ZEROS AND PERIODICITY OF FUNCTIONS OF INFINITE MATRICES

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1. Introduction.

Certain functions of infinite matrices are known to exist.† This gives rise to the following questions :

1. Whether the power series of matrices

$$f(A)=\sum_{r=0}^{\infty} a_r A^r$$

has a zero in the field ‡ of infinite matrices, and

2. If $f(A)$ exists for a certain infinite matrix A, is there an infinite matrix B such that

$$f(A+B)=f(A)\ ?.$$

In other words, is there a *matrix period* for $f(A)$?

In this paper theorems concerning zeros and periodicity of functions of *semi block infinite matrices* § (defined below) are established.

2. Definitions

SEMI BLOCK INFINITE MATRICES

Let S be the set of square matrices, including 1×1 matrices, and $\{S_r\}$ a sequence of matrices of S of orders w_r, $r=0, 1, 2, \ldots$.

Let A be an infinite matrix formed from the sequence $\{S_r\}$ arranged along its leading diagonal, and from arbitrary elements to the left, while all elements to the right of the S_r are zero. We shall call such a matrix a *lower semi block* matrix.||

The leading minors of A of orders w_0, w_0+w_1, $w_0+w_1+w_2$, ... form a sequence of square matrices A_r each containing the previous members of the sequence, and having the property that all elements to the right of A_r are zero. Given any element a_{ij} of A, the first matrix in the sequence A_r which contains a_{ij} will be called the *carrier* of a_{ij}, and will be denoted by $A(i, j)$. The latent roots of A_r are those of $S_0, S_1, \ldots, S_r$, since

$$\det(A_r-\lambda I)=\prod_{i=0}^{r} \det(S_i-\lambda I).$$

We shall call every latent root of every matrix S_r a *scalar root* of A ; the set of all latent roots λ_i of all the S_r form the set of scalar roots of A.

† See, e.g., Cooke (1), 14 ; see also *ibid.*, 38, Ex. 18, (ii), 270, Ex. 4, and Ibrahim (2).

‡ " Field " is not here used in the usual algebraic sense ; see Cooke (1), p. 26, footnote.

§ For further results about semi block infinite matrices, see Ibrahim (2) or (3).

|| I am indebted to Dr P. Vermes for putting the definition in the present form, which is much shorter and clearer than mine.

Upper semi block matrices are analogously defined. A lower semi block matrix A will be of the form

$$A=\left[\begin{array}{c|cc|c|c} S_0 & 0 & 0 & 0 & \dots \\ & 0 & 0 & 0 & \dots \\ \hline \cdot & & & 0 & \dots \\ & & & 0 & \dots \\ \cdot & & S_1 & 0 & \dots \\ & & & 0 & \dots \\ \hline \cdot & \cdot & \cdot & S_2 & \dots \\ \cdot & \cdot & \cdot & \cdot & \end{array}\right] = \left[\begin{array}{cccc|c|c} & & A_{(i,j)} & & 0 & \dots \\ & & & & 0 & \dots \\ & & a_{ij} & & 0 & \dots \\ & & & & 0 & \dots \\ \hline \cdot & \cdot & \cdot & \cdot & S_t & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{array}\right],$$

with arbitrary elements to the left of the S_r, giving

$$A^r=\left[\begin{array}{cccc|c|c} & & & & 0 & \dots \\ & & A^r_{(i,j)} & & 0 & \dots \\ & & & & 0 & \dots \\ & & a^{(r)}_{ij} & & 0 & \dots \\ \hline \cdot & \cdot & \cdot & & S^r_t & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{array}\right], \quad A^{-1}=\left[\begin{array}{cccc|c|c} & & & & 0 & \dots \\ & & A^{-1}_{(i,j)} & & 0 & \dots \\ & & & & 0 & \dots \\ & & a^{-1}_{ij} & & 0 & \dots \\ \hline \cdot & \cdot & \cdot & & S^{-1}_t & \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \dots \end{array}\right],$$

and generally for a function $f(A)$ of A,

$$[f(A)]_{k,l}=[f\{A(k,l)\}]_{k,l}.$$

Lemma 1. *If A is a lower (or upper) semi block matrix, then for every positive integer or zero (with $A^\circ=I$),*

