\left[(\xi-1)z^4+2z^2-4)\right].$$

Thus, to obtain the long-time asymptotics, we need to analyse the real part of $2itp^{(-)}$. We hope that after appropriately choosing triangular factorisations of the jump matrices and associated deformations of the original RH problem, the jumps remaining on $\mathbb{R}$ can become constant matrices (independent of

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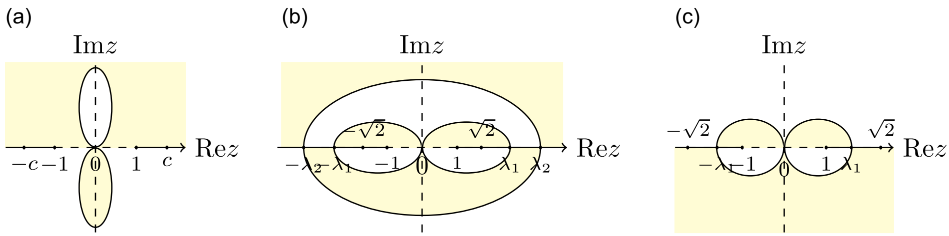

Figure 2. *In the white region,* $\mathrm{Im}[p^{(-)}] > 0$, *while in another region,* $\mathrm{Im}[p^{(-)}] < 0$. *(a)* $\xi < 3/4$; *(b)* $3/4 < \xi < 1$; *(c)* $1 \leq \xi < 3$.

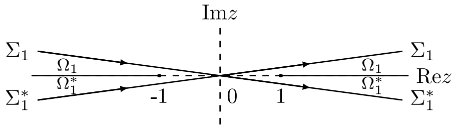

Figure 3. *Figure of curves* Σ_j *and domains* Ω_j, $j = 1, 2$, *in the case of* $\{(\xi, c) : \xi < 3/4\}$.

z, but dependent on ξ and c) of special structure or a jump matrices of solvable model, whereas the other jumps decay exponentially to the identity matrix. We introduce the g-function mechanism [17] to problems with step-like background in different regions. This mechanism is relevant when some entries of the jump matrix grow exponentially or oscillate as $t \to \infty$. The general idea consists in replacing the original phase function in the jump matrix. This new g-function needs to be analytic on $\mathbb{C}$ except some new cut (it is undetermined and do not must be $\Sigma_\pm$) and satisfies the above condition. Moreover, it must have same asymptotic properties as $z \to \infty$, $0 \in \mathbb{C}^+$ as $p^{(-)}$:

$$p^{(-)} = \frac{1-\xi}{2} z + \mathcal{O}(z^{-1}),\ \partial_z p^{(-)} = \frac{1-\xi}{2} + \mathcal{O}(z^{-2}),\ z \to \infty; \tag{2.33}$$

$$p^{(-)} = \frac{i}{z^2} - \frac{i\xi}{2} + \mathcal{O}(z),\ \partial_z p^{(-)} = \frac{-2i}{z^3} + \mathcal{O}(z),\ z \to 0 \in \mathbb{C}^+. \tag{2.34}$$

The structure of the limiting RH problem is such that the problem can be solved explicitly in terms of Riemann theta functions and Abel integrals on Riemann surfaces associated with the limiting RH problem [1, 4–7]. For different ranges of the parameter $\xi = y/t$, different Riemann surfaces may appear.

3. Region I: slow-decay background region

In this section, we will analyse the long-time asymptotics in the slow-decay background region. The signature table and stationary points of $p^{(-)}$ are shown in Figure 2.

(a) For the case $\xi < \frac{3}{4}$, there is no stationary point on $\mathbb{R}$;

(b) For the case $3/4 < \xi < 1$, there are four stationary points $\pm\lambda_1$ and $\pm\lambda_2$ on $\mathbb{R}$, where λ_1 is defined in (1.8) and $\lambda_2 := \lambda_2(\xi) = \left(\frac{1+\sqrt{4\xi-3}}{1-\xi}\right)^{1/2}$ where $\lambda_2(3/4) = 2$;

(c) For the case $1 \leq \xi < 3$, there are two stationary points $\pm\lambda_1$ on $\mathbb{R}$.

Therefore, the Region I contains the following three different cases:

(i) $\{(\xi, c) : \xi < 3/4\}$;

(ii) $\{(\xi, c) : 1 < c \leq \lambda_1,\ 3/4 < \xi < 1\}$;

(iii) $\{(\xi, c) : 1 < c \leq \lambda_1,\ 1 \leq \xi < 3\}$.

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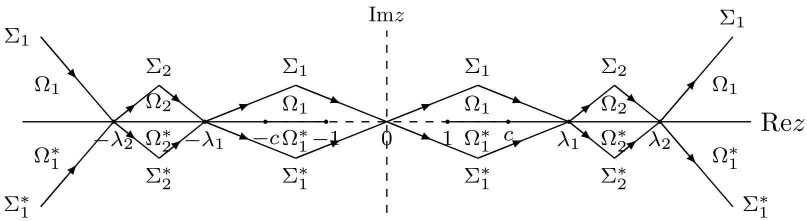

Figure 4. *Figure of curves* Σ_j *and domains* Ω_j*,* $j = 1, 2$*, in the case of* $\{(\xi, c) : 1 < c \leq \lambda_1,\ 3/4 < \xi < 1\}$.

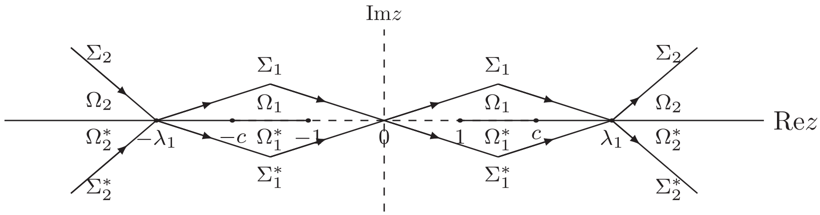

Figure 5. *Figure of curves* Σ_j *and domains* Ω_j*,* $j = 1, 2$*, in the case of* $\{(\xi, c) : 1 < c \leq \lambda_1,\ 1 \leq \xi < 3\}$.

Specially, in the case (ii) and case (iii), it follows that $c < \lambda_1$. In what follows, we introduce the curves $\Sigma_j := \Sigma_j(\xi, c)$ and domains $\Omega_j := \Omega_j(\xi, c)$, $j = 1, 2$, relying on (ξ, c), that is, it is different in cases $(i) - (iii)$. We will use the first decomposition of the jump matrix given in Subsection 2.3 on Ω_1 and Σ_1 while use the second decomposition on Ω_2 and Σ_2 to open the jump on $\mathbb{R}$.

(i) The case $\{(\xi, c) : \xi < 3/4\}$. In this region, there has no stationary point. Define

$$\Sigma_1 = e^{i\psi}\mathbb{R}^+ \cup e^{i(\pi-\psi)}\mathbb{R}^+, \qquad \Sigma_2 = \Omega_2 = \emptyset,$$

$$\Omega_1 = \{z : z = e^{\phi i}l,\ l \in \mathbb{R},\ 0 < \phi < \psi\} \cup \{z : z = e^{\phi i}l,\ l \in \mathbb{R},\ \pi - \psi < \phi < \pi\},$$

where ϕ is a small enough positive angle such that Ω_1 is non-intersect with the curve $\mathrm{Im}[p^{(-)}](z) = 0$ as shown in Figure 3.

(ii) The case $\{(\xi, c) : 1 < c \leq \lambda_1,\ 3/4 < \xi < 1\}$. Let curve $\Sigma_j, j = 1, 2$ as Figure 4 shown. It also admits that $\Omega_j, j = 1, 2$, is non-intersect with the curve $\mathrm{Im}[p^{(-)}](z) = 0$.

(iii) The case $\{(\xi, c) : 1 < c \leq \lambda_1,\ 1 \leq \xi < 3\}$. Let the curve $\Sigma_j, j = 1, 2$ as Figure 5 showing. The only difference from the case **(ii)** is that there is only two stationary points $\pm\lambda_1$.

To deal with the jump on $\mathbb{R}$, we denote a interval

$$I(\xi) = \begin{cases} \emptyset, & \text{as } \xi < \frac{3}{4}; \\ [-\lambda_2, -\lambda_1] \cup [\lambda_1, \lambda_2], & \text{as } \frac{3}{4} < \xi < 1; \\ (-\infty, -\lambda_1] \cup [\lambda_1, +\infty) & \text{as } \xi > 1; \end{cases} \tag{3.1}$$

and introduce an auxiliary function

$$\delta(z) := \delta(z; \xi, c) = \exp\left\{ \frac{1}{2\pi i} \int_{I(\xi)} \frac{\log(1 - r_1(s)r_2(s))}{s - z} ds \right\}, \tag{3.2}$$

We give the properties about $\delta(z)$ as follow without proof.

Proposition 2.

(a) As $z \to 0 \in \mathbb{C}^+$,

$$\delta(z) = \exp\left\{I_\delta^1\right\} \cdot \left(1 + zI_\delta^2\right) + \mathcal{O}(z^2),$$

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where

$$I_\delta^1 = \frac{1}{2\pi i}\int_{I(\xi)} \frac{\log(1-r_1(s)r_2(s))}{s}ds, \tag{3.3}$$

$$I_\delta^2 = \frac{1}{2\pi i}\int_{I(\xi)} \frac{\log(1-r_1(s)r_2(s))}{s^2}ds;$$

(b) $\delta_+(z) = \delta_-(z)(1-r_1r_2),\ z\in I(\xi),\ \delta_-(z)=\delta_+(z),\ z\in\mathbb{R}\setminus I(\xi)$;

(c) $\delta(z)\to 1$, *as* $z\to\infty\in\mathbb{C}\setminus I(\xi)$.
For the right endpoints λ *of* $I(\xi)$ (λ *may be* $-\lambda_1$, $+\lambda_2$ *here), there exists an analytic function* $\delta_\lambda(z)$ *on* $z\in U_\lambda\setminus I(\xi)$ *which is continuous to the boundary such that for* $\nu := \nu(z) = \log(1-r_1(z)r_2(z))/2\pi$,

$$\delta(z) = \delta_\lambda(z)(z-\lambda)^{-i\nu(\lambda)},\ \arg(z-\lambda)\in(-\pi,\pi), \tag{3.4}$$

with

$$|\delta_\lambda(z)-\delta_\lambda(\lambda)| \lesssim |z-\lambda|.$$

Via the function δ given in (3.2), we define a new matrix-valued function,

$$M^{(1)}(z) := M^{(1)}(z;\xi,c) = N(z;\xi,c)G(z;\xi,c)\delta^{\sigma_3}(z;\xi,c), \tag{3.5}$$

where $G(z) := G(z;\xi,c)$ is a piecewise matrix interpolation function

$$G(z) = \begin{pmatrix} 1 & -r_2e^{-2itp^{(-)}} \\ 0 & 1 \end{pmatrix},\quad z\in\Omega_1;\quad G(z) = \begin{pmatrix} 1 & 0 \\ -r_1e^{2itp^{(-)}} & 1 \end{pmatrix},\quad z\in\Omega_1^*;$$

$$G(z) = \begin{pmatrix} 1 & 0 \\ \frac{r_1e^{2itp^{(-)}}}{1-r_1r_2} & 1 \end{pmatrix},\quad z\in\Omega_2;\qquad G(z) = \begin{pmatrix} 1 & \frac{r_2e^{-2itp^{(-)}}}{1-r_1r_2} \\ 0 & 1 \end{pmatrix},\ \text{as } z\in\Omega_2^*; \tag{3.6}$$

$$G(z) = I\ \ z \text{ in elsewhere.}$$

Then $M^{(1)}$ satisfies the following RH problem.

RH problem 2.

1. *Analyticity:* $M^{(1)}(z)$ *is meromorphic in* $\mathbb{C}\setminus\Sigma^{(1)}$*, where*

$$\Sigma^{(1)} := \Sigma^{(1)}(\xi,c) = \left(\cup_{j=1}^2 \Sigma_j(\xi,c)\cup\Sigma_j(\xi,c)^*\right)\cup[-1,1]; \tag{3.7}$$

2. *Symmetry:* $M^{(1)}(z) = \sigma_2M^{(1)}(-z)\sigma_2^{-1} = \sigma_1\overline{M^{(1)}(\bar z)}\sigma_1$;
3. *Jump condition:* $M^{(1)}$ *has continuous boundary values* $M_\pm^{(1)}(z)$ *on* $\Sigma^{(1)}$ *and*

$$M_+^{(1)}(z) = M_-^{(1)}(z)V^{(1)}(z),\qquad z\in\Sigma^{(1)}, \tag{3.8}$$

where

$$V^{(1)}(z) = \begin{pmatrix} 1 & r_2\delta^{-2}e^{-2itp^{(-)}} \\ 0 & 1 \end{pmatrix},\ z\in\Sigma_1;\quad V^{(1)}(z) = \begin{pmatrix} 1 & 0 \\ -r_1\delta^2e^{2itp^{(-)}} & 1 \end{pmatrix},\ z\in\Sigma_1^*;$$

$$V^{(1)}(z) = \begin{pmatrix} 1 & 0 \\ \frac{-r_1\delta^2e^{2itp^{(-)}}}{1-r_1r_2} & 1 \end{pmatrix}\ z\in\Sigma_2;\quad V^{(1)}(z) = \begin{pmatrix} 1 & \frac{r_2\delta^{-2}e^{-2itp^{(-)}}}{1-r_1r_2} \\ 0 & 1 \end{pmatrix},\ z\in\Sigma_2^*;$$

$$V^{(1)}(z) = -i\sigma_1,\ z\in\Sigma_-$$

4. *Asymptotic behaviours:* $M^{(1)}(z) = I+\mathcal{O}(z^{-1}),\qquad z\to\infty$;
5. *Singularity:* $M^{(1)}(z)$ *has at most fourth root singularities at* $z=\pm1$.

Denote $U_{\pm\lambda_j}$ as a small neighbourhood of $\pm\lambda_j$ with $U_{\pm\lambda_j} = \left\{z : |z\pm\lambda_j|\le\varrho\right\}$, where ϱ is a small positive constant such that $\varrho < \min\left\{\frac{\lambda_1-c}{3}, \frac{\lambda_2-\lambda_1}{3}\right\}$.

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In the case (i), because of the absence of stationary point, $U_{\pm\lambda_j}=\emptyset$, $j=1,2$, while in case (ii), $U_{\pm\lambda_2}=\emptyset$.

The jump matrix exponentially decays to the identity matrix I as $t\to\infty$ away from stationary points, which inspires us to construct the solution $M^{(1)}(z)$ as follows

$$M^{(1)}(z)=\begin{cases} E(z;\xi,c)M^{mod1}(z;\xi,c), & z\notin U_{\pm\lambda_1}\cup U_{\pm\lambda_2},\\ E(z;\xi,c)M^{1,\pm}(z;\xi,c), & z\in U_{\pm\lambda_1},\\ E(z;\xi,c)M^{2,\pm}(z;\xi,c), & z\in U_{\pm\lambda_2}.\end{cases} \tag{3.9}$$

M^{mod1} is the global parametrix, which will be given later. E is the error function, whose existence is guaranteed by small norm theory in Subsection 3.2 and as $t\to\infty$, its asymptotic expansion can be computed. $M^{j,\pm}, j=1,2$ are the local parametrixs. Each $M^{j,\pm}, j=1,2$ admits jump $V^{(1)}$ only on $U_{\pm\lambda_j}$, and satisfies that $M^{j,\pm}\to I$, as $z\to\infty$. They can be approximated by a model RH problem whose solution is constructed explicitly in terms of parabolic cylinder functions and appears frequently in the literature of long-time asymptotic calculations for integrable nonlinear waves [1, 4–6, 17–20, 46].

Because of the symmetry $M^{(1)}(z)=\sigma_2M^{(1)}(-z)\sigma_2^{-1}$ in RH problem 2, the local RH problems should satisfy

$$M^{j,+}(z)=\sigma_2M^{j,-}(-z)\sigma_2^{-1}, z\in U_{+\lambda_j}.$$

It is sufficient to consider $M^{2,+}$ and $M^{1,-}$. Denote $\zeta^{j,\pm}=\sqrt{|2p^{(-)''}(\pm\lambda_j)|}(z\mp\lambda_j)$ as the local parameter in $U_{\pm\lambda_j}$, and

$$\begin{aligned} r_{+\lambda_2}=&r_2(\lambda_2)\delta_{+\lambda_2}(\lambda_2)e^{-2itp^{(-)}(\lambda_2)}(-2tp^{(-)''}(\lambda_2))^{2i\nu(\lambda_2)},\\ r_{-\lambda_1}=&r_2(-\lambda_1)\delta_{-\lambda_1}(-\lambda_1)e^{-2itp^{(-)}(-\lambda_1)}(-2tp^{(-)''}(-\lambda_1))^{2i\nu(\lambda_1)}.\end{aligned}$$

The Assumption 1 (analyticity) and (d) in the Proposition 2 imply that in the corresponding neighbourhood,

$$\left|r_{-\lambda_1}-r_2(z)\delta^{-2}_{-\lambda_1}(z)\right|\lesssim\left|\zeta^{1,-}\right|,\quad \left|r_{+\lambda_2}-r_2(z)\delta^{-2}_{+\lambda_2}(z)\right|\lesssim\left|\zeta^{2,+}\right|,$$

which satisfies the conditions of Theorem A.1 in [36] thanks to

$$-2p^{(-)}=-2p^{(-)}(\pm\lambda_j)+(\zeta^{j,\pm})^2+\mathcal{O}(\zeta^{j,\pm})^3.$$

Thus, $M^{2,+}$ and $M^{1,-}$ are well approximated by $P_1(\zeta^{2,+};r_{+\lambda_2})$ and $P_1(\zeta^{1,-};r_{-\lambda_1})$, respectively, which are defined in the Appendix B. As $t\to\infty$, the asymptotics of $M^{j,\pm}(z):=M^{j,\pm}(z;\xi,c)$ is given by the following proposition.

