J. Austral. Math. Soc. 22 (Series A) (1976), 371–379.

COMPATIBLE TIGHT RIESZ ORDERS ON $C(X)$

ELIZABETH LOCI

(Received 10 March 1975; revised 16 September 1975)

Abstract

The pointwise order makes the group $C(X)$ of continuous real-valued functions on a topological space X a lattice-ordered group. We give a characterization of the compatible tight Riesz orders on $C(X)$, and also of their maximal tangents, in terms of the zero-sets of X. The space of maximal tangents of a given compatible tight Riesz order T is studied, and consequently the concept of the T-radical of $C(X)$ is introduced, the T-radical being the intersection of all the maximal tangents of T.

Introduction

Given a topological space X we denote by $C(X)$ the set of continuous real-valued functions on X. If $C(X)$ is equipped with the following order

$$f \geqslant 0 \quad \text{if} \quad f(x) \geqq 0 \quad \text{for all} \quad x \in X$$

then it becomes an abelian lattice-ordered group with

$$f \wedge g(x) = \min\{f(x), g(x)\}$$

$$f \vee g(x) = \max\{f(x), g(x)\}.$$

In order to obtain the compatible tight Riesz orders on $C(X)$, we make use of the following result due essentially to Wirth (1973). A compatible tight Riesz order on a lattice-ordered group $(G, \leqslant)$ is determined by (and determines) a subset T of the positive set $G^+ = \{x \in G : 0 \leqslant x\}$ of G satisfying the following conditions:

(1) T is a proper dual-ideal of G^+

(2) T is normal in G

(3) $T = T + T$

(4) $0 \leqslant n\, x \leqslant y$ for all positive integers n, for all $y \in T$, implies $x = 0$.

In the case of $C(X)$ we can modify the above result, since, $C(X)$ abelian implies (2) holds for all subsets T of $C^+(X)$, $\mathbb{R}$ archimedean implies (4) holds for all subsets T of $C^+(X)$ and $C(X)$ divisible makes (3) easier to check. We have then that the compatible tight Riesz orders on $C(X)$ are determined by (and determine) proper dual-ideals T of $C^+(X)$ satisfying $T = T + T$. By an abuse of language we shall call each such dual-ideal T a compatible tight Riesz order on $C(X)$.

With each compatible tight Riesz order T on $C(X)$ we associate *tangents* [cf. Miller (1973)] i.e. convex sublattice subgroups of $C(X)$ not meeting T and *maximal tangents* i.e. convex sublattice subgroups of $C(X)$ that are maximal with respect to not meeting T. Each maximal tangent M of T satisfies the further condition — $f \wedge g \in M$ implies $f \in M$ or $g \in M$ — i.e. each maximal tangent is a *prime subgroup* of $C(X)$. Finally, if $C(X)$ is given the open-interval topology generated by the compatible tight Riesz order T, then every tangent of T is closed.

Zero-set characterization of compatible tight Riesz orders

We proceed in analogy to Gillman and Jerison (1960).

Given $f \in C(X)$, the set $\{x \in X : f(x) = 0\}$ is called the *zero-set* of f, and will be denoted by $Z(f)$. Any set that is a zero-set of some function in $C(X)$ is called a zero-set in X, and we denote the set of all zero-sets in X by $Z(X)$. Now

$$Z(f) \cup Z(g) = Z(|f|) \cup Z(|g|) = Z(|f| \wedge |g|)$$

and

$$Z(f) \cap Z(g) = Z(|f|) \cap Z(|g|) = Z(|f| \vee |g|)$$

whence $Z(X)$ is closed under finite unions and intersections. Thus $Z(X)$, ordered by inclusion, is a lattice, and we make the following (usual) definitions –

A non-empty subfamily $\mathscr{F}$ of $Z(X)$ is called a Z-*ideal* provided that:

(1) if $Z_1, Z_2 \in \mathscr{F}$ then $Z_1 \cup Z_2 \in \mathscr{F}$

(2) if $Z \in \mathscr{F}$, $Z' \in Z(X)$ and $Z \supset Z'$ then $Z' \in \mathscr{F}$

If in addition

(3) $X \notin \mathscr{F}$

then $\mathscr{F}$ is a *proper* Z-ideal.