$$(A^r)_{k,l}=\sum_{i=0}^{s}\left(A_{i,0}+A_{i,1}\frac{d}{d\lambda_i}+\dots+A_{i,l_i-1}\frac{d^{l_i-1}}{d\lambda_i^{l_i-1}}\right)_{k,l}\lambda_i^r,$$

where $\lambda_i(i=0, 1, 2, \dots, s)$ are the latent roots of the matrix $A(k, l)$ the carrier of $a_{k,l}$, and λ_i is repeated l_i times, so that $\sum_{i=0}^{s} l_i=n$, the order of the carrier $A(k,l)$. The

$$A_{i,j}(i=0, 1, 2, \dots, s\,;\, j=0, 1, \dots, l_i-1)$$

are polynomials in $A(k, l)$ of degrees $\leqslant n-1$.

Proof. As shown above, we have

$$A^r=\left[\begin{array}{ccc|c|c} & A^r_{(k,l)} & & 0 & \dots \\ & & & 0 & \dots \\ & a^{(r)}_{k,l} & & 0 & \dots \\ \hline \cdot & \cdot & \cdot & S^r_t & \dots \\ \cdot & \cdot & \cdot & & \dots \\ \cdot & \cdot & \cdot & \cdot & \dots \end{array}\right],$$

and since the carrier $A(k, l)$ satisfies the Hamilton-Cayley equation, the result follows immediately by a simple transformation of Wedderburn's exposition.†

† See Wedderburn (4), 25-30 ; see also Cooke (1), 13.

Remark. It is to be noted that the equation

$$A^r = \sum_{i=0}^{s} \left(A_{i,0} + A_{i,1} \frac{d}{d\lambda_i} + \ldots + A_{i,l_i-1} \frac{d^{l_i-1}}{d\lambda_i^{l_i-1}} \right) \lambda_i^r$$

without the suffixes k, l on both sides has no meaning, since the left-hand side is an infinite matrix, whereas the right-hand side is a sum of finite square matrices.

Lemma 2. *The exponential function exists for every lower (or upper) semi block matrix, and in particular for every lower semi-matrix.*†

Proof. By Lemma 1, and with its notation, we have

$$\begin{aligned}(e^A)_{k,l} &= \left[\sum_{r=0}^{\infty} \frac{A^r}{r!} \right]_{k,l} = \left[\sum_{r=0}^{\infty} \frac{A^r(k, l)}{r!} \right]_{k,l} \\ &= \sum_{i=0}^{s} \left(A_{i,0} + A_{i,1} \frac{d}{d\lambda_i} + \ldots + A_{i,l_i-1} \frac{d^{l_i-1}}{d\lambda_i^{l_i-1}} \right)_{k,l} \sum_{r=0}^{\infty} \frac{\lambda_i^r}{r!} \\ &= \sum_{i=0}^{s} (A_i)_{k,l} e^{\lambda_i},\end{aligned}$$

where the A_i are polynomials in $A(k, l)$ of degrees $\leqslant n-1$; i.e. e^A exists for every semi block matrix A, and hence for the special case, namely, every lower semi-matrix.

(a) *Zeros of functions of infinite matrices*

Theorem 1. *Let A be a semi block infinite matrix with scalar roots*

$$\lambda_i (i=0, 1, 2, \ldots),$$

and let l_i be the multiplicity of λ_i.

Let $f(z) = \sum_{r=0}^{\infty} a_r z^r$ be convergent in a circle D; then $f(A)=0$ (the zero matrix) if the λ_i are zeros of $f(z)$ with the same multiplicity l_i, and the λ_i are all in D.

Proof. Since λ_i is a zero of $f(z)$ of order l_i, we have

$$f(z) = (z - \lambda_i)^{l_i} \phi(z),$$

where $\phi(z)$ is analytic in the neighbourhood of $z = \lambda_i$; also $\phi(\lambda_i) \neq 0$. This shows that