Proposition 3. *For $z\in U_{\pm\lambda_j}\setminus\{\pm\lambda_j\}$, we have*

$$M^{j,\pm}(z)=I+t^{-1/2}\frac{A_{j,\pm}(\xi)}{z\mp\lambda_j}+\mathcal{O}(t^{-1}),\ j=1,2, \tag{3.10}$$

where

$$A_{j,\pm}(\xi)=\frac{1}{\sqrt{|2p^{(-)''}(\pm\lambda_j)|}}\begin{pmatrix}0 & \tilde{\beta}^{j,\pm}_{12}\\ \tilde{\beta}^{j,\pm}_{21} & 0\end{pmatrix},$$

and $\tilde{\beta}^{j,\pm}_{21}\tilde{\beta}^{j,\pm}_{12}=-\nu(\lambda_j)$ with

$$\begin{aligned}\tilde{\beta}^{1,+}_{12}=\tilde{\beta}^{1,-}_{21}=\frac{\sqrt{2\pi}e^{\frac12\pi\nu(\lambda_1)}e^{\frac{\pi i}{4}}}{r_{-\lambda_1}\Gamma(i\nu(\lambda_1))},\\ \tilde{\beta}^{2,-}_{12}=\tilde{\beta}^{2,+}_{21}=\frac{\sqrt{2\pi}e^{\frac12\pi\nu(\lambda_2)}e^{\frac{\pi i}{4}}}{r_{+\lambda_2}\Gamma(i\nu(\lambda_2))}.\end{aligned}$$

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3.1. A model RH problem on cuts

The global parametrix $M^{mod1}(z)$ is given by the following model RH problem:

RH problem 3.

1. *Analyticity:* $M^{mod1}(z)$ *is holomorphic in* $\mathbb{C}\setminus\Sigma_-$;
2. *Jump condition:* M^{mod1} *has continuous boundary values* $M_\pm^{mod1}(z)$ *on* Σ_- *with* $M_+^{mod1}(z)=M_-^{mod1}(z)V^{mod1}(z),\ z\in\Sigma_-$, *where*

$$V^{mod1}(z)=-i\sigma_1, z\in\Sigma_-; \tag{3.11}$$

3. *Asymptotic behaviours:* $M^{mod1}(z)=I+\mathcal{O}(z^{-1}),\qquad z\to\infty$;
4. *Singularity:* $M^{mod1}(z)$ *has at most fourth root singularities at* $z=\pm1$.

The solution of this model RH problem is given by

$$M^{mod1}(z)=D_1(\infty)D_1(z)^{-1}, \tag{3.12}$$

where D_1 is defined in (2.4). As $z\to 0\in\mathbb{C}^+$, it is adduced that

$$M^{mod1}(z)=\frac{1}{\sqrt{2}}(I-i\sigma_1)-\frac{zi}{2\sqrt{2}}(I+i\sigma_1)+\mathcal{O}(z^2). \tag{3.13}$$

3.2. The small norm RH problem for error function

In this subsection, we consider the error matrix-function $E(z):=E(z;\xi,c)$ in this region.

RH problem 4.

1. *Analyticity:* $E(z)$ *is analytical in* $\mathbb{C}\setminus\Sigma^E$, *where* $\Sigma^E=\partial U\cup\left[\Sigma^{(1)}\setminus(U\cup\Sigma_-)\right]$, *with* $U:=U(\xi,c)=\cup_{j=1,2}U_{\pm\lambda_j}$, *and* ∂U *is the boundary of* U;
2. *Asymptotic behaviours:* $E(z)\sim I+\mathcal{O}(z^{-1}),\qquad |z|\to\infty$;
3. *Jump condition:* $E(z)$ *has continuous boundary values* $E_\pm(z)$ *on* Σ^E *satisfying* $E_+(z)=E_-(z)V^E(z)$, *where the jump matrix* $V^E(z)$ *is given by*

$$V^E(z)=\begin{cases}M^{mod1}(z)V^{(1)}(z)M^{mod1}(z)^{-1}, & z\in\Sigma^E\setminus\partial U,\\ M^{j,\pm}(z)M^{mod1}(z)^{-1}, & z\in\partial U,\end{cases} \tag{3.14}$$

Out of U, the jump V^E admits the following estimates

$$\|V^E-I\|_p\lesssim\exp\left\{-tK_p\right\}, z\in\Sigma^E\setminus U,\ p\in[1,\infty], \tag{3.15}$$

for positive K_p relying on p. For $z\in\partial U$, $M^{mod1}(z)$ is bounded, so by (3.10), we find that

$$|V^E(z)-I|=\mathcal{O}(t^{-1/2}). \tag{3.16}$$

Therefore, the existence and uniqueness of the RH problem 4 is obtained via a small-norm RH problem [18, 19]. According to Beals–Coifman theory, the solution of the RH problem 4 can be given by

$$E(z)=I+\frac{1}{2\pi i}\int_{\Sigma^E}\frac{(I+\varpi(s))\,(V^E(s)-I)}{s-z}ds, \tag{3.17}$$

where the $\varpi\in L^\infty(\Sigma^E)$ is the unique solution of $(1-C_E)\varpi=C_E(I)$, and C_E is a integral operator: $L^\infty(\Sigma^E)\to L^2(\Sigma^E)$ defined by $C_E(f)(z)=C_-\left(f(V^E(z)-I)\right)$ with the usual Cauchy projection operator $C_-(f)(s)=\lim_{z\to\Sigma_-^E}\frac{1}{2\pi i}\int_{\Sigma^E}\frac{f(s)}{s-z}ds$.

In case (i), under the absence of $\lambda_j, j=1,2$, it appears that

$$E(z)=I+\mathcal{O}(e^{-C(\xi,c)t}) \tag{3.18}$$

where $C(\xi, c)$ is a positive constant relying on ξ and c. While in the case(ii) and (iii), the stationary points have contribution on $t \to \infty$. By (3.16), it adduced that

$$\|C_E\| \le \|C_-\| \|V^E(z) - I\|_2 \lesssim \mathcal{O}(t^{-1/2}), \tag{3.19}$$

which implies that $1 - C_E$ is invertible for sufficiently large t. So ϖ exists and is unique. Besides,

$$\|\varpi\|_{L^\infty(\Sigma^E)} \lesssim \frac{\|C_E\|}{1 - \|C_E\|} \lesssim t^{-1/2}. \tag{3.20}$$

In order to reconstruct the solution $u(y, t)$ of (1.1), we need the asymptotic behaviour of $E(z)$ as $z \to 0 \in \mathbb{C}^+$ and the long-time asymptotic behaviour of $E(0)$.

Proposition 4. *As $z \to 0 \in \mathbb{C}^+$, we have*

$$E(z) = E(0) + E_1 z + \mathcal{O}(z^2), \tag{3.21}$$

with long-time asymptotic behaviour

$$E(0) = I + t^{-1/2} H^{(0)} + \mathcal{O}(t^{-1}), \tag{3.22}$$

where

$$H^{(0)} = \sum_{p=\pm\lambda_j, j=1,2} \frac{M^{mod1}(p) A_{j,\pm}(\xi) M^{mod1}(p)^{-1}}{p}. \tag{3.23}$$

Here $A_{j,\pm}(\xi)$ is given by (3.10). And

$$E_1 = \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s))\,(V^E - I)}{s^2} ds = t^{-1/2} H^{(1)} + \mathcal{O}(t^{-1}),$$

where

$$H^{(1)} = \sum_{p=\pm\lambda_j, j=1,2} \frac{M^{mod}(p) A_{j,\pm}(\xi) M^{mod}(p)^{-1}}{p^2}. \tag{3.24}$$

Proof. Substituting the long-time asymptotic behaviour of V^E, $\varpi(s)$ and Proposition 3 into $2\pi i(E(0) - I)$, it is found that

$$\begin{aligned} &\int_{\Sigma^E} \frac{(I + \varpi(s))\,(V^E - I)}{s} ds \\ &= \int_{\partial U} \frac{M^{mod1}(s)(M^{j,\pm}(s) - I) M^{mod1}(s)^{-1}}{s} ds + \mathcal{O}(t^{-1}) \\ &= t^{-1/2} \int_{\partial U} \frac{M^{mod1}(s) A_{j,\pm}(\xi) M^{mod1}(s)^{-1}}{s(z \mp \lambda_j)} ds + \mathcal{O}(t^{-1}). \end{aligned} \tag{3.25}$$

Then by residue theorem we finally arrive at the result. □

4. Region II: fast-decay background region

The Region II is corresponding to the case $\{(\xi, c) : \xi > 1 + 2/c\}$. In this case, we introduce a new scalar function

$$X(z) = \sqrt{z^2 - c^2}, \quad \theta^{(+)}(z) = X(z) \left(\frac{\xi - 1}{2} - \frac{1}{cz^2} \right), \tag{4.1}$$

where $X(z)$ is analytic on $\mathbb{C} \setminus \Sigma_+$ and takes the single-valued analytic branch such that $X_+(z) \in i\mathbb{R}^+$ on Σ_+. In this region of ξ, we also define the stationary point of $\theta^{(+)}$ as $\lambda_1 = \sqrt{\frac{2}{c(\xi-1)}} \in (0, 1)$ satisfying

$$\frac{\xi - 1}{2} - \frac{1}{c\lambda_1^2} = 0.$$

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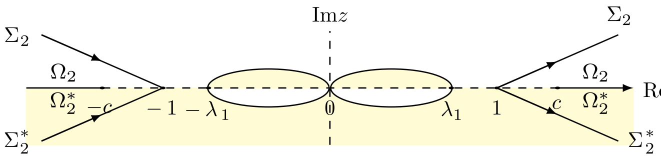

Figure 6. *The region $\Omega_2 \cup \Omega_2^*$ and curve $\Sigma_2 \cup \Sigma_2^*$. In this case $\{(\xi, c) : \xi > 1 + 2/c\}$, Im$[\theta^{(+)}](z) < 0$ in yellow region while Im$[\theta^{(+)}](z) > 0$ in white region. And critical line Im $[\theta^{(+)}](z) = 0$ is black solid line.*

Unlike Region I, the stationary point λ_1 is not the zero of $\partial_z\theta^{(+)} = 0$.The sign of the imaginary part Im$[\theta^{(+)}]$ is shown in Figure 6. Define the contour Σ_1 and the region Ω_1 as shown in Figure 6. Obviously,

$$p^{(-)} - \theta^{(+)} = \mathcal{O}\left(z^{-1}\right), \text{ as } z \to \infty, \tag{4.2}$$

$$p^{(-)} - \theta^{(+)} = \frac{ci}{2}(\xi - 1) + \frac{i}{2c^2} - \frac{i\xi}{2} + \mathcal{O}(z^2), \text{ as } z \to 0 \in \mathbb{C}^+. \tag{4.3}$$

So we can use $\theta^{(+)}$ to replace $p^{(-)}$ in the jump matrix. And we will utilise the factorisations of the jump matrix in Section 2.3 to deform the jump contours, so that the oscillating factor $e^{\pm 2it\theta^{(+)}}$ are decaying in corresponding region, respectively.

Similar to the above section, in this region of ξ, we introduce a piecewise matrix interpolation function

$$G(z) =: G(z; \xi, c) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{r_1 e^{2it\theta^{(+)}}}{1 - r_1 r_2} & 1 \end{pmatrix}, & \text{as } z \in \Omega_2; \\ \begin{pmatrix} 1 & \frac{r_2 e^{-2it\theta^{(+)}}}{1 - r_1 r_2} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \Omega_2^*; \\ I & \text{as } z \text{ in elsewhere}, \end{cases}. \tag{4.4}$$

Invoking that $1 - r_1(\pm c) r_2(\pm c) = 0$, the matrix function $G(z)$ brings a new $-\frac{1}{4}$-singularity on $z = \pm c$. To deal with the jump on $\mathbb{R}$, we introduce an auxiliary function $\delta(z) := \delta(z; \xi, c)$ defined by

$$\log\delta(z) = \frac{X(z)}{2\pi i}\left(\int_{-c}^{-1} + \int_{c}^{1}\right)\frac{\log(ir_{2,-}(s))}{(s - z)X_+(s)}ds + \frac{X(z)}{2\pi i}\int_{\mathbb{R}\setminus\Sigma_+}\frac{\log(1 - r_1(s)r_2(s))}{(s - z)X(s)}ds, \tag{4.5}$$

which relies on ξ and admits the following jump condition:

$$\delta_+(z) = \delta_-(z)(1 - r_1 r_2), \qquad z \in \mathbb{R} \setminus \Sigma_+;$$
$$\delta_-(z)\delta_+(z) = ir_{2,-}(z), \qquad z \in \Sigma_+ \setminus \Sigma_-;$$
$$\delta_-(z)\delta_+(z) = 1, \qquad z \in \Sigma_-.$$

Then we have the following proposition

Proposition 5. *The scalar function $\delta(z)$ satisfies the following properties*

(a) $\delta(z)$ is analytic on $\mathbb{C} \setminus \mathbb{R}$;

(b) $\delta(z)$ has singularity at $z = c, -c$ with

$$\delta(z) = \mathcal{O}\left((z - p)^{\mp 1/4}\right), \quad z \in \mathbb{C}^\pm \to p, \quad p = \pm c;$$

(c) As $z \to \infty \in \mathbb{C} \setminus \mathbb{R}$, $\delta(z)$ has limit $\delta(\infty)$ with

$$\log\delta(\infty) = -\frac{1}{2\pi i}\left(\int_{-c}^{-1} + \int_{c}^{1}\right)\frac{\log(ir_{2,-}(s))}{X_+(s)}ds - \frac{1}{2\pi i}\int_{\mathbb{R}\setminus\Sigma_+}\frac{\log(1 - r_1(s)r_2(s))}{X(s)}ds.$$

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(d) As $z \to 0 \in \mathbb{C}^+$,

$$\delta(z) = \exp\left\{I_\delta^1\right\} \cdot \left(1 + zI_\delta^2\right) + \mathcal{O}(z^2).$$

Here,

$$I_\delta^1 = \frac{c}{2\pi}\left(\int_{-c}^{-1} + \int_c^1\right)\frac{\log(ir_{2,-}(s))}{sX_+(s)}ds + \frac{c}{2\pi}\int_{\mathbb{R}\setminus\Sigma_+}\frac{\log(1 - r_1(s)r_2(s))}{sX(s)}ds, \tag{4.6}$$

$$I_\delta^2 = \frac{c}{2\pi}\left(\int_{-c}^{-1} + \int_c^1\right)\frac{\log(ir_{2,-}(s))}{s^2X_+(s)}ds + \frac{c}{2\pi}\int_{\mathbb{R}\setminus\Sigma_+}\frac{\log(1 - r_1(s)r_2(s))}{s^2X(s)}ds. \tag{4.7}$$

Proof. The proof of (a), (c) and (d) is trivial as [46]. And for (b), noting that the integral function $\frac{X(z)}{2\pi i}\left(\int_{-c}^{-1} + \int_c^1\right)\frac{\log(ir_{2,-}(s))}{(s-z)X_+(s)}ds$ is bounded as $z \to \pm c$, it remains to estimate the second integral $\frac{X(z)}{2\pi i}\int_{\mathbb{R}\setminus\Sigma_+}\frac{\log(1 - r_1(s)r_2(s))}{(s-z)X(s)}ds$. The proof is given by taking $z \to c$ as an example. There exists a constant c_r such that as $c < s \to c$

$$\log(1 - r_1r_2) = c_r + \frac{1}{2}\log(s - c) + \mathcal{O}\left((s-c)^{1/2}\right).$$

It follows from [40] that as $z \to c$ for $z \in \mathbb{C}^\pm$

$$\frac{X(z)}{2\pi i}\int_{\mathbb{R}\setminus\Sigma_+}\frac{c_r}{(s-z)X(s)}ds = \mathcal{O}(1).$$

On the other hand,

$$\frac{X(z)}{2\pi i}\int_{\mathbb{R}\setminus\Sigma_+}\frac{\frac{1}{2}\log(s-c)}{(s-z)X(s)}ds = \frac{1}{4}\log(z-c) + o(1),$$

from which we conclude the property (b). □

Define a new transformation via (4.4), (4.5),

$$M^{(1)}(z) := M^{(1)}(z;\xi,c) = \delta(\infty)^{-\sigma_3}N(z)e^{it(p^{(-)}-\theta^{(+)})\sigma_3}G(z)\delta(z)^{\sigma_3}, \tag{4.8}$$

which has continuous boundary values $M_\pm^{(1)}(z)$ on $\Sigma^{(1)} := \Sigma_+ \cup \Sigma_2 \cup \Sigma_2^*$ and

$$M_+^{(1)}(z) = M_-^{(1)}(z)V^{(1)}(z), \qquad z \in \Sigma^{(1)}, \tag{4.9}$$

where

$$V^{(1)}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{-r_1\delta^2e^{2it\theta^{(+)}}}{1-r_1r_2} & 1 \end{pmatrix}, & \text{as } z \in \Sigma_2, \\ \begin{pmatrix} 1 & \frac{r_2\delta^{-2}e^{-2it\theta^{(+)}}}{1-r_1r_2} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \Sigma_2^*, \\ -i\sigma_1, & \text{as } z \in \Sigma_+, \end{cases} \tag{4.10}$$

The jump matrix exponentially decays to the identity matrix I as $t \to \infty$ on $\Sigma_2 \cup \Sigma_2^*$, which finally leads to the model RH problem replaced 1 to c in RH problem 3 with solution

$$M^{modc}(z) := D_c(\infty)D_c(z)^{-1}, \tag{4.11}$$

where D_c is defined in (2.4). As $z \to 0 \in \mathbb{C}^+$, it is accomplished that

$$M^{modc}(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} - \frac{zi}{2\sqrt{2}c}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} + \mathcal{O}(z^2). \tag{4.12}$$

Consider the error function

$$E(z) := E(z;\xi,c) = M^{(1)}(z)(M^{modc}(z))^{-1}, \tag{4.13}$$

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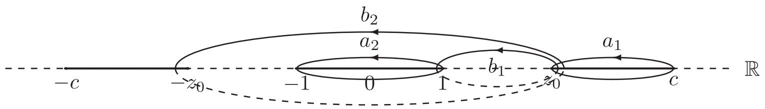

Figure 7. *The canonical homology basis $\{a_j, b_j\}_{j=1}^2$ of the genius 2 Riemann surface.*

which has jump matrix exponentially decaying to the identity matrix I as $t \to \infty$ on $\Sigma_2 \cup \Sigma_2^*$ including ± 1. Then its existence and uniqueness can be shown by a small-norm RH problem with

$$E = I + \mathcal{O}(e^{-Ct}), \tag{4.14}$$

for some constant $C > 0$.