A non-empty subfamily $\mathscr{G}$ of $Z(X)$ is called a Z-*filter* provided that:

(1) if $Z_1, Z_2 \in \mathscr{G}$ then $Z_1 \cap Z_2 \in \mathscr{G}$

(2) if $Z \in \mathscr{G}$, $Z' \in Z(X)$ and $Z' \supset Z$ then $Z' \in \mathscr{G}$.

If in addition

(3) $\square \notin \mathscr{G}$

then $\mathscr{G}$ is a *proper* Z-filter.

Throughout this paper we will assume all Z-ideals and Z-filters to be proper.

https://doi.org/10.1017/S1446788700014816 Published online by Cambridge University Press

THEOREM 1. (a) *If T is a proper dual-ideal of $C^+(X)$, then the family*

$$Z[T] = \{Z(f): f \in T\}$$

is a Z-ideal.

(b) *If $\mathscr{F}$ is a Z-ideal then the family*

$$Z \leftarrow [\mathscr{F}]^* = \{|f|: Z(f) \in \mathscr{F}\}$$

is a proper dual-ideal of $C^+(X)$.

PROOF. (a) 1. Let $Z_1, Z_2 \in Z[T]$. Choose $f_1, f_2 \in T$ satisfying $Z_1 = Z(f_1)$, $Z_2 = Z(f_2)$, then since $f_1, f_2 \in C^+(X)$ we have $Z(f_1) \cup Z(f_2) = Z(f_1 \wedge f_2)$, and since T is a dual-ideal, $f_1 \wedge f_2 \in T$. Thus $Z_1 \cup Z_2 \in Z[T]$.

2. Let $Z \in Z[T]$ and $Z' \in Z(X)$ with $Z \supset Z'$. Choose $f \in T$ and $f' \in C(X)$ satisfying $Z = Z(f)$, $Z' = Z(f') = Z(|f'|)$. Then $Z(f) \supset Z(|f'|)$ whence $Z(f \vee |f'|) = Z(f) \cap Z(|f'|) = Z(|f'|) = Z'$. But $f \leqslant f \vee |f'|$ implies $f \vee |f'| \in T$, thus $Z' \in Z[T]$.

3. T proper implies $0 \notin T$ implies $X \notin Z[T]$.

(b) 1. Let $f = |f|$, $g = |g| \in Z \leftarrow [\mathscr{F}]^*$. Then $f \wedge g = |f \wedge g|$ and $Z(f \wedge g) = Z(f) \cup Z(g) \in \mathscr{F}$ whence $f \wedge g \in Z \leftarrow [\mathscr{F}]^*$.

2. Let $f = |f| \in Z \leftarrow [\mathscr{F}]^*$ and let $g \in C(X)$ satisfy $f \leqslant g$, whence $g \in C^+(X)^+(X)$. Now $f, g \in C^+(X)$, $f \leqslant g$ imply $Z(f) \supset Z(g)$ whence $Z(g) \in \mathscr{F}$. Thus $g = |g| \in Z \leftarrow [\mathscr{F}]^*$.

3. $X \notin \mathscr{F}$ implies $0 \notin Z \leftarrow [\mathscr{F}]^*$.

It is worth noting that since $C(X)$ is divisible we have that $Z(f) = Z(f/2)$ for all $f \in C(X)$. In particular then, given $\mathscr{F}$ a Z-ideal and $f = |f| \in Z \leftarrow [\mathscr{F}]^*$ we have $f/2 \in Z \leftarrow [\mathscr{F}]^*$. Thus, Theorem 1 may be read with 'compatible tight Riesz order on $C(X)$' for 'proper dual-ideal of $C^+(X)$'.

Clearly we have the following relationships

$$Z[Z \leftarrow [\mathscr{F}]^*] = \mathscr{F}$$

and

$$Z \leftarrow [Z[T]]^* \supset T$$

for all Z-ideals $\mathscr{F}$ and compatible tight Riesz orders T. A compatible tight Riesz order T satisfying $Z \leftarrow [Z[T]]^* = T$ will be called an *algebraic* tight Riesz order.