$$f'(\lambda_i) = f''(\lambda_i) = \ldots = f^{l_i-1}(\lambda_i) = 0.$$

Now, by Lemma 1, and with its notation, we have

$$\begin{aligned}[f(A)]_{k,l} &= \left[\sum_{r=0}^{\infty} a_r A^r \right]_{k,l} \\ &= \left[\sum_{r=0}^{\infty} a_r A^r(k, l) \right]_{k,l} \\ &= \sum_{i=0}^{s} \left[A_{i,0} + A_{i,1} \frac{d}{d\lambda_i} + \ldots + A_{i,l_i-1} \frac{d^{l_i-1}}{d\lambda_i^{l_i-1}} \right]_{k,l} \sum_{r=0}^{\infty} a_r \lambda_i^r \\ &= \sum_{i=0}^{s} \left[A_{i,0} f(\lambda_i) + A_{i,1} f'(\lambda_i) + \ldots + A_{i,l_i-1} f^{l_i-1}_{(\lambda i)} \right]_{k,l} \\ &= 0.\end{aligned}$$

This proves the theorem.

† Compare Dienes' result, Cooke (1), 14, (1.7, I).

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Corollary. *In particular, if the* λ_i *are of multiplicity* 1, *then* $f(A)=0$ *if the* λ_i *are simple zeros of* $f(z)$.

For example, the semi block infinite matrix A, whose scalar roots λ_1 are all distinct and are multiples of 2π is such that $\sin A=0$ and $\cos A=I$.

It is clear that if A is a semi block infinite matrix, then, by definition,

$$\cos A=\sum_{r=0}^{\infty}(-1)^r\frac{A^{2r}}{2r!},\ \sin A=\sum_{r=0}^{\infty}(-1)^r\frac{A^{2r+1}}{(2r+1)!},\ \text{and } e^A=\sum_{r=0}^{\infty}\frac{A^r}{r!};$$

and we easily see, as in Lemma 2, that they all exist for every semi block infinite matrix. Also

$$e^{iA}=\cos A+i\sin A \text{ and } e^{-iA}=\cos A-i\sin A,$$

i.e.,

$$\sin A=\frac{1}{2i}(e^{iA}-e^{-iA}),\ \text{and } \cos A=\frac{1}{2}(e^{iA}+e^{-iA}).$$

Again, $e^A.e^B=e^{A+B}$, where B is another semi block infinite matrix which commutes with A.†

Hence

$$\begin{aligned}\sin^2 A+\cos^2 A&=-\tfrac{1}{4}(e^{2iA}-2I+e^{-2iA})+\tfrac{1}{4}(e^{2iA}+2I+e^{-2iA})\\&=I.\end{aligned}$$

(b) *Periodic Functions of Infinite Matrices*

Definition. The function $f(A)$ of the infinite matrix A is said to be *periodic* if there exists an infinite matrix B such that

$$f(A+B)=f(A).$$

The infinite matrix B is said to be the *matrix period* of $f(A)$.

Theorem 2. *The matrix iB, where B is a real semi block matrix whose scalar roots are all distinct and are multiples of* 2π *is a period of the function* e^A, *where A is another semi block matrix which commutes with B.*

Proof. Since A commutes with B, we have

$$e^{A+iB}=e^A.e^{iB}=e^A(\cos B+i\sin B).$$

But $\sin B=0$ and $\cos B=I$, as shown above. Therefore $e^{iB}=I$, and hence

$$e^{A+iB}=e^A.I=e^A,$$

which shows that iB is a period of the function e^A.

Corollary. *B, as defined in Theorem* 2, *is a matrix period of* sin A *and* cos A.

For, $e^{-iB}=\cos B-i\sin B=I$ and

$$\begin{aligned}\sin(A+B)&=\frac{1}{2i}(e^{iA+iB}-e^{-iA-iB})\\&=\frac{1}{2i}(e^{iA}.e^{iB}-e^{-iA}.e^{-iB})\\&=\frac{1}{2i}(e^{iA}-e^{-iA})=\sin A,\end{aligned}$$

and similarly for $\cos A$.

† See Cooke (1), 14.

Finally, we remark that certain elliptic functions can be shown to exist for semi block infinite matrices.

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REFERENCES

(1) R. G. Cooke, *Infinite Matrices and Sequence Spaces* (Macmillan), 1950.

(2) S. A. Ibrahim (Ph.D. Thesis, University of London, 1955).

(3) S. A. Ibrahim, Structure of infinite matrices, *J. London Math. Soc.*, **34** (1959), 281-288.

(4) J. H. M. Wedderburn, *Lectures on Matrices*, Amer. M.S. Colloquium Publications xvii (New York), 1934.

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