5. Region III: the first-type genus-2 elliptic wave region

In the Region III, we need to introduce a new g-function defined on genus 2 Riemann surface. Note that this region contains two cases

(i) $\{(\xi, c) : 2 < c^2 < 4,\ 1 - \frac{2(c^2-2)}{c^4} < \xi < 1\} \cup \{(\xi, c) : c^2 > 4,\ \xi_m < \xi < 1\}$;

(ii) $\{(\xi, c) : c^2 < 2,\ 1 + \frac{2(2-c^2)}{c^4} < \xi < 1 + 2/c\} \cup \{(\xi, c) : c^2 > 2,\ 1 \le \xi < 1 + 2/c\}$.

In this two different cases, although has same expression, g has different property. So after we proving the basic property of g, we will discuss this two different cases separately. Here, ξ_m is the critical point of ξ that the stationary point z_2 of the g-function given in Proposition 6 in case (i) merge c.

5.1. Constructing the g-function

To construct the g-function, we first introduce:

$$Y(z) := Y(z; z_0, c) = \left[\frac{z^2 - z_0^2}{(z^2-1)(z^2-c^2)}\right]^{1/2}, \tag{5.1}$$

where $z_0 \in (1, c)$. Its branch cut is

$$\Sigma_{mod} = [-c, -z_0] \cup \Sigma_- \cup [z_0, c]. \tag{5.2}$$

and the branch of the square root is chosen such that $Y_+(z) \in i\mathbb{R}^+$ for $z \in [z_0, c]$. And dg is the derivative of g-function given as follows

$$\mathrm{d}g = \frac{Y(z)}{z^3}\left[\frac{1-\xi}{2} z^4 - \frac{c}{z_0}\left(1 + \frac{1}{c^2} - \frac{1}{z_0^2}\right) z^2 + \frac{2c}{z_0}\right] \mathrm{d}z. \tag{5.3}$$

Here, dg is a meromorphic differential defined on the 2-genus Riemann surface $\mathcal{M}$, which has real branch points ± 1, $\pm c$ and $\pm z_0$ with $1 < z_0 < c$. And the canonical homology basis $\{a_j, b_j\}_{j=1}^2$ is shown in Figure 7. Simply calculation shows that

$$\partial_z g - \partial_z p^{(-)} = \mathcal{O}(z^{-2}),\ \text{as } z \to \infty;\ \partial_z g - \partial_z p^{(-)} = \mathcal{O}(z),\ \text{as } z \to 0 \in \mathbb{C}^+.$$

Thus the g-function is given by

$$g(z) := g(z; \xi, c) = \int_c^z \mathrm{d}g,\quad z \in \mathbb{C} \setminus \Sigma_{mod}. \tag{5.4}$$

Proposition 6. *There exists a real number $z_0 = z_0(\xi, c)$ in $(1, c)$ such that the function $g(z)$ defined above has the following properties*

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(a) The a-period of $g(z)$ is zero and the b-period of $g(z)$ is real;

(b) $g(z)$ satisfies the following jump conditions across Σ_+:

$$\begin{aligned} g_-(z) + g_+(z) &= 0, & z &\in (z_0, c), \\ g_-(z) - g_+(z) &= 0, & z &\in (1, z_0) \cup (-z_0, -1), \\ g_-(z) + g_+(z) &= B_1, & z &\in \Sigma_-, \\ g_-(z) + g_+(z) &= B_2, & z &\in (-c, -z_0), \end{aligned}$$

where $B_j = B_j(\xi, c) = \oint_{b_j} dg$ is real;

(c) $g(z)$ has another stationary point $z_1 = z_1(\xi) \in (z_0, c)$, which is one of the solutions of equation $\frac{\xi-1}{2}z^4 + \frac{c}{z_0}\left(1 + \frac{1}{c^2} - \frac{1}{z_0^2}\right) z^2 - \frac{2c}{z_0} = 0$;

(d) In Case (i), (ii) with $c^2 > 2$, $1 > \xi$, $g(z)$ has another stationary point $z_2 = z_2(\xi) \in (c, +\infty)$, $z_2 > z_1$, which also is a solution of equation $\frac{\xi-1}{2}z^4 + \frac{c}{z_0}\left(1 + \frac{1}{c^2} - \frac{1}{z_0^2}\right) z^2 - \frac{2c}{z_0} = 0$. When $c > 2$, as $\xi \to \xi_m < 1 - \frac{2(c^2-2)}{c^4}$, $z_2(\xi)$ decreases to c.

Proof. First, we give the existence of z_0. From the symmetry of dg, it is accomplished that a_2-period of $g(z)$ is zero. Rewrite the function $Y(z)$ as $Y(z; z_0, c)$. Let $F(s) := F(s; \xi, c)$ be a function defined on $\mathbb{R}$ with

$$F(s; \xi, c) = \int_s^c \frac{Y_+(z;s,c)}{z^3} \left[\frac{\xi - 1}{2} z^4 + \frac{c}{s}\left(1 + \frac{1}{c^2} - \frac{1}{s^2}\right) z^2 - \frac{2c}{s} \right] \mathrm{d}z.$$

Then it follows that $F(c) = 0$ and

$$F(1) = \int_1^c \frac{1}{z^3} \left[\left(z^2 - c^2\right)^{-1/2} \right]_+ \left(\frac{\xi - 1}{2} z^4 + \frac{z^2}{c} - 2c \right) \mathrm{d}z = -\theta_+^{(+)}(1),$$

with $\theta^{(+)}$ defined in (4.1). And we calculate the s-derivative of F at $s = c$,

$$\partial_s F(c) = -\frac{i}{c^3} \left(c^2 - 1\right)^{-1/2} \left(\frac{\xi - 1}{2} c^4 + c^2 - 2 \right).$$

In the case $\xi < 1$, obviously, $F(1) \in i\mathbb{R}^-$. Thus, when $c^2 > 2$, $\xi > -\frac{2(c^2-2)}{c^4} + 1$, $\partial_s F(c) \in i\mathbb{R}^-$. And in the case $1 \leq \xi < \frac{2}{c} + 1$, from the property of $\theta^{(+)}$ in above section, we have that $F(1) \in i\mathbb{R}^+$. While when $c^2 < 2$, $1 + \frac{2(2-c^2)}{c^4} < \xi < \frac{2}{c} + 1$ and $c^2 > 2$, $1 \leq \xi < \frac{2}{c} + 1$, it is adduced that $\partial_s F(c) \in i\mathbb{R}^+$. So there must exist $z_0 \in (1, c)$, such that $F(z_0) = 0$.

Moreover, there exists $z_1 \in (z_0, c)$ such that $f(z_1^2) = 0$ with

$$f(x) = \frac{\xi - 1}{2} x^2 + \frac{c}{z_0}\left(1 + \frac{1}{c^2} - \frac{1}{z_0^2}\right) x - \frac{2c}{z_0}.$$

By simply calculating the a_1-period of $g(z)$ is zero and the both b-period are real. Obviously, $f(0) < 0$. So in the $\xi > 1$ case, $f(x)$ only has one zero z_1 on $\mathbb{R}^+$. And in the $\xi < 1$ case, we denote another real solution of $f(z) = 0$ as z_2^2.

In addition, in the $\xi < 1$ case, simple calculation gives that

$$\partial_s F(s; \xi) = -\int_s^c \frac{z(1-\xi)}{2\sqrt{(z^2-1)(z^2-c^2)(z^2-s^2)}s^3} (s^2 - z_1^2)(s^2 - z_2^2) dz,$$

$$\partial_{(1-\xi)/2} F(s; \xi) = \int_s^c \frac{z\sqrt{z^2 - s^2}}{\sqrt{(z^2-1)(z^2-c^2)}} dz.$$

Thus, when ξ decreases from 1, $z_0(\xi)$ increases in $(1, c)$ while z_2 as a solution of $f(x) = 0$ decreases. When z_2 merges with c, we denote this critical condition as ξ_m. □

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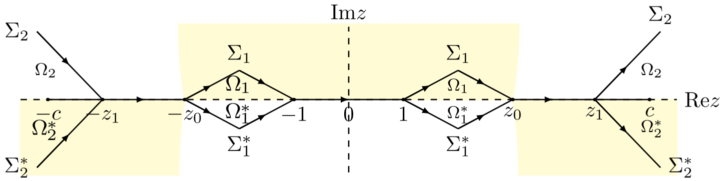

Figure 8. *The opened jump contours $\Sigma^{(1)}$ and opened domains $\Omega_j \cup \Omega_j^*$, $j = 1, 2$. The yellow region means Im$[g] < 0$, while white region means Im$[g] > 0$.*

Denote constant

$$g(\infty) = \lim_{z\to\infty} g(z) - p^{(-)}(z). \tag{5.5}$$

Next, because g will have different sign table in $\xi > 1$ and $\xi < 1$, we will discuss g-function separately in these two cases.

5.2. *Opening the jump in the region* $1 \le \xi < +\infty$

In this region, we give the signature table of Im$[g]$ in Figure 8. Let the open domains $\Omega_j^{\pm}$ be as in Figure 8. Now we use g to replace $p^{(-)}$ in the exponential function. In this region of (ξ, c), we introduce a piecewise matrix interpolation function $G(z) := G(z; \xi, c)$ with

$$\begin{aligned}
&G(z) = \begin{pmatrix} 1 & 0 \\ \frac{r_1 e^{2itg}}{1-r_1r_2} & 1 \end{pmatrix}, \ z \in \Omega_2; \quad G(z) = \begin{pmatrix} 1 & \frac{r_2 e^{-2itg}}{1-r_1r_2} \\ 0 & 1 \end{pmatrix}, \ z \in \Omega_2^*; \\
&G(z) = \begin{pmatrix} 1 & -r_2 e^{-2itg} \\ 0 & 1 \end{pmatrix}, \ z \in \Omega_1; \quad G(z) = \begin{pmatrix} 1 & 0 \\ -r_1 e^{2itg} & 1 \end{pmatrix}, \ z \in \Omega_1^*; \\
&\qquad G(z) = I, \ z \text{ in elsewhere.}
\end{aligned} \tag{5.6}$$

Same as above section, $G(z)$ brings a new $-\frac{1}{4}$-singularity on $z = \pm c$. To open the jump contour $\mathbb{R}$, we define

$$\begin{aligned}
\Sigma_2 &= \left\{ z = -z_1 + e^{\frac{3\pi i}{4}} \mathbb{R}^+ \right\} \cup \left\{ z = z_1 + e^{\frac{\pi i}{4}} \mathbb{R}^+ \right\}, \\
\Sigma_1 &= \left\{ z = -z_0 + e^{\psi i} l, \ l \in (0, \frac{z_0 - 1}{2\cos\psi}) \right\} \cup \left\{ z = z_0 + e^{(\pi-\psi)i} l, \ l \in (0, \frac{z_0 - 1}{2\cos\psi}) \right\} \\
&\quad \cup \left\{ z = 1 + e^{\psi i} l, \ l \in (0, \frac{z_0 - 1}{2\cos\psi}) \right\} \cup \left\{ z = -1 + e^{(\pi-\psi)i} l, \ l \in (0, \frac{z_0 - 1}{2\cos\psi}) \right\},
\end{aligned}$$

where $\psi < \pi/4$ is chosen as a small enough positive constant such that Σ_1 is contained in the region of Im$[g] > 0$.

We define a new matrix-valued function $M^{(1)}(z)$,

$$M^{(1)}(z) := M^{(1)}(z; \xi, c) = e^{itg(\infty)\sigma_3} N e^{it(p^{(-)} - g)\sigma_3} G, \tag{5.7}$$

which satisfies the following RH problem.

RH problem 5.

1. *Analyticity: $M^{(1)}(z)$ is meromorphic in $\mathbb{C} \setminus \Sigma^{(1)}$, where*

$$\Sigma^{(1)} = \left(\cup_{j=1}^2 \Sigma_j \cup \Sigma_j^* \right) \cup \mathbb{R}$$

is shown Figure 8;

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2. *Symmetry:* $M^{(1)}(z)=\sigma_2 M^{(1)}(-z)\sigma_2^{-1}=\sigma_1\overline{M^{(1)}(\bar z)}\sigma_1$;
3. *Asymptotic behaviours:* $M^{(1)}(z)=I+\mathcal{O}(z^{-1}),\qquad z\to\infty$;
4. *Singularity:* $M^{(1)}(z)$ *has singularity at* $z=\pm1,\pm c$ *with:*

$$M^{(1)}(z)\sim\mathcal{O}(z\mp1)^{-1/4},\ z\to\pm1 \text{ in } \mathbb{C}\setminus\Sigma^{(1)},$$
$$M^{(1)}(z)\sim\left(\mathcal{O}(1),\mathcal{O}(z\mp c)^{-1/2}\right),\ z\to\pm c \text{ in } \mathbb{C}^+,$$
$$M^{(1)}(z)\sim\left(\mathcal{O}(z\mp c)^{-1/2},\mathcal{O}(1)\right),\ z\to\pm c \text{ in } \mathbb{C}^-.$$

5. *Jump condition:* $M^{(1)}$ *has continuous boundary values* $M^{(1)}_\pm(z)$ *on the contour* $\Sigma^{(1)}$ *with* $M^{(1)}_+(z)=M^{(1)}_-(z)V^{(1)}(z)$, *where*

$$V^{(1)}(z)=\begin{pmatrix}1 & r_2e^{-2itg}\\ 0 & 1\end{pmatrix},\ z\in\Sigma_1;\quad V^{(1)}(z)=\begin{pmatrix}1 & 0\\ -r_1e^{2itg} & 1\end{pmatrix},\ z\in\Sigma_1^*;$$
$$V^{(1)}(z)=\begin{pmatrix}1 & 0\\ \frac{-r_1e^{2itg}}{1-r_1r_2} & 1\end{pmatrix},\ z\in\Sigma_2;\quad V^{(1)}(z)=\begin{pmatrix}1 & \frac{r_2e^{-2itg}}{1-r_1r_2}\\ 0 & 1\end{pmatrix},\ z\in\Sigma_2^*;$$
$$V^{(1)}(z)=(1-r_1r_2)^{-\sigma_3},\ z\in\mathbb{R}\setminus\Sigma_+;\quad V^{(1)}(z)=\begin{pmatrix}0 & -ie^{-itB_1}\\ -ie^{itB_1} & 0\end{pmatrix},\ z\in\Sigma_-;$$
$$V^{(1)}(z)=\begin{pmatrix}0 & r_{2,-}(z)e^{-itB_2}\\ -r_{1,+}(z)e^{itB_2} & 0\end{pmatrix},\ z\in[-c,-z_1];$$
$$V^{(1)}(z)=\begin{pmatrix}0 & r_{2,-}(z)e^{-itB_2}\\ -r_{1,+}(z)e^{itB_2} & e^{it(g_--g_+)}\end{pmatrix},\ z\in[-z_1,-z_0];$$
$$V^{(1)}(z)=\begin{pmatrix}0 & r_{2,-}(z)\\ -r_{1,+}(z) & e^{-2itg_+}\end{pmatrix},\ z\in[z_0,z_1];$$
$$V^{(1)}(z)=\begin{pmatrix}0 & r_{2,-}(z)\\ -r_{1,+}(z) & 0\end{pmatrix},\ z\in[z_1,c];$$

To deal with the jump on $\mathbb{R}$, we define $Y_3(z)=(z^2-1)(z^2-c^2)Y(z)$, and $\delta(z;\xi,c):=\delta(z)$ with

$$\log\delta(z)=\frac{Y_3(z)}{2\pi i}\left[\sum_{\pm}\mp\int_{\pm c}^{\pm z_0}\frac{\log(ir_{2,-}(s))-w_\pm}{(s-z)Y_{3,+}(s)}ds+\int_{\mathbb{R}\setminus\Sigma_+}\frac{\log(1-r_1(s)r_2(s))}{(s-z)Y_3(s)}ds\right],\tag{5.8}$$

where $w_\pm$ satisfy linear system as

$$\begin{pmatrix}\int_{-c}^{-z_0}\frac{ds}{Y_{3,+}(s)} & \int_{z_0}^{c}\frac{ds}{Y_{3,+}(s)}\\ \int_{-c}^{-z_0}\frac{sds}{Y_{3,+}(s)} & \int_{z_0}^{c}\frac{sds}{Y_{3,+}(s)}\end{pmatrix}\begin{pmatrix}w_-\\ w_+\end{pmatrix}=\begin{pmatrix}\int_{\Sigma_+\setminus[-z_0,z_0]}\frac{\log(ir_{2,-}(s))}{Y_{3,+}(s)}ds+\int_{\mathbb{R}\setminus\Sigma_+}\frac{\log(1-r_1(s)r_2(s))}{Y_3(s)}ds\\ \int_{\Sigma_+\setminus[-z_0,z_0]}\frac{s\log(ir_{2,-}(s))}{Y_{3,+}(s)}ds+\int_{\mathbb{R}\setminus\Sigma_+}\frac{s\log(1-r_1(s)r_2(s))}{Y_3(s)}ds\end{pmatrix}.$$

$\delta(z)$ admits the following jump condition:

$$\delta_+(z)=\delta_-(z)(1-r_1(z)r_2(z)),\qquad z\in\mathbb{R}\setminus\Sigma_+;$$
$$\delta_-(z)\delta_+(z)=ir_{2,-}(z)e^{-w_\pm},\qquad z\in\mp[\pm c,\pm z_0];$$
$$\delta_-(z)\delta_+(z)=1,\qquad z\in\Sigma_-.$$

By a similar way to Proposition 5, we obtain the following proposition

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Proposition 7. *The scalar function $\delta(z)$ satisfies the following properties*

(a) $\delta(z)$ *is analytic on* $\mathbb{C}\setminus((-\infty,-z_0)\cup\Sigma_-\cup(z_0,\infty))$;

(b) $\delta(z)$ *has singularity at* $z=\pm c$ *with:*

$$\delta(z)=\mathcal{O}(z-p)^{\mp 1/4},\quad z\in\mathbb{C}^{\pm}\to p,\quad p=\pm c.$$

(c) As $z\to\infty\in\mathbb{C}$, $\delta(z)\sim\delta_\infty:=\delta_\infty(\xi,c)$ *with*

$$\log\delta_\infty=-\frac{1}{2\pi i}\left(\int_{-c}^{-z_0}+\int_c^{z_0}\right)\frac{s^2\log(ir_{2,-}(s))}{Y_{3,+}(s)}ds-\frac{1}{2\pi i}\int_{\mathbb{R}\setminus\Sigma_+}\frac{s^2\log(1-r_1(s)r_2(s))}{Y_3(s)}ds.$$

(d) As $z\to 0\in\mathbb{C}^+$,

$$\delta(z)=\delta_+(0)\left(1+\delta^{(1)}z\right)+\mathcal{O}(z^2),$$

where

$$\delta^{(1)}=\frac{cz_0}{2\pi}\left(\int_{-c}^{-z_0}+\int_c^{z_0}\right)\frac{\log(ir_{2,-}(s))}{s^2Y_{3,+}(s)}ds+\frac{cz_0}{2\pi}\int_{\mathbb{R}\setminus\Sigma_+}\frac{\log(1-r_1(s)r_2(s))}{s^2Y_3(s)}ds.$$

By using $\delta(z)$ in (5.8), we define a new matrix function

$$M^{(2)}(z;\xi,c):=\delta_\infty(\xi,c)^{-\sigma_3}M^{(1)}(z;\xi,c)\delta(z;\xi,c)^{\sigma_3},\tag{5.9}$$

which then satisfies the following RH problem.