If $\mathscr{F}$ is a Z-ideal then $Z \leftarrow [\mathscr{F}]^*$ is an algebraic tight Riesz order. Every maximal compatible tight Riesz order is algebraic.

Using the term adjunction in the sense of MacLane (1971), we may restate our previous results as follows:

THEOREM 2. *There is an adjunction* $Z \vdash Z \leftarrow^*$ *from the set of compatible tight Riesz orders on* $C(X)$, *ordered by inclusion, to the set of* Z-*ideals of* $Z(X)$, *ordered by inclusion, such that the algebras for this adjunction are just the algebraic tight Riesz orders.*

COROLLARY 3. *The unique minimal algebraic tight Riesz order is* $T_0 = \{f \in C(X): f(x) > 0$ *for all* $x \in X\}$.

PROOF. The unique minimal Z-ideal is $\{\square\}$ and $Z \leftarrow [\{\square\}]^* = T_0$.

COROLLARY 4. T_0 *is contained in every algebraic tight Riesz order.*

Corollaries 3 and 4 are in fact special cases of a result of Wirth (1973, Lemma 4).

THEOREM 5. *If* T *is an algebraic tight Riesz order on* $C(X)$ *there is an adjunction* $Z^T \vdash Z \leftarrow^{*T}$ *from the set of convex sublattice subgroups of* $C(X)$ *not meeting* T, *ordered by inclusion, to the set of* Z-*filters of* $Z(X)$ *not meeting* $Z[T]$, *ordered by inclusion, such that the algebras for this adjunction include the maximal tangents of* T.

PROOF. Let T be an algebraic tight Riesz order and G a convex sublattice subgroup (vector lattice ideal) of $C(X)$ not meeting T. Consider $Z[G]$.
(1) $\square \notin Z[G]$, for suppose otherwise, then there exists $f \in G$ such that $f(x) \neq 0$ for all $x \in X$ i.e. $|f| \in G \cap T_0$, a contradiction, since by Corollary 4, $T_0 \subset T$.
(2) Let $Z_1, Z_2 \in Z[G]$. Choose $f, g \in G \cap C^+(X)$ such that $Z_1 = Z(f)$ and $Z_2 = Z(g)$. Then $Z_1 \cap Z_2 = Z(f \vee g) \in Z[G]$ since G is a sublattice. Thus $Z[G]$ is a filterbase and we denote the Z-filter generated by $Z[G]$ by $Z[G]^T$, i.e. $Z[G]^T = \{Z \in Z(X): Z \supset Z(f)$ for some $f \in G\}$. Suppose $Z[G]^T$ meets $Z[T]$. Then there exists $f \in T$, $g \in G$ such that $Z(f) \supset Z(g)$. But T an algebraic tight Riesz order and $Z(f) \supset Z(g)$ imply $|g| \in T$, i.e. $|g| \in G \cap T$, a contradiction. Thus $Z[G]^T$ is a Z-filter not meeting $Z[T]$.

Conversely, let $\mathscr{F}$ be a Z-filter not meeting $Z[T]$. Consider $N = Z \leftarrow [\mathscr{F}]^*$.
(1) Given $g \in N$ and $0 \leqslant f \leqslant g$ we have $Z(f) \supset Z(g)$ whence $Z(f) \in \mathscr{F}$, i.e. N is convex.
(2) Given $f, g \in N$ we have that $f \wedge g \in N$ since $Z(f \wedge g) = Z(f) \cup Z(g) \supset Z(g) \in \mathscr{F}$ and that $f \vee g \in N$ since $Z(f \vee g) = Z(f) \cap Z(g) \in \mathscr{F}$. Thus N is a sublattice.
(3) Moreover $(N, +)$ is a subsemigroup of $C(X)$, since given $f, g \in N$ we have $Z(f + g) = Z(f) \cap Z(g) \in \mathscr{F}$.
(4) $N \cap T = \square$ for if $f \in N \cap T$ then $Z(f) \in \mathscr{F} \cap Z[T]$, a contradiction. Thus N is a convex sublattice subsemigroup not meeting T. Remembering that every directed subgroup is generated by its positive elements we have that $Z \leftarrow [\mathscr{F}]^{*T} = \{f - g: f, g \in N\}$ is a convex sublattice subgroup not meeting T.

