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Projective groups in varieties

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A number of questions of Philip Hall concerning complemented normal subgroups of finite relatively free groups are considered.

1. Introduction

We here give answers of sorts to a number of questions of Philip Hall. In a well-known paper [3] Hall defines the concept of splitting group in a variety of groups. (The reader is referred to [3], or to §4 of Chapter 4 of Hanna Neumann's book [4] for definitions.) In §4 of that paper he finds various finite splitting groups in locally finite varieties. Since a group in a variety $\underline{\underline{V}}$ is a splitting group if and only if it is isomorphic to a complement of a normal subgroup in some free group of $\underline{\underline{V}}$, the problem of finding splitting groups is in this sense the same as that of finding complemented normal subgroups in free groups of $\underline{\underline{V}}$. Hence (paraphrasing (Q_1) of [3]) one asks,

(1.1) *What normal subgroups of relatively free groups are complemented?*

Let F be a (finite) free group in a locally finite variety $\underline{\underline{V}}$. Following [4] write, for any finite group G, $M(G)$ for some fixed term of the lower nilpotent series of G or some fixed term of a lower p-series of G. Then (Theorem 3 in [3]): $M(F)$ is complemented in F. Now put

(1.2) $$\underline{\underline{M}} = \{G \in \underline{\underline{V}} : G \text{ finite, } M(G) = 1\} .$$

It is easy to see that $\underline{\underline{M}}$ is subgroup closed, quotient group closed and (finite) direct product closed. Hence, using 15.73 in [4], there is a (unique) subvariety $\underline{\underline{U}}$ of $\underline{\underline{V}}$ whose class of finite groups $\underline{\underline{U}}^*$ is precisely $\underline{\underline{M}}$:

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$$\underline{\underline{U}}^* = \underline{\underline{M}} .$$

But then if $F_r(\underline{\underline{U}})$ is a free group of finite rank in $\underline{\underline{U}}$, there exists an onto homomorphism $\alpha : F_r(\underline{\underline{V}}) \to F_r(\underline{\underline{U}})$. Now $F_r(\underline{\underline{V}})$ is finite and one easily sees that $\ker\alpha = M(F_r(\underline{\underline{V}}))$; hence that $\ker\alpha$ is complemented in $F_r(\underline{\underline{V}})$ by Hall's Theorem mentioned above; and hence that

$$F_r(\underline{\underline{U}}) \text{ is a splitting group in } \underline{\underline{V}}$$

(Theorem 3 in [3]; 44.45 in [4]).

One then asks (paraphrasing (Q_2) of [3]):

(1.3) *What subvarieties of* $\underline{\underline{V}}$ *have the property that their free groups are splitting groups in* $\underline{\underline{V}}$ *?*

It is questions (1.1) and (1.3) that are considered here in the case of locally finite varieties. In fact much of what is done is in Bryant [1] (in spirit if not in fact), and what is not can be deduced from his Lemmas 2 and 3, though these deductions are hardly shorter than doing things directly. Accordingly we proceed from first principles. Many helpful comments from Dr Bryant I gratefully acknowledge.

2. A splitting theorem

Let G be a finite group and N a normal subgroup of G. Denote by $\Phi(G \div N)$ that subgroup of G defined by

$$\Phi(G \div N)/N = \Phi(G/N)$$

(the Frattini subgroup of G/N). Call N *Φ-minimal* in G if for every normal subgroup M of G contained in N, $\Phi(G \div M) = \Phi(G \div N)$ only if $M = N$.

A group is a splitting group in a variety if and only if it is projective and we shall from now on speak mainly of projective, rather than splitting, groups in a variety (see §4 of Chapter 4 of [4]). A group will be called projective if it is projective in some variety.

The theorem now stated provides an answer to (1.1).

THEOREM 2.1. *A normal subgroup of a finite projective group is*

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complemented if and only if it is Φ*-minimal.*

Proof. One way is easy: if N is complemented in any group G (never mind projective) suppose that M is normal in G and contained in N with

$$\Phi(G \div M) = \Phi(G \div N) .$$

Then

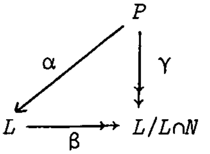
$$N/M \leq \Phi(G \div N)/M = \Phi(G \div M)/M = \Phi(G/M) .$$

But N/M is complemented in G/M so it follows that $N/M = 1$ or $M = N$, as required.

Conversely suppose that P is projective and that N is Φ-minimal in P. Let L be a minimal supplement of N in P, so that

(2.2) $$L \cap N \leq \Phi(L)$$

(or else a maximal subgroup of L would supplement N in P contradicting the minimality of L). Now $P/N = LN/N \cong L/L\cap N$. Let $\gamma : P \to L/L\cap N$ be the natural homomorphism and put $\beta = \gamma|L$. Then, since P is projective, there exists $\alpha : P \to L$ such that the diagram



commutes.