RH problem 6.

1. *Analyticity:* $M^{(2)}(z)$ *is meromorphic in* $\mathbb{C}\setminus\Sigma^{(2)}$ *with* $\Sigma^{(2)}=\left(\cup_{j=1}^2\Sigma_j\cup\Sigma_j^*\right)\cup\Sigma_{mod}$, *where* Σ_{mod} *is given in (5.2);*
2. *Symmetry:* $M^{(2)}(z)=\sigma_2M^{(2)}(-z)\sigma_2^{-1}=\sigma_1\overline{M^{(2)}(\bar z)}\sigma_1$;
3. *Asymptotic behaviours:* $M^{(2)}(z)=I+\mathcal{O}(z^{-1}),\quad z\to\infty$;
4. *Singularity:* $M^{(2)}(z)$ *has at most fourth root singularities at* $z=\pm c$;
5. *Jump condition:* $M^{(2)}$ *has continuous boundary values* $M_\pm^{(2)}(z)$ *on* $\Sigma^{(2)}$ *and*

$$M_+^{(2)}(z)=M_-^{(2)}(z)V^{(2)}(z),\qquad z\in\Sigma^{(2)},$$

where

$$V^{(2)}(z)=\begin{pmatrix}1 & r_2\delta^{-2}e^{-2itg}\\ 0 & 1\end{pmatrix},\ z\in\Sigma_1;\quad V^{(2)}(z)=\begin{pmatrix}1 & 0\\ -r_1\delta^2e^{2itg} & 1\end{pmatrix},\ z\in\Sigma_1^*;$$

$$V^{(2)}(z)=\begin{pmatrix}1 & 0\\ \frac{-r_1\delta^2e^{2itg}}{1-r_1r_2} & 1\end{pmatrix},\ z\in\Sigma_2;\quad V^{(2)}(z)=\begin{pmatrix}1 & \frac{r_2\delta^{-2}e^{-2itg}}{1-r_1r_2}\\ 0 & 1\end{pmatrix},\ z\in\Sigma_2^*;$$

$$V^{(2)}(z)=\begin{pmatrix}0 & -ie^{-itB_2+w_-}\\ -ie^{itB_2-w_-} & 0\end{pmatrix},\ z\in[-c,-z_1];\quad V^{(2)}(z)=\begin{pmatrix}0 & -ie^{w_+}\\ -ie^{-w_+} & 0\end{pmatrix},\ z\in[z_1,c];$$

$$V^{(2)}(z)=\begin{pmatrix}0 & -ie^{-itB_1}\\ -ie^{itB_1} & 0\end{pmatrix},\ z\in\Sigma_-;V^{(2)}(z)=\begin{pmatrix}0 & -ie^{w_+}\\ -ie^{-w_+} & \frac{\delta_-}{\delta_+}e^{-2itg_+}\end{pmatrix},\ z\in[z_0,z_1];$$

$$V^{(2)}(z)=\begin{pmatrix}0 & -ie^{-itB_2+w_-}\\ -ie^{itB_2-w_-} & \frac{\delta_-}{\delta_+}e^{itB_2}e^{-2itg_+}\end{pmatrix},\ z\in[-z_1,-z_0];$$

For $z\in\Sigma^{(2)}\setminus\mathbb{R}$, the jump $V^{(2)}(z)$ exponentially approaches the identity matrix as $t\to\infty$. So we expect to only consider the jump on $\mathbb{R}$. To arrive at this goal, in this case, we denote $U:=U(\xi,c)$ as the union set of neighbourhood of $\pm z_0$:

$$U=U_{+z_0}\cup U_{-z_0},\ U_{\pm z_0}=\{z:|z\mp z_0|\le\varrho\},\tag{5.10}$$

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where ϱ is a small positive constant such that $\varrho < \min\left\{\frac{z_0-1}{3}, \frac{z_1-z_0}{3}\right\}$. The stationary point $\pm z_1$ is on the cut with $\mathrm{Im}[g_+](z_1) < 0$, which means that the exponential function in $V^{(2)}(z)$ also decays exponentially on $\pm z_1$. In fact, it is also decays exponentially on $(z_0, z_1] \cup [-z_1, z_0)$. So the contribution of $\pm z_1$ is small as $t \to \infty$.

Thus, the jump matrix $V^{(2)}(z)$ uniformly goes to I on $\Sigma^{(2)} \setminus U$. So outside U there is only exponentially small error (in t) by completely ignoring the jump condition of $M^{(2)}(z)$. It enlightens us to construct the solution $M^{(2)}(z)$ as follow

$$M^{(2)}(z) = \begin{cases} E(z;\xi,c)M^{mod}(z;\xi,c), & z \notin U_{\pm z_0}, \\ E(z;\xi,c)M^{lo,+}(z;\xi,c), & z \in U_{+z_0}, \\ E(z;\xi,c)M^{lo,-}(z;\xi,c), & z \in U_{-z_0}, \end{cases} \tag{5.11}$$

where $M^{mod}(z)$ is the model RH problem on the Riemann surface, which solution is given by theta function in Subsection 5.2.1. $M^{lo,\pm}(z)$ are local model of $\pm z_0$ which solution can be expressed in terms of Airy functions shown in Subsection 5.2.2. And $E(z;\xi,c)$ is the error function, which will be discussed in subsection 5.2.3 by the small-norm RH problem theory.

5.2.1. *Model RH problem on Riemann surface*

We consider the following model RH problem with its jump matrix on $\mathbb{R}$.

RH problem 7.

1. *Analyticity: $M^{mod}(z)$ is analytical in $\mathbb{C} \setminus \Sigma_{mod}$, where Σ_{mod} is given in (5.2);*
2. *Asymptotic behaviours: $M^{mod}(z) \sim I + \mathcal{O}(z^{-1}), \quad |z| \to \infty$;*
3. *Jump condition: $M^{mod}(z)$ satisfies the jump relation*

$$M^{mod}_+(z) = M^{mod}_-(z)V^{mod}(z), \quad z \in \Sigma_{mod},$$

where the jump matrix $V^{mod}(z)$ is given by

$$V^{mod}(z) = \begin{cases} \begin{pmatrix} 0 & -ie^{-itB_2+w_-} \\ -ie^{itB_2-w_-} & 0 \end{pmatrix}, & \text{as } z \in [-c, -z_0], \\ \begin{pmatrix} 0 & -ie^{-itB_1} \\ -ie^{itB_1} & 0 \end{pmatrix}, & \text{as } z \in \Sigma_-, \\ \begin{pmatrix} 0 & -ie^{w_+} \\ -ie^{-w_+} & 0 \end{pmatrix}, & \text{as } z \in [z_0, c]. \end{cases} \tag{5.12}$$

4. *Singularity: $M^{mod}(z)$ has at most fourth root singularities at $z = \pm c, \ \pm 1, \ \pm z_0$.*

The solution M^{mod} of the model RH problem can be specifically characterised by Θ function on the Riemann surface with genus-2. We define

$$\mathcal{N}_1(z) = \frac{1}{2}\left(\kappa(z) + \kappa(z)^{-1}\right), \ \mathcal{N}_2(z) = \frac{1}{2}\left(\kappa(z)^{-1} - \kappa(z)\right),$$

where $\kappa(z)$ is analytic function for $z \in \mathbb{C} \setminus \Sigma_{mod}$,

$$\kappa(z) = \left[\frac{(z-c)(z-1)(z+z_0)}{(z-z_0)(z+1)(z+c)}\right]^{\frac{1}{4}},$$

and its branch is fixed by requiring that $\kappa(z) = 1 + \mathcal{O}(z^{-2}), \quad z \to \infty$. Let $\omega_i, \ i = 1, 2$ be the standard holomorphic differentials on the genus 2 Riemann surface $\mathcal{M}$ such that $\int_{a_i} \omega_j = \delta_{ij}, i, j = 1, 2$. And denote

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its b-period matrix $\tilde{B}=[\tilde{B}_{ij}]_{i,j=1,2}\in GL_2(\mathbb{C})$, with $\tilde{B}_{ij}=\oint_{b_j}\omega_i$, $i,j=1,2$. Define the Abel map

$$\mathcal{A}:\mathcal{M}\to\mathbb{C}^2/\tilde{B}M+N,\quad M,N\in\mathbb{Z}^2,$$
$$P\mapsto\left(\int_1^P\omega_1,\ \int_1^P\omega_2\right)^T.$$

The theta function Θ associated to $\tilde{B}$ is defined by

$$\Theta(\vec{u})=\sum_{\vec{l}\in\mathbb{Z}^g}\exp\left(\pi i<\tilde{B}\vec{l},\vec{l}>+2\pi i<\vec{l},\vec{u}>\right)$$

with

$$\Theta(\vec{u}\pm e_j)=\Theta(\vec{u}),\ \ \Theta(\vec{u}\pm\tilde{B}e_j)=\exp\left(\mp 2\pi i u_j-\pi i\tilde{B}_{jj}\right)\Theta(\vec{u}).$$

Let $C=(-\frac{tB_1-w_+}{2\pi},-\frac{tB_2+i(w_--w_+)}{2\pi})^T\in\mathbb{C}^2$ be a column vector. According to [23], there exists a constant $\mathcal{K}\in\mathbb{C}^2$ such that $\Theta(\mathcal{A}(P)-\mathcal{K})$, $\Theta(\mathcal{A}(P)+\mathcal{K})$ have the same zeros as $\mathcal{N}_1(z)$ and $\mathcal{N}_2(z)$, respectively. Define

$$M^{mod}(z)=e^{\frac{w_+}{2}\hat{\sigma}_3}\begin{pmatrix}N^{mod}_{11}(\infty)^{-1}\mathcal{N}_1N^{mod}_{11}(z) & N^{mod}_{11}(\infty)^{-1}\mathcal{N}_2N^{mod}_{12}(z)\\ N^{mod}_{22}(\infty)^{-1}\mathcal{N}_2N^{mod}_{21}(z) & N^{mod}_{22}(\infty)^{-1}\mathcal{N}_1N^{mod}_{22}(z)\end{pmatrix},\tag{5.13}$$

which is the solution of RH problem 7, where

$$N^{mod}(z)=\begin{pmatrix}\frac{\Theta(\mathcal{A}(z)-\mathcal{K}+C)}{\Theta(\mathcal{A}(z)-\mathcal{K})} & \frac{\Theta(-\mathcal{A}(z)-\mathcal{K}+C)}{\Theta(-\mathcal{A}(z)-\mathcal{K})}\\ \frac{\Theta(\mathcal{A}(z)+\mathcal{K}+C)}{\Theta(\mathcal{A}(z)+\mathcal{K})} & \frac{\Theta(-\mathcal{A}(z)+\mathcal{K}+C)}{\Theta(-\mathcal{A}(z)+\mathcal{K})}\end{pmatrix},\ N^{mod}(\infty)=\lim_{z\to\infty}N^{mod}(z).$$

As $z\to 0\in\mathbb{C}^+$, it follows that

$$M^{mod}(z)=M^{mod}_+(0)+M^{mod}_1z+\mathcal{O}(z^2),$$

where

$$M^{mod}_+(0)=\frac{\sqrt{2}}{2}e^{\frac{w_+}{2}\hat{\sigma}_3}\begin{pmatrix}N^{mod}_{11}(\infty)^{-1}N^{mod}_{11,+}(0) & -iN^{mod}_{11}(\infty)^{-1}N^{mod}_{12,+}(0)\\ -iN^{mod}_{22}(\infty)^{-1}N^{mod}_{21,+}(0) & N^{mod}_{22}(\infty)^{-1}N^{mod}_{22,+}(0)\end{pmatrix},$$

$$M^{mod}_1=\frac{\sqrt{2}}{2}\begin{pmatrix}N^{mod}_{11}(\infty)^{-1} & 0\\ 0 & N^{mod}_{22}(\infty)^{-1}\end{pmatrix}$$
$$e^{\frac{w_+}{2}\hat{\sigma}_3}\left[\begin{pmatrix}\partial_zN^{mod}_{11,+}(0) & -i\partial_zN^{mod}_{12,+}(0)\\ -i\partial_zN^{mod}_{21,+}(0) & \partial_zN^{mod}_{22,+}(0)\end{pmatrix}+\left(1-\frac{1}{z_0}+\frac{1}{c}\right)\begin{pmatrix}-iN^{mod}_{11,+}(0) & N^{mod}_{12,+}(0)\\ N^{mod}_{21,+}(0) & -iN^{mod}_{22,+}(0)\end{pmatrix}\right].$$

5.2.2. Localised RH problem near stationary points

Using local approximations, appropriate error estimates as well as higher-order asymptotics beyond the $\mathcal{O}(1)$ term can be derived. In this subsection, we only give the details of the model around z_0. We consider $M^{lo,+}(z)$ here as an example. First, we denote $P(z_0)$ as the neighbourhood of z_0 in the Riemann surface $\mathcal{M}$ corresponding to dg. It is an analytic homeomorphism. It follows form the definition of dg in (5.1)–(5.3) that there exist a holomorphic function f_+ (we hope the subscript $\pm$ indicates $\pm z_0$) on $P(z_0)$ such that

$$g=-\frac{2i}{3}f_+^3.$$

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Here, on the complex plane, because $g_+(z) \subset i\mathbb{R}^-$ when $z \in (z_0, c)$, we can choose $f_+ \in \mathbb{R}^+$ on $(z_0, c) \cap U_{+z_0}$. Let

$$\lambda_+ = t^{\frac{2}{3}} f_+^2. \tag{5.14}$$

Because of $g_+(z) = -g_-(z)$, $\lambda_+ := \lambda_+(z)$ is a holomorphic homeomorphism from U_{+z_0} to a neighbourhood of zero and

$$\frac{4}{3}\lambda_+^{\frac{3}{2}} = 2itg_+, \quad z \in (z_0, c) \cap U_{+z_0},$$

where the branch cut of $(\cdot)^{\frac{3}{2}}$ is chosen same as in the Airy model in the Appendix A.