Recall that $Z^{\leftarrow}[Z[T']]^* \supset T'$ for all dual-ideals T' of $C^+(X)$. Similarly $Z^{\leftarrow}[Z[G]]^{*T} \supset G$ for all convex sublattice subgroups of $C(X)$ not meeting T. Thus $Z^{\leftarrow}[Z[M]]^{*T} = M$ for all maximal tangents of T.

COROLLARY 6. *If T is an algebraic tight Riesz order on $C(X)$ there is a one-one correspondence between maximal tangents of T and Z-filters maximal with respect to not meeting $Z[T]$.*

The following result is due to Gillman and Jerison (1960),

THEOREM 7. *If $\mathscr{F}$ is a Z-filter, then the family $Z^{\leftarrow}[\mathscr{F}] = \{f: Z(f) \in \mathscr{F}\}$ is a ring-ideal of $C(X)$.*

Using this criterion for obtaining ring-ideals of $C(X)$ *we prove the following*:

THEOREM 8. *If T is an algebraic tight Riesz order then each maximal tangent of T is a ring-ideal of $C(X)$.*

PROOF. Let T be an algebraic tight Riesz order and let M be a maximal tangent of T. Corollary 6 tells us that $Z[M]^T = \{Z \in Z(X): Z \supset Z(f)$ for some $f \in M\}$ is a Z-filter and moreover that $Z(g) = Z(f)$ for some $f \in M$ implies $|g| \in M$.

Consider $Z^{\leftarrow}[Z[M]^T] = \{g: Z(g) \supset Z(f)$ for some $f \in M\}$. Now

$$\begin{aligned} g \in Z^{\leftarrow}[Z[M]^T] &\Rightarrow Z(|g|) \cap Z(|f|) = Z(f) \text{ for some } f \in M \\ &\Rightarrow |g| \vee |f| \in M \\ &\Rightarrow |g| \text{ whence } g \in M,\ M \text{ a convex subgroup.} \end{aligned}$$

Thus $Z^{\leftarrow}[Z[M]^T] \subset M$. The converse is trivially true, so by Theorem 7, M is a ring-ideal of $C(X)$.

Given T an algebraic tight Riesz order on $C(X)$ we define the T-*radical* of $C(X)$ to be the intersection of all the maximal tangents of T. Clearly the T-radical of $C(X)$ is a ring-ideal of $C(X)$.

The space of maximal tangents of an algebraic tight Riesz order

Let T be an algebraic tight Riesz order. The set of all maximal tangents of T is denoted by $\mathrm{Max}(T)$. Given $f \in C^+(X)$ define

$$U(f) = \{M \in \mathrm{Max}(T): f \notin M\}.$$

Then we have the following:

LEMMA 9. *$\{U(f): f \in C^+(X)\}$ is a base topology, say U, on* $\mathrm{Max}(T)$.

https://doi.org/10.1017/S1446788700014816 Published online by Cambridge University Press

PROOF. (1) $f \in T$ implies $U(f) = \text{Max}(T)$, so that $\text{Max}(T) = \cup\{U(f): f \in C^+(X)\}$.

(2) Let $M \in \text{Max}(T)$, and $f_1, f_2 \in C^+(X)$. Now $f_1 \in M$ or $f_2 \in M$ implies $f_1 \wedge f_2 \in M$ (M convex) whence $f_1 \wedge f_2 \notin M$ implies $f_1 \notin M$ and $f_2 \notin M$, i.e. $U(f_1 \wedge f_2) \subseteq U(f_1) \cap U(f_2)$. Conversely, $f_1 \wedge f_2 \in M$ implies $f_1 \in M$ or $f_2 \in M$ (M prime) whence $f_1 \notin M$ and $f_2 \notin M$ imply $f_1 \wedge f_2 \notin M$, i.e. $U(f_1) \cap U(f_2) \subset U(f_1 \wedge f_2)$.
Thus $U(f_1) \cap U(f_2) = U(f_1 \wedge f_2)$.

