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ASYMPTOTICALLY LINEAR ELLIPTIC SYSTEMS WITH PARAMETERS

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Abstract. In this paper, we show that the semi-linear elliptic systems of the form

$$\begin{cases} -\Delta u - \mu\Delta v = g(x, v), \quad -\Delta v - \lambda\Delta u = f(x, u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega \end{cases} \tag{0.1}$$

possess at least one non-trivial solution pair $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, λ and μ are non-negative numbers, $f(x, t)$ and $g(x, t)$ are continuous functions on $\Omega \times \mathbb{R}$ and asymptotically linear at infinity.

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1. Introduction. In this paper, we consider the existence of non-trivial solutions of non-linear elliptic systems

$$\begin{cases} -\Delta u - \mu\Delta v = g(x, v), \quad -\Delta v - \lambda\Delta u = f(x, u), & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, λ and μ are non-negative numbers, $f(x, t)$ and $g(x, t)$ are continuous functions on $\Omega \times \mathbb{R}$ and asymptotically linear at infinity for t.

In the case of $\lambda = \mu = 0$, in recent years, much attention has been paid to the existence of non-trivial solutions of problem (1.1) for the case that f and g are superlinear, see **[1]**, **[2]**, **[3]**, **[7]** and references therein. In **[4]**, G. Li and J.Yang considered the asymptotically linear elliptic systems

$$-\Delta u + u = g(x, v), \quad -\Delta v + v = f(x, u), x \in \mathbb{R}^N;$$

it obtained a positive solution by using linking theorem under the Cerami compactness condition.

If $\lambda, \mu \neq 0$, the problem has some new features. First, by the Pohozaev-type identity, the parameters λ and μ affect the sub-critical range of the growth of non-linear terms at infinity. Second, if $\lambda\mu < 1$, the decomposition of the space in the framework involves the parameters, see **[5, 6]**. Moreover, f and g are superlinear in **[5]** and are asymptotically linear in **[6]**.

In this paper, we will consider asymptotically linear elliptic systems (1.1) in $E = H_0^1(\Omega) \times H_0^1(\Omega)$ with parameters λ, μ satisfies $\lambda\mu > 1$, which allow us to define an equivalent norm on E. In fact, let E be equipped with the norm

$$\|z\|_E = \left(\int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) dx \right)^{\frac{1}{2}},$$

where $z = (u, v)$. Since $\lambda\mu > 1$, then there exists a real number $l > 0$ such that $\lambda > l > \frac{1}{\mu}$ and we have

$$\max\left\{\frac{1+\lambda}{2}, \frac{1+\mu}{2}\right\}(|\nabla u|^2 + |\nabla v|^2) \geq \nabla u \nabla v + \frac{\lambda}{2}|\nabla u|^2 + \frac{\mu}{2}|\nabla v|^2$$
$$\geq \min\left\{\frac{\lambda - l}{2}, \frac{\mu}{2} - \frac{1}{2l}\right\}(|\nabla u|^2 + |\nabla v|^2). \quad (1.2)$$

Then we may introduce a new inner product on E by the formula

$$\langle (u, v), (\varphi, \psi) \rangle = \int_\Omega (\lambda \nabla u \nabla \varphi + \nabla u \nabla \psi + \nabla v \nabla \varphi + \mu \nabla v \nabla \psi)\, dx, \quad (1.3)$$

and the corresponding norm is

$$\|z\| = (\langle z, z \rangle)^{\frac{1}{2}} = \left(\int_\Omega (\lambda |\nabla u|^2 + 2\nabla u \nabla v + \mu |\nabla v|^2)\, dx \right)^{\frac{1}{2}}, \ \forall z = (u, v) \in E. \quad (1.4)$$

The norms $\|\cdot\|$ and $\|\cdot\|_E$ are then equivalent if $\lambda\mu > 1$ by (1.2).

We assume that f and g satisfy

$(H1)$ $f, g \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$, $f(x, t) = g(x, t) = 0$ if $t \leq 0$.

$(H2)$ $\lim_{t\to 0} (f(x, t)/t) = \lim_{t\to 0} (g(x, t)/t) = 0$ uniformly with respect to $x \in \Omega$ and $f(x, t) > 0$, $g(x, t) > 0$ for $t > 0$, $x \in \Omega$.