From the definition of λ_+, it follows that

$$\frac{4}{3}\lambda_+^{\frac{3}{2}} = 2itg, \quad \text{Im}z > 0, \quad \frac{4}{3}\lambda_+^{\frac{3}{2}} = -2itg, \quad \text{Im}z < 0.$$

Then we explicitly define $M^{lo,+}(z)$ which should have the same jump condition with $M^{(2)}(z)$ locally in U_{+z_0} as follow via the Airy parametrix. Similar to [6, 25, 29], it is shown that

$$M^{lo,+}(z) = \frac{\sqrt{2}}{2} M^{mod} H(z; z_0)^{-1} (I - i\sigma_1) \lambda_+^{\frac{\sigma_3}{4}} m^{Ai}(\lambda_+) H(z; z_0), \tag{5.15}$$

where

$$H(z; z_0) = \begin{cases} \delta^{\sigma_3} r_1^{\frac{\sigma_3}{2}}, & z - z_0 \in \mathbb{C}^+ \cap U_{+z_0}, \\ \sigma_3\sigma_1\delta^{\sigma_3} r_2^{-\frac{\sigma_3}{2}}, & z - z_0 \in \mathbb{C}^- \cap U_{+z_0}. \end{cases}$$

Moreover, for convenience, denote

$$F^+(z) := F^+(z; \xi, c) = \frac{\sqrt{2}}{2} M^{mod}(z; \xi, c) H(z; \pm z_0)^{-1} (I - i\sigma_1) f_+^{\frac{1}{2}\sigma_3}, \tag{5.16}$$

which thereby is an analytic and invertible function in U_{+z_0}. Similarly,

$$M^{lo,-}(z) = \frac{\sqrt{2}}{2} M^{mod} H(z; -z_0)^{-1} (I - i\sigma_1) \lambda_-^{\frac{\sigma_3}{4}} m^{Ai}(\lambda_-) H(z; -z_0). \tag{5.17}$$

with

$$H(z; -z_0) = \begin{cases} e^{\frac{itB_2}{2}\sigma_3} \sigma_3 \delta^{\sigma_3} r_1^{\frac{\sigma_3}{2}}, & z - z_0 \in \mathbb{C}^+ \cap U_{-z_0}, \\ e^{\frac{itB_2}{2}\sigma_3} \sigma_1 \delta^{\sigma_3} r_2^{-\frac{\sigma_3}{2}}, & z - z_0 \in \mathbb{C}^- \cap U_{-z_0}, \end{cases}$$

$$g = -\frac{2i}{3} f_-^3 + \frac{B_2}{2}, \qquad \lambda_- = t^{\frac{2}{3}} f_-^2,$$

where $f_- \in \mathbb{R}+$ on $z \in (-c, -z_0) \cap U_{-z_0}$ and

$$F^- := M^{mod} H(z; -z_0)^{-1} (I - i\sigma_1) \lambda_-^{\frac{\sigma_3}{4}} \tag{5.18}$$

is an analytic and invertible function in U_{-z_0}. Moreover, as $z \to z_0$ in $\mathbb{C} \setminus [z_0, c]$, λ_+ has expansion

$$\lambda_+ = (\tilde{A}_+)^{\frac{2}{3}}(z - z_0) + \frac{2\tilde{B}_+}{3\tilde{A}_+^{\frac{1}{3}}}(z - z_0)^2 + \mathcal{O}((z - z_0)^3),$$

$$\lambda_- = (\tilde{A}_-)^{\frac{2}{3}}(z + z_0) + \frac{2\tilde{B}_-}{3\tilde{A}_-^{\frac{1}{3}}}(z + z_0)^2 + \mathcal{O}((z + z_0)^3),$$

where

$$\tilde{A}_+ = \frac{t}{2}\left(\frac{2z_0}{(z_0^2-1)(c^2-z_0^2)}\right)^{\frac{1}{2}}\left(\frac{\xi-1}{2}z_0+\frac{c}{z_0^2}(1+\frac{1}{c^2})-\frac{3c}{z_0^4}\right),$$

$$\tilde{B}_+ = \frac{3t}{10}\left[-\frac{1}{2}(\frac{2z_0}{(z_0^2-1)(c^2-z_0^2)})^{-\frac{1}{2}}\frac{-7z_0^4+3(1+c^2)z_0^2+c^2}{(z_0^2-1)^2(c^2-z_0^2)^2}\left(\frac{\xi-1}{2}z_0+\frac{c}{z_0^2}\left(1+\frac{1}{c^2}\right)-\frac{3c}{z_0^4}\right)\right.$$
$$\left.+\left(\frac{2z_0}{(z_0^2-1)(c^2-z_0^2)}\right)^{\frac{1}{2}}\left(\frac{\xi-1}{2}-\frac{c}{z_0^3}\left(1+\frac{1}{c^2}-\frac{1}{z_0^2}\right)+\frac{6c}{z_0^5}\right)\right],$$

$$\tilde{A}_- = -\frac{it}{2}\left(\frac{2z_0}{(z_0^2-1)(c^2-z_0^2)}\right)^{\frac{1}{2}}\left(-\frac{\xi-1}{2}z_0+\frac{c}{z_0^2}\left(1+\frac{1}{c^2}\right)-\frac{3c}{z_0^4}\right),$$

$$\tilde{B}_- = \frac{3t}{10}\left[-\frac{i}{2}\left(\frac{2z_0}{(z_0^2-1)(c^2-z_0^2)}\right)^{-\frac{1}{2}}\frac{-7z_0^4+3(1+c^2)z_0^2+c^2}{(z_0^2-1)^2(c^2-z_0^2)^2}\left(-\frac{\xi-1}{2}z_0+\frac{c}{z_0^2}\left(1+\frac{1}{c^2}\right)-\frac{3c}{z_0^4}\right)\right.$$
$$\left.-i\left(\frac{2z_0}{(z_0^2-1)(c^2-z_0^2)}\right)^{\frac{1}{2}}\left(\frac{\xi-1}{2}+\frac{c}{z_0^3}\left(1+\frac{1}{c^2}-\frac{1}{z_0^2}\right)-\frac{6c}{z_0^5}\right)\right]. \tag{5.19}$$

5.2.3. The small norm RH problem for error function

In this subsection, we consider the error matrix-function $E(z) := E(z;\xi,c)$ in this region.

RH problem 8.

1. *Analyticity: $E(z)$ is analytical in $\mathbb{C}\setminus\Sigma^E$, where*
$$\Sigma^E = \partial U\cup\left[\Sigma^{(2)}\setminus(U\cup[-c,-z_1]\cup\Sigma_-\cup[z_1,c])\right]$$
with U defined in (5.10);
2. *Asymptotic behaviours: $E(z)\sim I+\mathcal{O}(z^{-1})$, $\quad |z|\to\infty$;*
3. *Jump condition: $E(z)$ has continuous boundary values $E_\pm(z)$ on Σ^E satisfying $E_+(z)=E_-(z)V^E(z)$, where the jump matrix $V^E(z)$ is given by*
$$V^E(z)=\begin{cases}M^{mod}(z)V^{(2)}(z)M^{mod}(z)^{-1}, & z\in\Sigma^E\setminus\partial U,\\ M^{lo,\pm}(z)M^{mod}(z)^{-1}, & z\in\partial U_{\pm z_0},\end{cases} \tag{5.20}$$
which is shown in Figure 9.

By (5.15), (5.17) and (A.5), the jump matrix of above RH problem satisfies

$$\| V^E(z)-I\|_2 \lesssim \mathcal{O}(t^{-1}).$$

Similar to the discussion in Subsection 3.2, the RH problem 8 admits a unique solution given by

$$E(z)=I+\frac{1}{2\pi i}\int_{\Sigma^E}\frac{(I+\varpi(s))\,(V^E(s)-I)}{s-z}ds, \tag{5.21}$$

where the $\varpi\in L^\infty(\Sigma^E)$ is the unique solution of $(1-C_E)\varpi=C_E\,(I)$.

In order to reconstruct the solution $u(y,t)$ of (1.1), we need the asymptotic behaviour of $E(z)$ as $z\to 0\in\mathbb{C}^+$ and the long-time asymptotic behaviour of $E(0)$.

Proposition 8. *As $z\to 0\in\mathbb{C}^+$, we have*

$$E(z)=E(0)+E_1 z+\mathcal{O}(z^2),$$

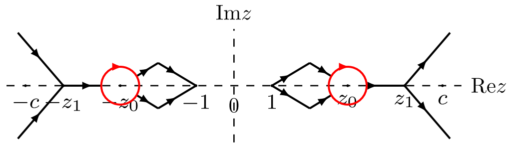

Figure 9. *The jump contour Σ^E for the $E(z)$. The red circles are ∂U.*

with long-time asymptotic behaviour

$$E(0) = I + t^{-1}H^{(0)} + \mathcal{O}(t^{-2}). \tag{5.22}$$

And

$$H^{(0)} := H^{(0)}(\xi, c) = \sum_{p=\pm z_0} \partial_z \left(\frac{1}{z} F^{\pm} \begin{pmatrix} 0 & -\frac{5i}{48}\tilde{A}_{\pm}^{-\frac{4}{3}} \\ 0 & 0 \end{pmatrix} (F^{\pm})^{-1} \right)(p)$$
$$+ \sum_{p=\pm z_0} \frac{1}{p} \left(F^{\pm} \begin{pmatrix} 0 & \frac{5}{36}\tilde{B}_{\pm}\tilde{A}_{\pm}^{-\frac{7}{3}} \\ -\frac{7i}{48}\tilde{A}_{\pm}^{-\frac{2}{3}} & 0 \end{pmatrix} (F^{\pm})^{-1} \right)(p).$$

Here, $\tilde{B}_{\pm}$, $\tilde{A}_{\pm}$ are given in (5.19). And

$$E_1 := E_1(\xi, c) = \frac{1}{2\pi i} \int_{\Sigma^E} \frac{(I + \varpi(s))\,(V^E - I)}{s^2} ds,$$

satisfies long-time asymptotic behaviour condition

$$E_1 = t^{-1}H^{(1)} + \mathcal{O}(t^{-2}), \tag{5.23}$$

where

$$H^{(1)} := H^{(1)}(\xi, c) = \sum_{p=\pm z_0} \partial_z \left(\frac{1}{z^2} F^{\pm} \begin{pmatrix} 0 & -\frac{5i}{48}\tilde{A}_{\pm}^{-\frac{4}{3}} \\ 0 & 0 \end{pmatrix} (F^{\pm})^{-1} \right)(p)$$
$$+ \sum_{p=\pm z_0} \frac{1}{p^2} \left(F^{\pm} \begin{pmatrix} 0 & \frac{5}{36}\tilde{B}_{\pm}\tilde{A}_{\pm}^{-\frac{7}{3}} \\ -\frac{7i}{48}\tilde{A}_{\pm}^{-\frac{2}{3}} & 0 \end{pmatrix} (F^{\pm})^{-1} \right)(p).$$

Proof. By using expansion of V^E and $\varpi(s) = \mathcal{O}(t^{-1})$, we have

$$\int_{\Sigma^E} \frac{(I + \varpi(s))\,(V^E - I)}{s} ds = \frac{1}{t} \int_{\partial U_{\pm z_0}} \frac{M^{mod} H(s; \pm z_0)^{-1} m_1^{Ai} H(s; \pm z_0)(M^{mod})^{-1}}{s f_{\pm}^3} ds + \mathcal{O}(t^{-2}).$$

From the definition of $F^{\pm}$ in (5.16) and (5.18), we rewrite

$$M^{mod} H(s; \pm z_0)^{-1} m_1^{Ai} H(s; \pm z_0)(M^{mod})^{-1}$$
$$= \frac{1}{2} F^{\pm} f_{\pm}^{-\frac{\sigma_3}{2}} \,(I + i\sigma_1)\, m_1^{Ai}\, (I - i\sigma_1)\, f_{\pm}^{\frac{\sigma_3}{2}} (F^{\pm})^{-1}.$$

Here, from (5.19) and m_1^{Ai} in Appendix A, as $z \to z_0 \in \mathbb{C}^+$, we have

$$\frac{1}{2} f_{\pm}^{-3} f_{\pm}^{-\frac{\sigma_3}{2}} \,(I + i\sigma_1)\, m_1^{Ai}\, (I - i\sigma_1)\, f_{\pm}^{\frac{\sigma_3}{2}} =$$
$$\frac{1}{(z \mp z_0)^2} \begin{pmatrix} 0 & -\frac{5i}{48}\tilde{A}_{\pm}^{-\frac{4}{3}} \\ 0 & 0 \end{pmatrix} + \frac{1}{(z \mp z_0)} \begin{pmatrix} 0 & \frac{5}{36}\tilde{B}_{\pm}\tilde{A}_{\pm}^{-\frac{7}{3}} \\ -\frac{7i}{48}\tilde{A}_{\pm}^{-\frac{2}{3}} & 0 \end{pmatrix} + \mathcal{O}(1).$$

Then by residue theorem, we finally arrive at the result. □

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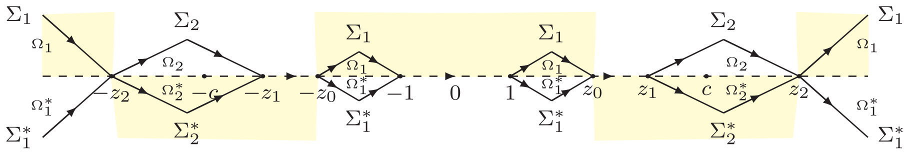

Figure 10. *The domains Ω_j and curves Σ_j, $j=1,2$. The yellow region means Im$[g](z)<0$ while white region means Im$[g](z)>0$.*

5.3. Opening the jump in the region $3/4<\xi<1$

This region includes two cases:

(1) $\left\{(\xi,c): 2<c^2<4, 1-\frac{2(c^2-2)}{c^4}<\xi<1\right\}$;

(2) $\left\{(\xi,c): c^2>4, \xi_m<\xi<1\right\}$.

In both cases we introduce the same g function by

$$dg=\frac{Y(z)}{z^3}\left[\frac{1-\xi}{2}z^4-\frac{c}{z_0}\left(1+\frac{1}{c^2}-\frac{1}{z_0^2}\right)z^2+\frac{2c}{z_0}\right]dz$$

which will have another zero on $\mathbb{R}$ except on cut. It means that g has three pairs of stationary points on $\mathbb{R}$, which will gives additional contribution as $t\to\infty$. The stationary points of g are $\pm z_0$ and the zeros of the equation

$$\frac{1-\xi}{2}z^4-\frac{c}{z_0}\left(1+\frac{1}{c^2}-\frac{1}{z_0^2}\right)z^2+\frac{2c}{z_0}=0.$$

It has two pairs of zeros on $\mathbb{R}$: $\pm z_1\in(z_0,c)$, $\pm z_2$. Note that, $z_1^2z_2^2=\frac{4c}{z_0(1-\xi)}$. For a given c, when ξ decreases from 1, z_2 as a function of ξ decreases from $+\infty$. We denote ξ_m as the critical condition of ξ that stationary point z_2 merge c. Under this case, the sign table of Img is shown in Figure 10.

Similar to the above section, we define the contour Σ_j and closed region Ω_j relying on (ξ,c) as in Figure 10. Here the angle of Σ_j is a small enough positive constant such that Σ_j, $j=1,2$ are contained in the region of Im$g>0$.

In this region of (ξ,c), we introduce a piecewise matrix interpolation function

$$G(z):=G(z;\xi,c)=\begin{cases}\begin{pmatrix}1 & -r_2e^{-2itg}\\ 0 & 1\end{pmatrix}, & \text{as } z\in\Omega_1;\\ \begin{pmatrix}1 & 0\\ -r_1e^{2itg} & 1\end{pmatrix}, & \text{as } z\in\Omega_1^*;\\ \begin{pmatrix}1 & 0\\ \frac{r_1e^{2itg}}{1-r_1r_2} & 1\end{pmatrix}, & \text{as } z\in\Omega_2;\\ \begin{pmatrix}1 & \frac{r_2e^{-2itg}}{1-r_1r_2}\\ 0 & 1\end{pmatrix}, & \text{as } z\in\Omega_2^*;\\ I & \text{as } z \text{ in elsewhere},\end{cases}\tag{5.24}$$

Same as above section, $G(z)$ brings a new singularity. To deal with the jump on $\mathbb{R}$, we introduce an auxiliary function $\delta(z):=\delta(z;\xi,c)$. Define $Y_3(z)=(z^2-1)(z^2-c^2)Y(z)$, and

$$\log\delta(z)=\frac{Y_3(z)}{2\pi i}\sum_{\pm}\mp\int_{\pm c}^{\pm z_0}\frac{\log(ir_{2,-}(s))-w_\pm}{(s-z)Y_{3,+}(s)}ds+\frac{Y_3(z)}{2\pi i}\left(\int_{-z_2}^{-c}+\int_c^{z_2}\right)\frac{\log(1-r_1(s)r_2(s))}{(s-z)Y_3(s)}ds,$$

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where

$$\begin{pmatrix} \int_{-c}^{-z_0} \frac{ds}{Y_{3,+}(s)} & \int_{z_0}^{c} \frac{ds}{Y_{3,+}(s)} \\ \int_{-c}^{-z_0} \frac{sds}{Y_{3,+}(s)} & \int_{z_0}^{c} \frac{sds}{Y_{3,+}(s)} \end{pmatrix} \begin{pmatrix} w_- \\ w_+ \end{pmatrix} = \begin{pmatrix} \int_{\Sigma_+\setminus[-z_0,z_0]} \frac{\log(ir_{2,-}(s))}{Y_{3,+}(s)} ds + \int_{[-z_2,z_2]\setminus\Sigma_+} \frac{\log(1-r_1(s)r_2(s))}{Y_3(s)} ds \\ \int_{\Sigma_+\setminus[-z_0,z_0]} \frac{s\log(ir_{2,-}(s))}{Y_{3,+}(s)} ds + \int_{[-z_2,z_2]\setminus\Sigma_+} \frac{s\log(1-r_1(s)r_2(s))}{Y_3(s)} ds \end{pmatrix}.$$

$\delta(z)$ admits the following jump condition

$$\begin{aligned} &\delta_+(z) = \delta_-(z)(1-r_1r_2), && z \in [-z_2, z_2] \setminus \Sigma_+; \\ &\delta_-(z)\delta_+(z) = ir_{2,-}(z)e^{-w_\pm}, && z \in \mp[\pm c, \pm z_0]; \\ &\delta_-(z)\delta_+(z) = 1, && z \in \Sigma_-. \end{aligned}$$

and the following proposition

Proposition 9. *The scalar function $\delta(z)$ satisfies the following properties*

(a) $\delta(z)$ *is analytic on* $\mathbb{C} \setminus ((-z_2, -z_0) \cup \Sigma_- \cup (z_0, z_2))$;

(b) $\delta(z)$ *has singularity at* $z = \pm c$ *with*

$$\delta(z) = \mathcal{O}(z-p)^{\mp 1/4}, \quad z \in \mathbb{C}^\pm \setminus \mathbb{R} \to p, \quad p = c, -c. \tag{5.25}$$

(c) As $z \to \infty \in \mathbb{C} \setminus \mathbb{R}$, $\delta(z) \sim \delta_\infty$ *with*

$$\log \delta_\infty = -\frac{1}{2\pi i} \int_{\Sigma_+\setminus[-z_0,z_0]} \frac{s^2 \log(ir_{2,-}(s))}{Y_{3,+}(s)} ds - \frac{1}{2\pi i} \int_{[-z_2,z_2]\setminus\Sigma_+} \frac{s^2 \log(1-r_1(s)r_2(s))}{Y_3(s)} ds.$$

(d) As $z \to 0 \in \mathbb{C}^+$,

$$\delta(z) = \delta_+(0)\left(1 + \delta^{(1)} z\right) + \mathcal{O}(z^2), \tag{5.26}$$

where

$$\delta^{(1)} = \frac{cz_0}{2\pi} \left(\int_{-c}^{-z_0} + \int_{z_0}^{c} \right) \frac{\log(ir_{2,-}(s))}{s^2 Y_{3,+}(s)} ds + \frac{cz_0}{2\pi} \left(\int_{-z_2}^{-c} + \int_{c}^{z_2} \right) \frac{\log(1-r_1(s)r_2(s))}{s^2 Y_3(s)} ds;$$

(e) For z_2, *there exists an analytic function* $\delta_{+z_2}(z)$ *on* $z \in U_{+z_2} \setminus (c, z_2)$ *which is continuous to the boundary such that for* $\nu(z_2) = \frac{1}{2\pi} \log(1 - r_1(z_2)r_2(z_2))$,

$$\delta(z) = \delta_{\pm z_2}(z)(z-z_2)^{-i\nu(z_2)}, \quad \arg(z-z_2) \in (-\pi, \pi), \tag{5.27}$$

and

$$\left|\delta_{+z_2}(z) - \delta_{+z_2}(z_2)\right| \lesssim |z - z_2|.$$

Through $\delta(z)$ and $G(z)$, in this region of (ξ, c), same as above subsection, we give series of transformations:

$$N \xrightarrow{(5.7)} M^{(1)} \xrightarrow{(5.9)} M^{(2)} = \delta_\infty^{-\sigma_3} e^{itg(\infty)\sigma_3} N e^{it(p_- - g)\sigma_3} \delta^{\sigma_3},$$

which then satisfies the following RH problem.