Similarly to (2) in the above proof we can show $U(f_1) \cup U(f_2) = U(f_1 \vee f_2)$.

PROPOSITION 10. *$(\text{Max}(T), U)$ is a T_1-space.*

PROOF. Let M_1 and M_2 be distinct members of $\text{Max}(T)$. Then there exist $f_1 \in (M_2 \cap C^+(X))\backslash M_1$ and $f_2 \in (M_1 \cap C^+(X))\backslash M_2$, i.e. $M_1 \in U(f_1)$, $M_2 \notin U(f_1)$ and $M_2 \in U(f_2)$, $M_1 \notin U(f_2)$.

PROPOSITION 11. *$(\text{Max}(T), U)$ is compact.*

PROOF. Basic closed sets being complements of basic open sets are of the form $V(f) = \{M \in \text{Max}(T): f \in M\}$, $f \in C^+(X)$. Let $\{V(f_\lambda): \lambda \in \Lambda\}$ be a collection of basic closed sets with the finite intersection property, i.e. $V(f_{\lambda_1}) \cap \cdots \cap V(f_{\lambda_n}) \neq \Box$ for all finite subsets $\{\lambda_1, \cdots, \lambda_n\}$ of Λ.

Consider $I \cdots$ the ideal generated by $\{f_\lambda\}_{\lambda \in \Lambda}$. I does not meet T, for if so there exists $g \in T$ such that $g \leqslant f_{\lambda_1} \vee \cdots \vee f_{\lambda_n}$ for some $\lambda_1, \cdots, \lambda_n \in \Lambda$. But then $f_{\lambda_1} \vee \cdots \vee f_{\lambda_n} \in T$ so that $V(f_{\lambda_1} \vee \cdots \vee f_{\lambda_n}) = \Box$. However $V(f_{\lambda_1} \vee \cdots \vee f_{\lambda_n}) = V(f_{\lambda_1}) \cap \cdots \cap V(f_{\lambda_n}) \neq \Box$ so that I is a proper l-ideal containing $\{f_\lambda\}_{\lambda \in \Lambda}$ and not meeting T.

Suppose $\bigcap_{\lambda \in \Lambda} V(f_\lambda) = \Box$. This says that there exists no l-ideal containing all the f_λ's and not meeting T. This however is clearly false, since I meets all these requirements. Thus $\bigcap_{\lambda \in \Lambda} V(f_\lambda) \neq \Box$, whence $(\text{Max}(T), U)$ is compact.

PROPOSITION 12. *Let T and T' be algebraic tight Riesz orders such that $T \subseteq T'$. Then each maximal tangent of T' is contained in a unique maximal tangent of T.*

PROOF. Let $M' \in \text{Max}(T')$. Then M' is a prime subgroup not meeting T. Since the class of convex sublattice subgroups lying above a prime subgroup is totally-ordered by inclusion [Holland (1963)] we have that M' is contained in a unique maximal tangent M of T.

THEOREM 13. *Let T and T' be algebraic tight Riesz orders such that $T \subseteq T'$. If $(\text{Max}(T'), U')$ is Hausdorff then the map $m: (\text{Max}(T'), U') \to (\text{Max}(T), U)$*

given by $m(M') = M$ *— the unique maximal tangent of* T *containing* M' *— is continuous.*

PROOF. Given a basic open set $U(f) = \{M \in \operatorname{Max}(T): f \notin M\}$ we want to see that $S = \{M' \in \operatorname{Max}(T'): m(M') \in U(f)\}$ is open in $(\operatorname{Max}(T'), U')$, and we do so by seeing that $\operatorname{Max}(T')\backslash S$ is compact.