Note that

$$L\alpha\beta = L\gamma = L/L\cap N ,$$

so that

$$L = (L\alpha)\ker\beta = L\alpha(L\cap N) = L\alpha ,$$

since $L \cap N$ is Frattini in L . Being onto, $\alpha|L$ is therefore one-to-one:

(2.3) $$\ker\alpha \cap L = 1 ;$$

and if $x \in P$, there exists $l \in L$ such that $x\alpha = l\alpha$ whence $l^{-1}x \in \ker\alpha$, or

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(2.4) $$L\ker\alpha = P .$$

Finally, $\ker\alpha \leq N$ so that

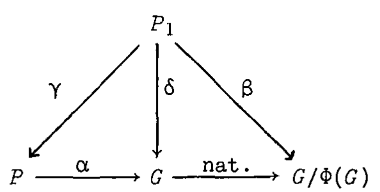
$$\Phi(P \div \ker\alpha) = \ker\alpha\Phi(L) = \ker\alpha(L\cap N)\Phi(L) = N\Phi(L) = N\Phi(L \div L\cap N) = \Phi(P \div N) .$$

The Φ-minimality of N then ensures that $\ker\alpha = N$ and (2.3) and (2.4) that L complements N.

COROLLARY 2.5. *Let P be a finite projective group in a variety $\underline{\underline{V}}$ with N normal in P. Every minimal supplement of N in P is projective in $\underline{\underline{V}}$, and, if N is Φ-minimal, every minimal supplement of N is a complement.*

Proof. Up to (2.4) in the proof above, N was arbitrary.

COROLLARY 2.6 (Bryant [1]). *Let $\underline{\underline{V}}$ be a locally finite variety and G a finite group in it. There exists a finite projective P in $\underline{\underline{V}}$ (the* projective cover *of G) such that $P/\Phi(P) \cong G/\Phi(G)$, and a homomorphism $\alpha : P \to G$ onto G. Moreover if P_1 is projective in $\underline{\underline{V}}$ and $\beta : P_1 \to G/\Phi(G)$ is onto then there exist onto homomorphisms $\gamma : P_1 \to P$, $\delta : P_1 \to G$ such that the diagram*



commutes.

Proof. Choose a finite free group F of $\underline{\underline{V}}$ and a homomorphism $\mu : F \to G$ onto G. Let $N \leq \ker\mu$ be Φ-minimal with

$$\Phi(F \div N) = \Phi(F \div \ker\mu) .$$

By Theorem 2.1 N is complemented in F so $F/N = P$ (say) is projective in $\underline{\underline{V}}$, and there exists a homomorphism $\alpha : P \to G$ onto G; note that $\ker\alpha \leq \Phi(P)$. The rest of the proof follows from the projectivity of P_1 and the non-generator property of Frattini subgroups.

If N_1, N_2 are normal in the finite group G, and $\Phi(G \div N_1) = \Phi(G \div N_2)$ then, as is easy to see, every supplement of one is a supplement of the other. Moreover, then every minimal supplement of one

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is a minimal supplement of the other. If, further, G is projective and N_1, N_2 are Φ-minimal then every complement of one is a complement of the other: if C complements N_2 it is, by the preceding remark, a minimal supplement for N_1, whence a complement by Theorem 2.1. That is

LEMMA 2.7. *If P is projective and N_1, N_2 are Φ-minimal normal subgroups of P such that $\Phi(P \div N_1) = \Phi(P \div N_2)$, then they have common complements.*

COROLLARY 2.8 (Bryant [1]). *A finite group P in a locally finite variety $\underline{\underline{V}}$ is projective if and only if it is maximal among finite groups of $\underline{\underline{V}}$ whose Frattini factor groups are isomorphic to $P/\Phi(P)$.*

Proof. Suppose that P is maximal in this sense. Let F be free in $\underline{\underline{V}}$ and $\alpha : F \to P$ a homomorphism onto P. It is clear that $\ker\alpha$ is Φ-minimal, hence complemented by Theorem 2.1, so P is projective in $\underline{\underline{V}}$.

Conversely suppose that P is projective in $\underline{\underline{V}}$ and that $H \in \underline{\underline{V}}$ with

$$H/\Phi(H) \cong P/\Phi(P) .$$

Choose F free in $\underline{\underline{V}}$ of large enough rank. Then there exist homomorphisms β, γ of F onto H, P respectively such that

$$\Phi(F \div \ker\beta) = \Phi(F \div \ker\gamma) .$$

Let $N \leq \ker\beta$ be Φ-minimal with $\Phi(F \div N) = \Phi(F \div \ker\beta)$.