RH problem 9.

1. *Analyticity:* $M^{(2)}(z)$ *is meromorphic in* $\mathbb{C} \setminus \Sigma^{(2)}$ *with*

$$\Sigma^{(2)} = \cup_{j=1}^{2} \left(\Sigma_j \cup \Sigma_j^*\right) \cup \Sigma_{mod},$$

where Σ_{mod} *is given in (5.2)*

2. *Symmetry:* $M^{(2)}(z) = \sigma_2 M^{(2)}(-z)\sigma_2^{-1} = \sigma_1 \overline{M^{(2)}(\bar{z})}\sigma_1$;
3. *Asymptotic behaviours:* $M^{(2)}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty$;

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4. *Jump condition: $M^{(2)}$ has continuous boundary values $M^{(2)}_\pm(z)$ on $\Sigma^{(2)}$ and*

$$M^{(2)}_+(z) = M^{(2)}_-(z)V^{(2)}(z), \qquad z \in \Sigma^{(2)}, \tag{5.28}$$

where

$$V^{(2)}(z) = \begin{pmatrix} 1 & r_2\delta^{-2}e^{-2itg} \\ 0 & 1 \end{pmatrix}, \ z \in \Sigma_1; \quad V^{(2)}(z) = \begin{pmatrix} 1 & 0 \\ -r_1\delta^2 e^{2itg} & 1 \end{pmatrix}, \ z \in \Sigma_1^*;$$

$$V^{(2)}(z) = \begin{pmatrix} 1 & 0 \\ \frac{-r_1\delta^2 e^{2itg}}{1-r_1r_2} & 1 \end{pmatrix}, \ z \in \Sigma_2; \quad V^{(2)}(z) = \begin{pmatrix} 1 & \frac{r_2\delta^{-2}e^{-2itg}}{1-r_1r_2} \\ 0 & 1 \end{pmatrix}, \ z \in \Sigma_2^*;$$

$$V^{(2)}(z) = \begin{pmatrix} 0 & -ie^{-itB_2+w_-} \\ -ie^{itB_2-w_-} & 0 \end{pmatrix}, \ z \in [-c, -z_1]; \quad V^{(2)}(z) = \begin{pmatrix} 0 & -ie^{w_+} \\ -ie^{-w_+} & 0 \end{pmatrix}, \ z \in [z_1, c];$$

$$V^{(2)}(z) = \begin{pmatrix} 0 & -ie^{-itB_1} \\ -ie^{itB_1} & 0 \end{pmatrix}, \ z \in \Sigma_-; \quad V^{(2)}(z) = \begin{pmatrix} 0 & -ie^{w_+} \\ -ie^{-w_+} & \frac{\delta_-}{\delta_+}e^{-2itg_+} \end{pmatrix}, \ z \in [z_0, z_1];$$

$$V^{(2)}(z) = \begin{pmatrix} 0 & -ie^{-itB_2+w_-} \\ -ie^{itB_2-w_-} & \frac{\delta_-}{\delta_+}e^{itB_2}e^{-2itg_+} \end{pmatrix}, \ z \in [-z_1, -z_0];$$

5. *Singularity: $M^{(2)}(z)$ has at most fourth root singularities at $z = \pm c, \ \pm 1$.*

Except the cut away from $\mathbb{R}$, the jump $V^{(2)}(z)$ exponentially approaches the identity matrix as $t \to \infty$. So we expect to only consider the jump on $\mathbb{R}$. However, different from above section, in this region, g has another pair of stationary points on $\mathbb{R}$. So in this case, we denote $U := U(\xi, c)$ as the union set of neighbourhood of $\pm z_0$ and $\pm z_2$

$$U = U_{\pm z_0} \cup U_{\pm z_2}, \ U_{\pm z_j} = \left\{ z : |z \mp z_j| \le \varrho \right\}, \ j = 0, 2. \tag{5.29}$$

Here, ϱ is a small positive constant such that $\varrho < \min\left\{ \frac{z_0-1}{3}, \frac{z_1-z_0}{3}, \frac{z_2-c}{3}, \epsilon \right\}$. Outside U there is only exponentially small error (in t) by completely ignoring the jump condition of $M^{(2)}(z)$. This proposition enlightens us to construct the solution $M^{(2)}(z)$ as follow

$$M^{(2)}(z) = \begin{cases} E(z;\xi,c)M^{mod}(z;\xi,c), & z \notin U, \\ E(z;\xi,c)M^{0,\pm}(z;\xi,c), & z \in U_{\pm z_0}, \\ E(z;\xi,c)M^{mod}(z;\xi,c)M^{2,\pm}(z;\xi,c), & z \in U_{\pm z_2}, \end{cases}$$

where same as $M^{mod}(z;\xi,c)$ is the model RH problem on the Riemann surface, which solution is given by theta function in Subsection 5.2.1. The difference is $M^{j,\pm}(z;\xi,c)$ are local model of $\pm z_j$, $j = 0, 2$. When $j = 2$, its solution can be expressed in terms of parabolic cylinder parametrix. When $j = 0$, same as above subsection, its solution can be expressed in terms of Airy parametrix shown in Subsection 5.2.2. Its contribution will be higher-order term of the local parametrix of $M^{2,\pm}(z;\xi,c)$

Similar to discussion of the Proposition 3 in the Section 3, let

$$\zeta^{2,+} = 2\sqrt{g''(\pm z_2)}(z \mp z_2), \ r_{+z_2} = r_2(+z_2)\delta^2_{+z_2}(\pm z_2)e^{-2itg(\pm z_2)}(4tg''(\pm z_2))^{2i\nu(z_2)}.$$

One can find that $M^{2,+}(z)$ is well approximated by $P_1(\zeta^{2,+}; r^+_{z_2})$ and $M^{2,+}(z) = \sigma_2 M^{2,-}(-z)\sigma_2^{-1}$. Therefore, similar to the Proposition 3, we give the approximation below.

Proposition 10. *$M^{2,\pm}(z)$ admits the following asymptotic expansion*

$$M^{2,\pm}(z) = I + t^{-1/2}\frac{A_\pm(\xi)}{z \mp z_2} + \mathcal{O}(t^{-1}), \quad t \to +\infty, \tag{5.30}$$

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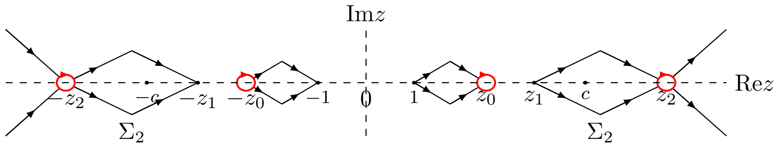

Figure 11. *The jump contour Σ^E for the $E(z)$. The red circles are ∂U.*

where

$$A_\pm(\xi, c) = \frac{1}{2\sqrt{|g''(\pm z_2)|}} \begin{pmatrix} 0 & \tilde{\beta}_{12}^\pm \\ \tilde{\beta}_{21}^\pm & 0 \end{pmatrix}, \tag{5.31}$$

$$\tilde{\beta}_{12}^- = \tilde{\beta}_{21}^+ = \frac{\sqrt{2\pi}\, e^{\frac{\pi\nu(z_2)}{2}} e^{\frac{\pi i}{4}}}{r_{+z_2}\Gamma(i\nu(z_2))}, \quad \tilde{\beta}_{21}^\pm \tilde{\beta}_{12}^\pm = -\nu(z_2),$$

where δ_{+z_2} and $\nu(z_2)$ are defined in (5.27).

5.3.1. The small norm RH problem for error function

In this subsection, we consider the error matrix-function $E(z) := E(z; \xi, c)$ in this region.

RH problem 10.

1. *Analyticity: $E(z)$ is analytical in $\mathbb{C} \setminus \Sigma^E$, where*

$$\Sigma^E = \partial U \cup \left[\Sigma^{(2)} \setminus (U \cup [-c, -z_1] \cup \Sigma_- \cup [z_1, c])\right];$$

2. *Asymptotic behaviours: $E(z) \sim I + \mathcal{O}(z^{-1}), \qquad |z| \to \infty;$*
3. *Jump condition: $E(z)$ has continuous limit $E_\pm(z)$ on Σ^E satisfying $E_+(z) = E_-(z)V^E(z)$, where the jump matrix $V^E(z)$ is given by*

$$V^E(z) = \begin{cases} M^{mod}(z)V^{(2)}(z)M^{mod}(z)^{-1}, & z \in \Sigma^E \setminus \partial U, \\ M^{lo,\pm}(z)M^{mod}(z)^{-1}, & z \in \partial U_{\pm z_0}, \\ M^{mod}(z)M^{2,\pm}(z)M^{mod}(z)^{-1}, & z \in \partial U_{\pm z_2}, \end{cases} \tag{5.32}$$

which is shown in Figure 11.

Similar to the discussion in Section 3.2, the RH problem 10 satisfies

$$|V^E(z) - I| = \mathcal{O}(t^{-1/2}). \tag{5.33}$$

and admits a unique solution, which can be given by

$$E(z) = I + \frac{1}{2\pi i}\int_{\Sigma^E} \frac{(I + \varpi(s))\,(V^E(s) - I)}{s - z} ds, \tag{5.34}$$

where the $\varpi \in L^\infty(\Sigma^E)$ is the unique solution of the equation $(1 - C_E)\varpi = C_E\,(I)$.

In order to reconstruct the solution $u(y, t)$ of (1.1), we need the asymptotic behaviour of $E(z)$ as $z \to 0 \in \mathbb{C}^+$ and the long-time asymptotic behaviour of $E(0)$, which is obtained from Proposition 4.

Proposition 11. *As $z \to 0 \in \mathbb{C}^+$, we have*

$$E(z) = E(0) + E_1 z + \mathcal{O}(z^2), \tag{5.35}$$

with long-time asymptotic behaviour

$$E(0) = I + t^{-1/2}H^{(0)} + \mathcal{O}(t^{-2}). \tag{5.36}$$

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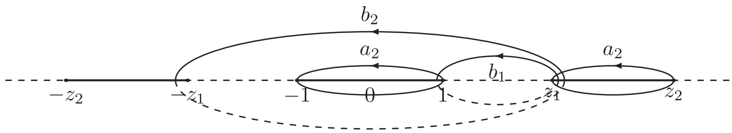

Figure 12. *The canonical homology basis $\{a_j, b_j\}_{j=1}^2$ of the genius 2 Riemann surface.*

And

$$H^{(0)} := H^{(0)}(\xi, c) = \sum_{p=\pm z_2} \frac{M^{mod}(p)A_\pm(\xi)M^{mod}(p)^{-1}}{p}. \tag{5.37}$$

Here, $\tilde{B}_\pm$, $\tilde{A}_\pm$ are shown in (5.19). And

$$E_1 = \frac{1}{2\pi i}\int_{\Sigma^E} \frac{(I+\varpi(s))\,(V^E - I)}{s^2} ds, \tag{5.38}$$

satisfies long-time asymptotic behaviour condition

$$E_1 = t^{-1/2}H^{(1)} + \mathcal{O}(t^{-1}), \tag{5.39}$$

where

$$H^{(1)} := H^{(1)}(\xi, c) = \sum_{p=\pm z_2} \frac{M^{mod}(p)A_\pm(\xi)M^{mod}(p)^{-1}}{p^2}. \tag{5.40}$$

6. Region IV: the second-type genus-2 elliptic wave region

The Region IV is corresponding to the case $\frac{3}{4} < \xi < \xi_m$, $c > 2$. Here, as denoted in the above section, ξ_m is the critical condition that stationary point z_2 merge c. Similarly, we need to construct new g-functions defined on genus 2 Riemann surface which has real branch points ± 1, $\pm z_1$ and $\pm z_2$ with $z_1 < z_2$. And the canonical homology basis $\left\{a_j, b_j\right\}_{j=1}^2$ is shown in Figure 12.

6.1. Constructing the g-function

To construct the g-function, we introduce

$$Z(z) = \left[\frac{(z^2 - z_1^2)(z^2 - z_2^2)}{(z^2-1)}\right]^{1/2}, \tag{6.1}$$

whose branch cut is

$$\Sigma_{mod} := \Sigma_{mod}(\xi, c) = [-z_2, -z_1] \cup \Sigma_- \cup [z_1, z_2], \tag{6.2}$$

and the branch of the square root is chosen such that $Z_+(z) \in i\mathbb{R}^+$ for $z \in [z_1, z_2]$. And z_1, z_2 admit:

$$\frac{1-\xi}{2} = \frac{1}{z_1 z_2}\left(1 - \frac{1}{z_1^2} - \frac{1}{z_2^2}\right). \tag{6.3}$$

Denote

$$\mathrm{d}g = \frac{Z(z)}{z^3}\left[\frac{1-\xi}{2}z^2 - \frac{2}{z_1 z_2}\right]\mathrm{d}z. \tag{6.4}$$

dg is a meromorphic differential defined on the 2-genus Riemann surface, with dg on the upper sheet and $-$dg on the lower sheet. Similarly, the g-function is given by

$$g(z) = \int_{z_2}^{z} \mathrm{d}g, \ z \in \mathbb{C} \setminus \Sigma_{mod}. \tag{6.5}$$

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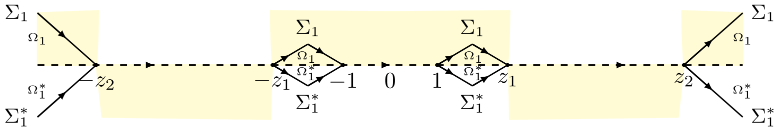

Figure 13. *The region of $\Omega_1 \cup \Omega_1^*$ and the contour $\Sigma_1 \cup \Sigma_1^*$. The shaded region means Im$[g](z) < 0$, while white region means Im$[g](z) > 0$.*

Proposition 12. *There exist a pair of real numbers $z_1 := z_1(\xi, c)$, $z_2 := z_2(\xi, c)$ in $(1, c)$ such that the function $g(z)$ defined above has the following properties*

(a) The a-period of $g(z)$ is zero and the b-period of $g(z)$ is in $\mathbb{R}$;

(b) The sign of Im$[g]$ has the same property in Figure 13;

(c) $g(z)$ satisfies the following jump conditions across $[-z_2, z_2]$:

$$
\begin{aligned}
&g_-(z) + g_+(z) = 0, && z \in (z_1, z_2),\\
&g_-(z) - g_+(z) = 0, && z \in (1, z_1) \cup (-z_1, -1),\\
&g_-(z) + g_+(z) = B_1, && z \in (-1, 1),\\
&g_-(z) + g_+(z) = B_2, && z \in (-z_2, -z_1),
\end{aligned}
$$

here, $B_j = B_j(\xi) = \oint_{b_j} dg$ is real;

(d) $g(z)$ has another stationary point $z_0 = z_0(\xi) \in (z_1, z_2)$, which is the solution of equation $\frac{\xi-1}{2} z^2 - \frac{2}{z_1 z_2} = 0$.

(e) As $\xi \to \xi_m$, we have $z_2 \to c$, while as $\xi \to \frac{3}{4}$, $z_1, z_2 \to 2$.