Let $\{\operatorname{Max}(T')\backslash U'(f_\lambda) \cap \operatorname{Max}(T')\backslash S : \lambda \in \Lambda\}$ be a collection of basic closed subsets of $\operatorname{Max}(T')\backslash S$ with the finite intersection property, where $U'(f_\lambda) = \{M' \in \operatorname{Max}(T'): f_\lambda \notin M'\}$ for $f_\lambda \in C^+(X)$. Then for each finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of Λ we have

$$S \cup U'(f_{\lambda_1}) \cup \cdots \cup U'(f_{\lambda_n}) \neq \operatorname{Max}(T') \tag{1}$$

If $f \vee f_{\lambda_1} \vee \cdots \vee f_{\lambda_n} \in T$ for some finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of Λ then $U(f \vee f_{\lambda_1} \vee \cdots \vee f_{\lambda_n}) = U(f) \cup U(f_{\lambda_1}) \cup \cdots \cup U(f_{\lambda_n}) = \operatorname{Max}(T)$. By assumption (equation (1)) there exists $M' \in \operatorname{Max}(T')$ such that $M' \notin S$ and $M' \notin U'(f_{\lambda_i})$, $i = 1, 2, \cdots, n$. Then $m(M') \notin U(f)$ so that $m(M') \in U(f_{\lambda_i})$ for some i, which implies $M' \in U'(f_{\lambda_i})$ — a contradiction. Thus, for every finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of Λ, $f \vee f_{\lambda_1} \vee \cdots \vee f_{\lambda_n} \notin T$. Similarly, for every finite subset $\{\lambda_1, \cdots, \lambda_n\}$ of Λ, $f_{\lambda_1} \vee \cdots \vee f_{\lambda_n} \notin T'$. Hence, since by both $(\operatorname{Max}(T), U)$ and $(\operatorname{Max}(T'), U')$ are compact (Proposition 11), there is an $M' \in \operatorname{Max}(T')$ containing all $f_\lambda : \lambda \in \Lambda$, and an $M \in \operatorname{Max}(T)$ containing M' and f. Then $m(M') = M$, so we have $M' \notin S$ and $M' \notin U'(f_\lambda)$, for all $\lambda \in \Lambda$, i.e. $\operatorname{Max}(T')\backslash S$ is compact.

In proving the above theorem we made use of the fact that $(\operatorname{Max}(T'), U')$ was Hausdorff. We now consider necessary and sufficient conditions for such a space to be Hausdorff.

PROPOSITION 14. $(\operatorname{Max}(T), U)$ *is Hausdorff if and only if given* M_1 *and* M_2 *distinct members of* $\operatorname{Max}(T)$ *there exist* $f_1 \in C^+(X)\backslash M_1$ *and* $f_2 \in C^+(X)\backslash M_2$ *such that* $f_1 \wedge f_2 \in$ *T-radical of* $C(X) = \cap\{M : M \in \operatorname{Max}(T)\}$.
The proof is obvious.

THEOREM 15. M_1 *and* M_2 *distinct members of* $\operatorname{Max}(T)$ *can be Hausdorff separated for* U *if either* M_1 *or* M_2 *is minimal prime.*

PROOF. M_1 and M_2 distinct implies that there exist $f_1 \in (M_2 \cap C^+(X))\backslash M_1$ and $f_2 \in (M_1 \cap C^+(X))\backslash M_2$. Suppose M_2 is minimal prime. Then there exists $f \in C^+(X)\backslash M_2$ such that $f_1 \wedge f = 0$. Moreover $f \notin M_2$ implies $f \wedge f_2 \in M_1\backslash M_2 \cap C^+(X)$ (primality). Thus $U(f_1) \cap U(f \wedge f_2) = U(f_1 \wedge f \wedge f_2) = U(0) = \square$. A similar argument holds if M_1 is minimal prime.

THEOREM 16. *If* T *is dual-prime then* $(\operatorname{Max}(T), U)$ *is a singleton.*

PROOF. Let $M \in \operatorname{Max}(T)$. Then

$$f \notin M \Leftrightarrow |f| \vee |g| \in T \text{ for some } g \in M$$
$$\Leftrightarrow |f| \in T \qquad (T \text{ dual-prime})$$

i.e. there is but one maximal tangent of T.