Both N and $\ker\gamma$ are Φ-minimal, and

$$\Phi(F \div N) = \Phi(F \div \ker\gamma) .$$

By Lemma 2.7 therefore, N and $\ker\gamma$ have a common complement, C say, isomorphic to P. But H is a homomorphic image of C and therefore of P.

3. More projectives

We take up the question (1.3). Suppose that $\underline{\underline{X}}$ is a class of groups in a variety $\underline{\underline{V}}$, and call $\underline{\underline{X}}$ *saturated in* $\underline{\underline{V}}$ if for all finite G in $\underline{\underline{V}}$, $G/\Phi(G)$ is in $\underline{\underline{X}}$ only if G is in $\underline{\underline{X}}$.

LEMMA 3.1. *If $\underline{\underline{X}}$ is quotient group closed and saturated in $\underline{\underline{V}}$ and if P is a finite projective group in $\underline{\underline{V}}$ and N a normal subgroup of P minimal with respect to $P/N \in \underline{\underline{X}}$, then N is complemented in P.*

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Proof. For then N is Φ-minimal in P .

THEOREM 3.2. *A subvariety* $\underline{\underline{U}}$ *of a locally finite variety* $\underline{\underline{V}}$ *has the property that its free groups of finite rank are projective in* $\underline{\underline{V}}$ *if and only if* $\underline{\underline{U}}$ *is saturated in* $\underline{\underline{V}}$.

Proof. If $\underline{\underline{U}}$ is saturated in $\underline{\underline{V}}$ the result follows from Lemma 3.1. Conversely suppose each free group of finite rank in $\underline{\underline{U}}$ is projective in $\underline{\underline{V}}$ and that G in $\underline{\underline{V}}$ is finite with $G/\Phi(G) \in \underline{\underline{U}}$. Let P be the projective cover of $G/\Phi(G)$ in $\underline{\underline{U}}$. Since P is isomorphic to a complement of a normal subgroup of a free group of finite rank in $\underline{\underline{U}}$, it follows that P is projective in $\underline{\underline{V}}$. Hence by Corollary 2.6, G is a homomorphic image of P and therefore $G \in \underline{\underline{U}}$, so $\underline{\underline{U}}$ is saturated in $\underline{\underline{V}}$.

4. Remarks

1. The results of Hall mentioned in §1 are covered by Lemma 3.1 and Theorem 3.2, for the classes $\underline{\underline{M}}$ of groups in (1.2) are saturated (and quotient group closed of course): for soluble groups this is well known to formation theorists, and the proof of Lemma 3 in [3] (or of (44.44) in [4]) can be read to give this in the general case. The crux of these proofs is the Frattini argument, which is not surprising as in essence one must show that a local formation is saturated, which is itself proved by the Frattini argument. In this context it is worth drawing attention to §5 of [3] and in particular to its last paragraph, the last sentence of which is simply the observation (though not in these terms) that the classes $\underline{\underline{M}}$ above are saturated.

2. Lemma 3.1 above suggests that in seeking projectives it is not natural to look for sub-*varieties* of $\underline{\underline{V}}$ whose free groups are projective. For example, since in a locally finite variety $\underline{\underline{V}}$ there is a bound on the order of r-generator groups, it is easy to see that any sub-formation $\underline{\underline{F}}$ of $\underline{\underline{V}}^*$ has an r-generator free group, and that these free groups are projective in $\underline{\underline{V}}$ if and only if $\underline{\underline{F}}$ is saturated in $\underline{\underline{V}}$. On a different note it is natural to ask: are the free groups of finite rank in a subvariety $\underline{\underline{U}}$ of $\underline{\underline{V}}$ projective in $\underline{\underline{V}}$ if and only if the free group of countably infinite rank in $\underline{\underline{U}}$ is also projective in $\underline{\underline{V}}$?

3. A stricter interpretation than (1.3) of (Q_2) on p. 351 of [3] can be made, namely: for fixed r what subvarieties $\underline{\underline{U}}$ of $\underline{\underline{V}}$ have the

property that $F_r(\underline{\underline{U}})$ is projective in $\underline{\underline{V}}$? Clearly an answer can be given to this in terms of saturation for (at most) r-generator groups.

4. Lemma 3.1 can be used to prove a well known result of Shult [5] and Carter and Hawkes [2] in the form: if G is finite and soluble, $\underline{\underline{F}}$ a saturated formation with $G^{\underline{\underline{F}}}$ (the smallest normal subgroup of G whose factor group is in $\underline{\underline{F}}$) abelian then every minimal supplement of $G^{\underline{\underline{F}}}$ is a complement.

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