Proof. Denote $\eta = -\frac{2}{z_1 z_2}$. Thus, (6.3) gives

$$
z_1^2 + z_2^2 = \frac{4}{\eta^2}\left(1 + \frac{1-\xi}{\eta}\right).
$$

Then the a_2-period of g equals to zero if and only if $F(\eta, \xi) = 0$ with

$$
F(\eta, \xi) = \int_{z_1}^{z_2} \frac{Z(z)}{z^3}\left[\frac{1-\xi}{2} z^2 - \frac{2}{z_1 z_2}\right] \mathrm{d}z.
$$

When $\xi = \frac{3}{4}$, $F(\eta, \xi) = 0$ has solution $(-\frac{1}{2}, \frac{3}{4})$, and on the other end $\xi = \xi_m$, $F(\eta, \xi) = 0$ has solution as shown in Proposition 6: $(-\frac{2}{c z_0(\xi_m)}, \xi_m)$. And

$$
\partial_\eta F(\eta, \xi) = \int_{z_1}^{z_2} \frac{z}{\sqrt{(z^2-1)(z^2-z_1^2)(z^2-z_2^2)}}\left[1 + (1-\xi)\left(\frac{2}{\eta^3} + \frac{2}{\eta^4}(1-\xi)\right)\right] \mathrm{d}z.
$$

Consider the function $f(x) = x^4 + 2(1-\xi)x + 3(1-\xi)^2$. It is noticed that (6.3) implies $-\eta > 1 - \xi$, so simple calculation gives that $f(\eta) > f(\xi - 1) > 0$. So $\partial_\eta F \neq 0$, which gives the existence of solution η. □

6.2. Opening the jump contour

Similar to the above section, we define the following contour Σ_1 and closed region Ω_1 relying on (ξ, c) as in Figure 13 to open the jump on $\mathbb{R}$. In this region of (ξ, c), we introduce a piecewise matrix interpolation function

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$$G(z) := G(z;\xi,c) = \begin{cases} \begin{pmatrix} 1 & -r_2 e^{-2itg} \\ 0 & 1 \end{pmatrix}, & \text{as } z \in \Omega_1; \\ \begin{pmatrix} 1 & 0 \\ -r_1 e^{2itg} & 1 \end{pmatrix}, & \text{as } z \in \Omega_1^*; \\ I & \text{as } z \text{ in elsewhere}, \end{cases} \tag{6.6}$$

Same as above sections, $G(z)$ brings a new singularity. To deal with the jump on $\mathbb{R}$, we give a introduction of an auxiliary function $\delta(z) := \delta(z;\xi,c)$. Define

$$Z_3(z) = (z^2-1)Z(z), \quad \log \delta(z) = \frac{Z_3(z)}{2\pi i} \sum_{\pm} \mp \int_{\pm z_2}^{\pm z_0} \frac{\log(ir_{2,-}(s)) - w_\pm}{(s-z)Z_{3,+}(s)} ds. \tag{6.7}$$

where $w_\pm$ satisfy linear system as

$$\begin{pmatrix} \int_{-z_2}^{-z_0} \frac{ds}{Z_{3,+}(s)} & \int_{z_0}^{z_2} \frac{ds}{Z_{3,+}(s)} \\ \int_{-z_2}^{-z_0} \frac{sds}{Z_{3,+}(s)} & \int_{z_0}^{z_2} \frac{sds}{Z_{3,+}(s)} \end{pmatrix} \begin{pmatrix} w_- \\ w_+ \end{pmatrix} = \begin{pmatrix} \int_{[-z_2,z_2]\setminus[-z_0,z_0]} \frac{\log(ir_{2,-}(s))}{Z_{3,+}(s)} ds \\ \int_{[-z_2,z_2]\setminus[-z_0,z_0]} \frac{s\log(ir_{2,-}(s))}{Z_{3,+}(s)} ds \end{pmatrix}.$$

$\delta(z)$ admits the following jump condition

$$\delta_-(z)\delta_+(z) = ir_{2,-}e^{-w_-}, \qquad z \in [-z_2,z_2] \setminus [-z_1,z_1];$$
$$\delta_-(z)\delta_+(z) = 1, \qquad z \in \Sigma_-,$$

and the following proposition

Proposition 13. *The scalar function $\delta(z)$ satisfies the following properties*

(a) $\delta(z)$ is analytic on $\mathbb{C} \setminus \Sigma_{mod}$;

(b) $\delta(z)$ has singularity at $z = c, -c$ with:

$$\delta(z) = \mathcal{O}\big((z-p)^{\mp 1/4}\big), \quad z \in \mathbb{C}^\pm \setminus \mathbb{R} \to p, \quad p = c, -c; \tag{6.8}$$

(c) As $z \to \infty \in \mathbb{C} \setminus \mathbb{R}$, $\delta(z) \sim \delta_\infty(z)$ with δ_∞ defined by

$$\log \delta_\infty = -\frac{1}{2\pi i}\left(\int_{-z_2}^{-z_1} + \int_{z_1}^{z_2}\right) \frac{s^2 \log(ir_{2,-}(s))}{Z_{3,+}(s)} ds;$$

(d) As $z \to 0 \in \mathbb{C}^+$,

$$\delta(z) = \delta_+(0)\left(1 + \delta^{(1)} z\right) + \mathcal{O}(z^2),$$

where

$$\delta^{(1)} = \frac{cz_1}{2\pi}\left(\int_{-z_2}^{-z_1} + \int_{z_2}^{z_1}\right) \frac{\log(ir_{2,-}(s))}{s^2 Z_{3,+}(s)} ds.$$

Through $\delta(z)$ and $G(z)$, in this region of (ξ, c), same as above subsection, we give the same transformation in this case:

$$N(z) \xrightarrow{(5.7)} M^{(1)}(z) \xrightarrow{(5.9)} M^{(2)}(z) = \begin{cases} E(z;\xi,c)M^{mod}(z;\xi,c) & z \notin U \\ E(z;\xi,c)M^{j,pm}(z;\xi,c) & z \in U_{\pm z_j},\ j = 1,2, \end{cases}$$

where $M^{mod}(z;\xi,c)$ is the model RH problem on the Riemann surface, which solution is given by theta function in Subsection 6.3. For $j = 1,2$, $M^{j,pm}(z;\xi,c)$ are local model of $\pm z_j$ which solution can be expressed in terms of Airy functions similarly in Subsection 5.2.2. And $E(z;\xi,c)$ is the error function, which has jump contour in Figure 14, and it is similarly in subsection 5.2.3. We obtain its asymptotic property directly as follow

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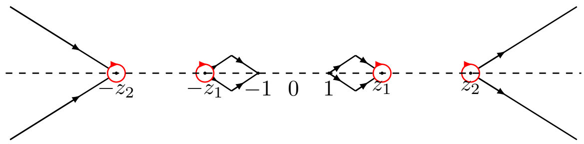

Figure 14. *The jump contour Σ^E for the $E(z;\xi,c)$. The red circles are ∂U.*

Proposition 14. *As $z \to 0 \in \mathbb{C}^+$, we have*

$$E(z;\xi,c) = E(0) + E_1 z + \mathcal{O}(z^2), \tag{6.9}$$

with long-time asymptotic behaviour

$$E(0) = I + t^{-1}H^{(0)} + \mathcal{O}(t^{-2}). \tag{6.10}$$

And

$$H^{(0)} := H^{(0)}(\xi,c) = \sum_{p=\pm z_j, j=1,2} \partial_z \left(\frac{1}{z} F^{\pm} \begin{pmatrix} 0 & -\frac{5i}{48}[\tilde{A}^j_{\pm}]^{-\frac{4}{3}} \\ 0 & 0 \end{pmatrix} (F^{\pm})^{-1} \right)(p)$$
$$+ \sum_{p=\pm z_j, j=1,2} \frac{1}{p} \left(F^{\pm} \begin{pmatrix} 0 & \frac{5}{36}\tilde{B}^j_{\pm}[\tilde{A}^j_{\pm}]^{-\frac{7}{3}} \\ -\frac{7i}{48}[\tilde{A}^j_{\pm}]^{-\frac{2}{3}} & 0 \end{pmatrix} (F^{\pm})^{-1} \right)(p),$$

where $\tilde{B}^j_{\pm}$ and $\tilde{A}^j_{\pm}$ are shown in (6.14). Further, E_1 admits the following asymptotic expansion

$$E_1 = t^{-1}H^{(1)} + \mathcal{O}(t^{-2}), \tag{6.11}$$

where

$$H^{(1)} := H^{(1)}(\xi,c) = \sum_{p=\pm z_j, j=1,2} \partial_z \left(\frac{1}{z^2} F^{\pm} \begin{pmatrix} 0 & -\frac{5i}{48}[\tilde{A}^j_{\pm}]^{-\frac{4}{3}} \\ 0 & 0 \end{pmatrix} (F^{\pm})^{-1} \right)(p)$$
$$+ \sum_{p=\pm z_j, j=1,2} \frac{1}{p^2} \left(F^{\pm} \begin{pmatrix} 0 & \frac{5}{36}\tilde{B}^j_{\pm}[\tilde{A}^j_{\pm}]^{-\frac{7}{3}} \\ -\frac{7i}{48}[\tilde{A}^j_{\pm}]^{-\frac{2}{3}} & 0 \end{pmatrix} (F^{\pm})^{-1} \right)(p).$$

Here we denote

$$F^{\pm}(z) := M^{mod} H(z; \pm z_0)^{-1} N^{-1} f_{\pm}^{\frac{\sigma_3}{2}}, \tag{6.12}$$

$$\frac{3}{2} itg(z) = \tilde{A}^1_{+}(z-z_1)^{3/2} + \tilde{B}^1_{+}(z-z_1)^2 + \mathcal{O}((z-z_1)^{5/2}), \tag{6.13}$$

where the definition of the square root is mapping $\mathbb{R}^+$ to $\mathbb{R}^+$, and $\tilde{A}^1_{-}, \tilde{B}^1_{-}, \tilde{A}^2_{+}, \tilde{B}^2_{+}, \tilde{A}^2_{-}, \tilde{B}^2_{-}$ are as the similar definition on $-z_1, z_2, -z_2$, respectively.

$$\tilde{A}^1_{+} = \frac{t}{2} \left(\frac{2z_1(z_2^2 - z_1^2)}{(z_1^2 - 1)} \right)^{\frac{1}{2}} \left(\frac{1-\xi}{z_1} - \frac{4}{z_1^4 z_2} \right), \tag{6.14}$$

$$\tilde{B}^1_{+} = \frac{3t}{10} \left(\frac{2z_1(z_2^2 - z_1^2)}{(z_1^2 - 1)} \right)^{\frac{1}{2}} \left[-\frac{1-\xi}{z_1^2} + \frac{12}{z_1^5 z_2} + \left(\frac{1-\xi}{z_1} - \frac{4}{z_1^4 z_2} \right) \left(\frac{1}{4z_1} + \frac{z_1}{z_1^2 - z_2^2} - \frac{2z_1}{z_1^2 - 1} \right) \right],$$

$$\tilde{A}^2_{+} = \frac{it}{2} \left(\frac{2z_2(z_2^2 - z_1^2)}{(z_2^2 - 1)} \right)^{\frac{1}{2}} \left(\frac{1-\xi}{z_2} - \frac{4}{z_1 z_2^4} \right),$$

$$\tilde{B}^2_{+} = \frac{3it}{10} \left(\frac{2z_1(z_2^2 - z_1^2)}{(z_2^2 - 1)} \right)^{\frac{1}{2}} \left[-\frac{1-\xi}{z_2^2} + \frac{12}{z_1 z_2^5} + \left(\frac{1-\xi}{z_2} - \frac{4}{z_1 z_2^4} \right) \left(\frac{1}{4z_2} + \frac{z_2}{z_2^2 - z_1^2} - \frac{2z_2}{z_2^2 - 1} \right) \right],$$

$$\tilde{A}^1_{-} = -i\tilde{A}^1_{+}, \quad \tilde{B}^1_{-} = i\tilde{B}^1_{+}, \quad \tilde{A}^2_{-} = i\tilde{A}^2_{+}, \quad \tilde{B}^2_{-} = -i\tilde{B}^2_{+}.$$

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6.3. Model RH problem on Riemann surface

Similarly to Subsection 5.2.1, we arrive at the following model RH problem

RH problem 11.

1. *Analyticity:* $M^{mod}(z)$ *is analytical in* $\mathbb{C}\setminus\Sigma_{mod}$ *with* Σ_{mod} *defined in (6.2).*
2. *Asymptotic behaviours:* $M^{mod}(z)\sim I+\mathcal{O}(z^{-1}), \qquad |z|\to\infty;$
3. *Jump condition:* $M^{mod}(z)$ *satisfies the jump relation*

$$M_+^{mod}(z)=M_-^{mod}(z)V^{mod}(z), \quad z\in\Sigma_{mod},$$

where the jump matrix $V^{mod}(z)$ *is given by*

$$V^{mod}(z)=\begin{cases}\begin{pmatrix}0 & -ie^{-itB_2+w_-}\\ -ie^{itB_2-w_-} & 0\end{pmatrix}, & as\ z\in[-z_2,-z_1],\\ \begin{pmatrix}0 & -ie^{-itB_1}\\ -ie^{itB_1} & 0\end{pmatrix}, & as\ z\in\Sigma_-,\\ \begin{pmatrix}0 & -ie^{w_+}\\ -ie^{-w_+} & 0\end{pmatrix}, & as\ z\in[z_1,z_2];\end{cases} \tag{6.15}$$

4. *Singularity:* $M^{mod}(z)$ *has at most fourth root singularities at* $z=\pm z_2,\ \pm 1,\ \pm z_1$.

M^{mod} can be derived by the Θ function on the Riemann surface of genus 2. To construct the model RH problem M^{mod}, we further let for $z\in\mathbb{C}\setminus\Sigma_{mod}$,

$$\kappa(z)=\left[\frac{(z-z_2)(z-1)(z+z_1)}{(z-z_1)(z+1)(z+z_2)}\right]^{\frac{1}{4}}, \ \mathcal{N}_1(z)=\frac{1}{2}(\kappa+\kappa^{-1}), \mathcal{N}_2(z)=\frac{1}{2}(\kappa^{-1}-\kappa),$$

where the branch of the forth root is chosen such that $\kappa(z)=1+\mathcal{O}(z^{-1}), z\to\infty$. Let ω_i denote the standard holomorphic differentials on this genus-2 Riemann surface $\mathcal{M}$ such that $\int_{a_i}\omega_j=\delta_{ij}$, $i,j=1,2$. Denote matrix $\tilde{B}\in GL_2(\mathbb{C})$, $\tilde{B}_{ij}=\oint_{b_j}\omega_i, i,j=1,2$. Considering the Abel map

$$\mathcal{A}:\mathcal{M}\to\mathbb{C}^2/\tilde{B}M+N, M,N\in\mathbb{Z}^2 \tag{6.16}$$

$$P\mapsto\left(\int_c^P\omega_i\right)_{i=1}^2, \tag{6.17}$$

and the Θ function

$$\Theta(\vec{u})=\sum_{\vec{l}\in\mathbb{Z}^g}\exp\left(\pi i<\tilde{B}\vec{l},\vec{l}>+2\pi i<\vec{l},\vec{u}>\right) \tag{6.18}$$

Let $C=(-\frac{tB_1-iw_+}{2\pi},-\frac{tB_2+i(w_--w_+)}{2\pi})^T\in\mathbb{C}^2$ be a column vector. Then there exists a constant $\mathcal{K}\in\mathbb{C}^2$ such that $\Theta(\mathcal{A}(P)-K)$, $\Theta(\mathcal{A}(P)+K)$ have the same zeros as $\mathcal{N}_{11}$ and $\mathcal{N}_{12}$, respectively. Then

$$M^{mod}(z)=\frac{1}{2}e^{\frac{w_+}{2}\hat{\sigma}_3}\mathrm{diag}\left(\frac{\Theta(\mathcal{A}(\infty)-K)}{\Theta(\mathcal{A}(\infty)-K+C)},\frac{\Theta(\mathcal{A}(\infty)-K)}{\Theta(\mathcal{A}(\infty)-K-C)}\right)\begin{pmatrix}\mathcal{N}_1N_{11}^{mod} & \mathcal{N}_2N_{12}^{mod}\\ \mathcal{N}_2N_{21}^{mod} & \mathcal{N}_1N_{22}^{mod}\end{pmatrix}, \tag{6.19}$$

is the solution of RH problem 11, where

$$N^{mod}(z)=\begin{pmatrix}\frac{\Theta(\mathcal{A}(z)-\mathcal{K}+C)}{\Theta(\mathcal{A}(z)-\mathcal{K})} & \frac{\Theta(-\mathcal{A}(z)-\mathcal{K}+C)}{\Theta(-\mathcal{A}(z)-\mathcal{K})}\\ \frac{\Theta(\mathcal{A}(z)+\mathcal{K}+C)}{\Theta(\mathcal{A}(z)+\mathcal{K})} & \frac{\Theta(-\mathcal{A}(z)+\mathcal{K}+C)}{\Theta(-\mathcal{A}(z)+\mathcal{K})}\end{pmatrix},$$

As $z\to 0\in\mathbb{C}^+$, it follows that

$$M^{mod}(z)=M_+^{mod}(0)+M_1^{mod}z+\mathcal{O}(z^2). \tag{6.20}$$

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7. Long-time asymptotics for the mCH equation

In this section, we give the proof of Theorem 1.1.

For the region I, we have a sequence of transformations for z in a small neighbourhood of 0

$$N(z) \xrightarrow{(3.5)} M^{(1)}(z) \xrightarrow{(3.9)} E(z)M^{mod1}(z). \tag{7.1}$$

Together with the definition of G in (3.6), Propositions 2 and 4, it is adduced that as $z \to 0$ in $\mathbb{C}^+$,

$$\begin{aligned}
N(z) =& E(z)M^{mod1}(z)\delta^{-\sigma_3}G(z)^{-1} = (I + H^{(0)}t^{-\frac{1}{2}})M_+^{mod1}(0)\exp\left(-I_\delta^1\sigma_3\right) \\
&+ \frac{z}{2\sqrt{2}}\left((I - i\sigma_1)\, e^{-I_\delta^1\sigma_3} + (iI - \sigma_1)\, e^{-I_\delta^1\sigma_3} I_\delta^2\sigma_3\right) \\
&+ \frac{zt^{-\frac{1}{2}}}{2\sqrt{2}} H^{(0)}\left((\sigma_1 - iI)\, e^{-I_\delta^1\sigma_3} - (I - i\sigma_1)\, e^{-I_\delta^1\sigma_3} I_\delta^2\sigma_3\right) \\
&+ \frac{zt^{-\frac{1}{2}}}{2\sqrt{2}} H^{(1)}\,(I - i\sigma_1)\, e^{-I_\delta^1\sigma_3} + \mathcal{O}(z^2) + \mathcal{O}(t^{-1}),
\end{aligned}$$

where the I_δ^1 and I_δ^2, $H^{(0)}$ and $H^{(1)}$ are given by (3.3), Proposition 4, respectively. Substituting above estimates into reconstruction formula (2.29) and (2.31) leads to the result in (1.9) and (1.10) with

$$\begin{aligned}
u^{(1)}(\xi) =& -(1 + iI_\delta^2)H_{22}^{(0)} - 2i(H_{21}^{(0)} + H_{12}^{(0)}) - (1 - iI_\delta^2)H_{11}^{(0)} \\
&- H_{21}^{(1)} - H_{12}^{(1)} - iH_{11}^{(1)} - iH_{22}^{(1)}
\end{aligned} \tag{7.2}$$

$$y^{(1)}(\xi) = H_{11}^{(0)} - H_{21}^{(0)}i. \tag{7.3}$$

For the region II, we have done the sequence of transformations for z in a small neighbourhood of 0

$$N(z) \xrightarrow{(4.8)} M^{(1)}(z) \xrightarrow{(4.13)} E(z)M^{modc}(z). \tag{7.4}$$

To reconstruct $u(x,t)$ by using (2.31), we take $z \to 0$ in $\mathbb{C}^+$ of N. Together with Proposition 5, (4.3), (4.4), (4.12) and (4.14), we obtain that

$$\begin{aligned}
N(z) =& \delta(\infty)^{\sigma_3}E(z)M^{modc}(z)\delta(z)^{-\sigma_3}G(z)^{-1}e^{-it(p^{(-)}-\theta^{(+)})\sigma_3} \\
=& \delta(\infty)^{\sigma_3}\left(\frac{1}{\sqrt{2}}(I - i\sigma_1) - \frac{zi}{2\sqrt{2}c}(I + i\sigma_1)\right) e^{-I_\delta^1\sigma_3}(I - I_\delta^2\sigma_3 z)e^{-it(p_+^{(-)}-\theta_+^{(+)})(0)\sigma_3} \\
&+ \mathcal{O}(t^{-2}) + \mathcal{O}(z^2)
\end{aligned}$$

Substituting above estimates into (2.29) and (2.31) leads to the result in (1.11) and (1.12).