The quotient space $C(X)/A$

Thoughout this section we assume X to be a compact Hausdorff space and T to be an algebraic tight Riesz order on $C(X)$. We denote the T-radical of $C(X)$ by A i.e. $A = \cap\{M : M \in \operatorname{Max}(T)\}$.

As a result of Proposition 14, we see that A plays an important role in determining whether or not $(\operatorname{Max}(T), U)$ is Hausdorff. For this reason we make a brief study of A and consequently of the quotient space $C(X)/A$.

Being an intersection of maximal tangents of T, A is a tangent — hence an l-ideal (not necessarily prime), and so we may consider $C(X)/A$ as the factor group of $C(X)$ with respect to the l-ideal A. Then $C(X)/A$ is a lattice-ordered group and the canonical mapping ρ: $C(X) \to C(X)/A$ preserves the order relation and lattice operations, (Fuchs (1963)). We use the same symbol $\preccurlyeq$ to denote the lattice-order in both $C(X)$ and $C(X)/A$, and we denote $\{f + A : 0 + A \preccurlyeq f + A\}$ by $C(X)/A^+$, where we have $0 + A \preccurlyeq f + A$ if and only if $0 \preccurlyeq f + a$ for some $a \in A$.

We consider the action of the canonical mapping ρ: $C(X) \to C(X)/A$.

THEOREM 17. *ρT is a compatible tight Riesz order on $C(X)/A$.*

PROOF. This follows immediately from Theorem 8° of Miller (1973).

THEOREM 18. *Let M be a maximal tangent of T then ρM is a maximal tangent of ρT.*

PROOF. Put $M' = \rho M = M + A$, $T' = \rho T = T + A$. Then

(1) Since ρ preserves the order relation and lattice operations we have immediately that M' is a convex sublattice of $C(X)/A$.

(2) M' is non-empty since M is non-empty. Moreover, it is straightforward to show that M' is closed under addition and that each element in M' has an additive inverse in M'. Thus M' is a subgroup of $C(X)/A$.

(3) Suppose $f + A \in M' \cap T'$. Now $f + A \in M + A$ implies $f + A = m + A$ for some $m \in M$ i.e. $f - m \in A \subset M$ for some $m \in M$. In other words $f \in M$. Similarly $f + A \in T + A$ implies $f - t \in M$ for some $t \in T$. Then $f - (f - t) = t \in M$ for some $t \in T$, a contradiction. Thus $M' \cap T' = \square$.

(4) Suppose $M' \subset N'$ where N' is a convex sublattice subgroup of $C(X)/A$ not meeting T'. Put $N = \{f : f + A \in N'\}$. Then N is a convex sublattice subgroup of

https://doi.org/10.1017/S1446788700014816 Published online by Cambridge University Press

$C(X)$ not meeting T. Moreover $M \subset N$. Thus $M = N$ and $M' = N'$ since M is a maximal tangent of T.

In other words, M' is a maximal tangent of the compatible tight Riesz order T'.

COROLLARY 19. *Let* $\mathcal{M}$ *be the set of maximal tangents of* ρT, *then* $\cap \{M; M \in \mathcal{M}\} = 0 + A$.

PROOF. This follows since ρ preserves intersections.

References

L. Fuchs (1963), *Partially Ordered Algebraic Systems* (Pergamon Press, Oxford, 1963).

L. Gillman and M. Jerison (1960), *Rings of Continuous Functions* (Van Nostrand, Princeton, 1960).

C. Holland (1963), 'The lattice-ordered group of automorphisms of an ordered set', *Mich. Math J.* **10,** 399–408.

Saunders Maclane (1971), *Categories for the Working Mathematician* (Springer-Verlag, New York, 1971).

J. B. Miller (1973), 'Quotient groups and realization of tight Riesz groups', *J. Austral. Math. Soc.* **16,** 416–430.

A. Wirth (1973), 'Compatible tight Riesz orders', *J. Austral. Math Soc.* **15,** 105–111.

Department of Mathematics,
La Trobe University,
Victoria 3083,
Australia.