For the regions III and IV, their solving processes are similar. Therefore, we take $\xi > 1$ part of region III as an example to give the proof. For z in a small neighbourhood of 0, we have done a sequence of transformations of RH problem

$$N(z) \xrightarrow{(5.7)} M^{(1)}(z) \xrightarrow{(5.9)} M^{(2)}(z) \xrightarrow{(5.11)} E(z)M^{mod}(z).$$

Taking $z \to 0$ in $\mathbb{C}^+$, from the asymptotic expansion of g in (5.5), Proposition 6, 7 and equations (5.6), (5.13), it follows that

$$\begin{aligned}
N(z) =& e^{-itg(\infty)\sigma_3}\delta_\infty(z)^{\sigma_3}E(z)M^{mod}(z)\delta(z)^{-\sigma_3}G(z)^{-1}e^{-it(p^{(-)}-g)\sigma_3}\\
=& e^{-itg(\infty)\sigma_3}\delta_\infty(0)^{\sigma_3}M^{mod}_+(0)\delta_+(0)^{-\sigma_3}e^{-it(p^{(-)}_+-g_+)(0)\sigma_3}\\
&+t^{-1}e^{-itg(\infty)\sigma_3}\delta_\infty(0)^{\sigma_3}H^{(0)}M^{mod}_+(0)\delta_+(0)^{-\sigma_3}e^{-it(p^{(-)}_+-g_+)(0)\sigma_3}\\
&+ze^{-itg(\infty)\sigma_3}\delta_\infty(0)^{\sigma_3}\left(\delta^{(1)}_\infty\sigma_3M^{mod}_+(0)\delta_+(0)^{-\sigma_3}+M^{mod}_1\delta_+(0)^{-\sigma_3}\right.\\
&\left.-M^{mod}_+(0)\delta_+(0)^{-\sigma_3}\delta^{(1)}\sigma_3\right)e^{-it(p^{(-)}_+-g_+)(0)\sigma_3}\\
&+zt^{-1}e^{-itg(\infty)\sigma_3}\delta_\infty(0)^{\sigma_3}H^{(1)}M^{mod}_+(0)\delta_+(0)^{-\sigma_3}e^{-it(p^{(-)}_+-g_+)(0)\sigma_3}+\mathcal{O}(t^{-2})+\mathcal{O}(z^2).
\end{aligned}$$

Substituting above equation into (2.29) and (2.31) leads to

$$u(x,t)=u(y(x,t),t)=u^{(3)}(y,t;\xi)+t^{-1}\mathcal{E}(\xi)+\mathcal{O}(t^{-2}),$$

where

$$\begin{aligned}
u^{(3)}(y,t;\xi)=&-e^{-2itg(\infty)}\delta_\infty(0)^2M^{mod}_{12,+}(0)\left[\left(\delta^{(1)}_\infty-\delta^{(1)}\right)M^{mod}_{11,+}(0)+M^{mod}_{1,11}\right]\\
&-e^{2itg(\infty)}\delta_\infty(0)^{-2}M^{mod}_{12,+}(0)^{-1}\left[\left(\delta^{(1)}-\delta^{(1)}_\infty\right)M^{mod}_{22,+}(0)+M^{mod}_{1,22}\right],
\end{aligned}\tag{7.5}$$

and

$$\begin{aligned}
\mathcal{E}(\xi)=&-e^{-2itg(\infty)}\delta_\infty(0)^2M^{mod}_{12,+}(0)\left[H^{(1)}_{11}M^{mod}_{11,+}(0)+H^{(1)}_{12}M^{mod}_{21,+}(0)\right]\\
&-e^{-2itg(\infty)}\delta_\infty(0)^2\left(H^{(0)}_{11}M^{mod}_{12,+}(0)+H^{(0)}_{12}M^{mod}_{22,+}(0)\right)\\
&\times\left[\left(\delta^{(1)}_\infty-\delta^{(1)}\right)M^{mod}_{11,+}(0)+M^{mod}_{1,11}\right]\\
&-e^{2itg(\infty)}\delta_\infty(0)^{-2}\frac{H^{(1)}_{21}M^{mod}_{12,+}(0)+H^{(1)}_{22}M^{mod}_{22,+}(0)}{M^{mod}_{12,+}(0)}\\
&+e^{2itg(\infty)}\delta_\infty(0)^{-2}\frac{H^{(0)}_{11}M^{mod}_{12,+}(0)+H^{(0)}_{12}M^{mod}_{22,+}(0)}{M^{mod}_{12,+}(0)^2}\\
&\times\left[\left(\delta^{(1)}-\delta^{(1)}_\infty\right)M^{mod}_{22,+}(0)+M^{mod}_{1,22}\right].
\end{aligned}\tag{7.6}$$

Moreover, it is accomplished that

$$\begin{aligned}
x(y,t)=&y-2\ln\left(-ie^{-itg(\infty)+it(p^{(-)}-g)(0)}\delta_\infty(0)\delta_+(0)M^{mod}_{12,+}(0)\right)\\
&+2i\frac{H^{(0)}_{11}M^{mod}_{12,+}(0)+H^{(0)}_{12}M^{mod}_{22,+}(0)}{M^{mod}_{12,+}(0)}t^{-1}+\mathcal{O}(t^{-2}),
\end{aligned}\tag{7.7}$$

where $H^{(0)}$ and $H^{(1)}$ is in Proposition 8 and 11 corresponding to different case of $\xi>1$ and $\xi<1$ parts of region III. Finally, by summarising the above results, we present our main Theorem 1.1 in this paper.

Acknowledgements. The authors are grateful to the anonymous reviewers for their careful reading of the manuscript, as well as their insightful comments and suggestions.

Financial support. The work of Fan was partially supported by NSFC under grants 12271104. The work of Yang was partially supported by Innovation Center for Mathematical Analysis of Fluid and Chemotaxis (Chongqing University).

Competing interests. The authors have no conflicts to disclose.

Data availability statement. No data was used for the research described in the article.

https://doi.org/10.1017/S0956792525000178 Published online by Cambridge University Press

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Appendix A. The RH model for Airy function

In this appendix, we recall the standard model RH problem of Airy function [6], which has been used in our paper. Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \subset \mathbb{C}$ be the rays

$$\Gamma_1 := \{ze^{\frac{2i\pi}{3}} : z \in \mathbb{R}^+\},\ \Gamma_2 := \{z : z \in \mathbb{R}^-\},\ \Gamma_3 := \{ze^{\frac{4i\pi}{3}} : z \in \mathbb{R}^+\},\ \Gamma_4 := \{z : z \in \mathbb{R}^+\}.$$

The corresponding open sectors are given as follows

$$S_1 = \{z : \arg z \in (0, 2\pi/3)\}, S_2 = \{z : \arg z \in (2\pi/3, \pi)\}, \tag{A.1}$$

$$S_3R = \{z : \arg z \in (\pi, 4\pi/3)\}, S_4 = \{z : \arg z \in (4\pi/3, 2\pi)\}. \tag{A.2}$$

Let $\chi = e^{\frac{2i\pi}{3}}$ and the function $m^{Ai}(z)$ for $z \in \mathbb{C} \setminus \Gamma$ by

$$m^{Ai}(z) = \mathcal{A}(z) \times \begin{cases} e^{\frac{2}{3}z^{\frac{3}{2}}\sigma_3}, & z \in S_1 \cup S_4, \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} e^{\frac{2}{3}z^{\frac{3}{2}}\sigma_3}, & z \in S_2, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} e^{\frac{2}{3}z^{\frac{3}{2}}\sigma_3}, & z \in S_3, \end{cases} \tag{A.3}$$

where

$$\mathcal{A}(z) = \begin{cases} \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\chi^2 z) \\ \mathrm{Ai}'(z) & \chi^2 \mathrm{Ai}'(\chi^2 z) \end{pmatrix} e^{-\frac{\pi}{6}\sigma_3}, & \mathrm{Im} z > 0, \\ \begin{pmatrix} \mathrm{Ai}(z) & -\chi^2 \mathrm{Ai}(\chi z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\chi z) \end{pmatrix} e^{-\frac{\pi}{6}\sigma_3}, & \mathrm{Im} z < 0, \end{cases} \tag{A.4}$$

and $\mathrm{Ai}(z)$ is Airy function. In addition, $m^{Ai} : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$ is a matrix valued analytic function and satisfies the jump condition

$$m_+^{Ai}(z) = m_-^{Ai}(z) v^{Ai}(z),$$

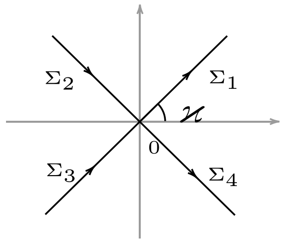

Figure 15. *Orient of* $\Sigma_k, k = 1, 2, 3, 4$.

where

$$v^{Ai}(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ -e^{\frac{4}{3}z^{\frac{3}{2}}} & 1 \end{pmatrix}, & z \in \Gamma_1 \cup \Gamma_3, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & z \in \Gamma_2, \\ \begin{pmatrix} 1 & -e^{\frac{4}{3}z^{\frac{3}{2}}} \\ 0 & 1 \end{pmatrix}, & z \in \Gamma_4. \end{cases}$$

The asymptotic behaviour of $m^{Ai}(z)$ as $z \to \infty$ is

$$\frac{\sqrt{2}}{2}(I - i\sigma_1)\, z^{\frac{\sigma_3}{4}} m^{Ai}(z) = I + \sum_{j=1}^{\infty} \frac{m_j^{Ai}}{z^{\frac{3j}{2}}}, \tag{A.5}$$

where

$$m_j^{Ai} = \frac{e^{\frac{i\pi}{4}}}{2}(I - i\sigma_1)\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}(\frac{3}{2})^j \begin{pmatrix} (-1)^j u_j & u_j \\ -(-1)^j v_j & v_j \end{pmatrix} e^{-\frac{i\pi}{4}\sigma_3}, \tag{A.6}$$

$$u_j = \frac{(2j+1)(2j+3)\ldots(6j-1)}{(216)^j j!}, \quad v_j = \frac{6j+1}{1-6j}u_j.$$

Appendix B. The RH model for parabolic cylinder function

RH problem 12. *For any non-zero complex constant* $|r_0| < 1$, $\nu(r_0) = \frac{1}{2\pi}\log\left(1 - |r_0|^2\right)$*, small positive angle* $0 < \varkappa < \frac{\pi}{4}$*, find a holomorphic function* $P_1(\zeta; r_0)$ *on* $\zeta \in \mathbb{C} \setminus \cup_{k=1}^4 \Sigma_k$, $\Sigma_1 = e^{i\varkappa}\mathbb{R}^+$, $\Sigma_2 = e^{-i\varkappa}\mathbb{R}^-$, $\Sigma_3 = e^{i\varkappa}\mathbb{R}^-$, $\Sigma_4 = e^{-i\varkappa}\mathbb{R}^+$ *oriented as Figure 15, satisfying the following conditions:*

(i) $P_1(\zeta; r_0) = I + \mathcal{O}(\frac{1}{\zeta})$, $\zeta \to \infty$.

(ii) On $\zeta \in \Sigma_k$, $P_{1,+}(\zeta; r_0) = P_{1,-}(\zeta; r_0)V_1^{pc}(\zeta; r_0)$*, where*

$$V_1^{pc}(\zeta; r_0) = \begin{cases} \begin{pmatrix} 1 & r_0\zeta^{2i\nu(r_0)}e^{\frac{i}{2}\zeta^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \Sigma_1, \\ \begin{pmatrix} 1 & 0 \\ -\frac{\bar{r}_0}{1-|r_0|^2}\zeta^{-2i\nu(r_0)}e^{-\frac{i}{2}\zeta^2} & 1 \end{pmatrix}, & \zeta \in \Sigma_2, \\ \begin{pmatrix} 1 & \frac{r_0}{1-|r_0|^2}\zeta^{2i\nu(r_0)}e^{\frac{i}{2}\zeta^2} \\ 0 & 1 \end{pmatrix}, & \zeta \in \Sigma_3, \\ \begin{pmatrix} 1 & 0 \\ -\bar{r}_0\zeta^{-2i\nu(r_0)}e^{-\frac{i}{2}\zeta^2} & 1 \end{pmatrix}, & \zeta \in \Sigma_4, \end{cases} \tag{B.1}$$

where the branch of the logarithmic function is chosen such that $\arg\zeta \in (-\pi, \pi)$.

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For any complex constant a, $D_a(\zeta)$ denotes the parabolic cylinder function which satisfies the Weber's equation

$$D_{a,\zeta\zeta}+\left(-\frac{\zeta^2}{4}+a+\frac{1}{2}\right)D_a=0, \tag{B.2}$$

with large ζ behaviour

$$D_a(\zeta)\sim\begin{cases}\zeta^a e^{-\frac{\zeta^2}{4}}, & -\frac{3\pi}{4}<\arg\zeta<\frac{3\pi}{4},\\ \zeta^a e^{-\frac{\zeta^2}{4}}-\frac{\sqrt{2\pi}}{\Gamma(-a)}e^{ia\pi}\zeta^{-a-1}e^{\frac{\zeta^2}{4}}, & \frac{\pi}{4}<\arg\zeta<\frac{5\pi}{4},\\ \zeta^a e^{-\frac{\zeta^2}{4}}-\frac{\sqrt{2\pi}}{\Gamma(-a)}e^{-ia\pi}\zeta^{-a-1}e^{\frac{\zeta^2}{4}}, & -\frac{5\pi}{4}<\arg\zeta<-\frac{\pi}{4}.\end{cases} \tag{B.3}$$

The unique solution of RH problem 12 is given by

$$P_1(\zeta;r_0)=\Phi_1(\zeta;r_0)S_1(\zeta;r_0)\zeta^{-i\nu(r_0)\sigma_3}e^{-\frac{i\zeta^2}{4}\sigma_3}, \tag{B.4}$$

where

$$\Phi_1=\begin{pmatrix} e^{\frac{-\pi\nu}{4}}D_{i\nu}(\zeta e^{-\frac{i\pi}{4}}) & -\frac{i\nu}{\beta_{21}}e^{\frac{3\pi(\nu-i)}{4}}D_{-i\nu-1}(\zeta e^{-\frac{3i\pi}{4}})\\ \frac{i\nu}{\beta_{12}}e^{\frac{-\pi(\nu+i)}{4}}D_{i\nu-1}(\zeta e^{-\frac{i\pi}{4}}) & e^{\frac{3\pi\nu}{4}}D_{-i\nu}(\zeta e^{-\frac{3i\pi}{4}})\end{pmatrix}, \qquad \mathrm{Im}\zeta>0,$$

$$\Phi_1=\begin{pmatrix} e^{\frac{3\pi\nu}{4}}D_{i\nu}(\zeta e^{\frac{3i\pi}{4}}) & -\frac{i\nu}{\beta_{21}}e^{\frac{\pi(-\nu+i)}{4}}D_{-i\nu-1}(\zeta e^{\frac{i\pi}{4}})\\ \frac{i\nu}{\beta_{12}}e^{\frac{3\pi(\nu+i)}{4}}D_{i\nu-1}(\zeta e^{\frac{3i\pi}{4}}) & e^{\frac{\pi\nu}{4}}D_{i\nu}(\zeta e^{\frac{i\pi}{4}})\end{pmatrix}, \qquad \mathrm{Im}\zeta<0,$$

$$\Phi_1\sim\zeta^{i\nu(r_0)\sigma_3}e^{\frac{i\zeta^2}{4}\sigma_3},\ \zeta\to\infty,$$

with $\beta_{21}=\frac{\sqrt{2\pi}e^{\frac{\pi\nu}{2}+\frac{i\pi}{4}}}{r_0\Gamma(i\nu)}$, $\beta_{12}=-\frac{\nu(r_0)}{\beta_{21}}$ and

$$S_1=\begin{pmatrix}1 & r_0\\ 0 & 1\end{pmatrix},\ 0<\arg\zeta<\varkappa,\ S_1=\begin{pmatrix}1 & 0\\ -\frac{\bar r_0}{1-|r_0|^2} & 1\end{pmatrix},\ \pi-\varkappa<\arg\zeta<\pi,$$

$$S_1=\begin{pmatrix}1 & \frac{r_0}{1-|r_0|^2}\\ 0 & 1\end{pmatrix},\ \pi<\arg\zeta<\pi+\varkappa,\ S_1=\begin{pmatrix}1 & 0\\ -\bar r_0 & 1\end{pmatrix},\ 2\pi-\varkappa<\arg\zeta<2\pi,$$

$$S_1=I,\ \arg\zeta\in(\varkappa,\pi-\varkappa)\cup(\pi+\varkappa,2\pi-\varkappa).$$

Moreover, the solution $P_1(\zeta;r_0)$ satisfies

$$P_1(\zeta;r_0)=I+\frac{P_1^{(1)}(r_0)}{\zeta}+\mathcal{O}(\frac{1}{\zeta^2}),\ \zeta\to\infty,$$

where

$$P_1^{(1)}(r_0)=\begin{pmatrix}0 & \beta_{12}\\ \beta_{21} & 0\end{pmatrix}.$$

Cite this article: Fan E., Li G. and Yang Y. On the long-time asymptotics of the modified Camassa–Holm equation with step-like initial data. *European Journal of Applied Mathematics*, https://doi.org/10.1017/S0956792525000178