$(H3)$ $\lim_{t\to\infty} (f(x, t)/t) = l > 0$, $\lim_{t\to\infty} (g(x, t)/t) = m > 0$ uniformly in $x \in \Omega$.

$(H4)$ $f(x, t)/t$ and $g(x, t)/t$ are non-decreasing in $t \geq 0$ for $x \in \Omega$.

Let λ_1 be the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ and $\varphi_1 > 0$ be the corresponding eigenfunction. Define $A = \min\{\frac{l}{1+\lambda}, \frac{m}{1+\mu}, \frac{m\lambda + \mu l - \sqrt{(m\lambda - \mu l)^2 + 4ml}}{2(\lambda\mu - 1)}\}$.

The main result of this paper is as follows:

THEOREM 1.1. *Suppose $(H1) - (H4)$ hold. If $\lambda\mu > 1$ and $\lambda_1 < A$, then the problem (1.1) possesses at least one non-trivial solution pair $z = (u, v) \in E$. Furthermore, problem (1.1) possesses the least energy non-trivial solution pair $z = (u, v) \in E$.*

We will use Mountain Pass theorem to prove Theorem 1.1. As a by-product, we show that

$$I^\infty = \inf\left\{ I(z) : \ I'(z) = 0, \ z = (u, v) \in E \backslash \{0\} \right\}$$

is achieved by some $z_0 = (u_0, v_0)$ with $u_0 \not\equiv 0$, $v_0 \not\equiv 0$.

Theorem 1.1 will be proved in Section 2.

2. Existence results. Suppose in this section λ, μ satisfies $\lambda\mu > 1$ and $\lambda_1 < A$. By $(H_1) - (H_3)$, it is easy to see that there is a $2 < p < 2N/(N-2)$ if $N > 2$ and $2 < p < +\infty$ if $N \leq 2$ and that for any $\epsilon > 0$ there is a $c_\epsilon > 0$ such that for $\forall (x, t) \in \Omega \times \mathbb{R}$,

$$|f(x, t)|, |g(x, t)| \leq \epsilon |t| + c_\epsilon |t|^{p-1}. \quad (2.1)$$

So the corresponding energy function

$$I(u, v) = \frac{1}{2}\|z\|^2 - \int_\Omega F(x, u)\,dx - \int_\Omega G(x, v)\,dx \tag{2.2}$$

is well defined on E and class $C^1(E, \mathbb{R})$, where $F(x, t) = \int_0^t f(x, s)\,ds$ and $G(x, t) = \int_0^t g(x, s)\,ds$. Moreover, the Fréchet derivative I' satisfying

$$\begin{aligned}\langle I'(u, v), (\varphi, \psi)\rangle = &\int_\Omega [\nabla u \nabla \psi + \nabla v \nabla \varphi + \lambda \nabla u \nabla \varphi + \mu \nabla v \nabla \psi]\,dx \\ &- \int_\Omega f(x, u)\varphi\,dx - \int_\Omega g(x, v)\psi\,dx \end{aligned}\tag{2.3}$$

for $\forall(\varphi, \psi) \in E$.

Sequence $\{z_n\} \subset E$ is called the Palais–Smale sequence of a C^1 function I on E at level c ($(PS)_c$-sequence for short) if $I(z_n) \to c$ and $I'(z_n) \to 0$ as $n \to \infty$. To get a $(PS)_c$-sequence, we will use the Mountain Pass theorem cited in [**8**].

PROPOSITION 2.1. *Let E be a Hilbert space, $I \in C^1(E, \mathbb{R})$, $e \in E$ and $r > 0$ such that $\|e\| > r$ and $b := \inf_{\|z\|=r} I(z) > I(0) \geq I(e)$. Let c be characterised by $c := \inf_{\gamma\in\Gamma} \max_{\tau\in[0,1]} I(\gamma(\tau))$, where $\Gamma := \{\gamma \in C([0, 1], E) : \gamma(0) = 0,\ \gamma(1) = e\}$. Then, there exists a sequence $\{z_n\} \subset E$ such that $I(z_n) \to c$ and $I'(z_n) \to 0$ as $n \to \infty$.*

LEMMA 2.1. *Let $(H_1) - (H_3)$ hold. Then we have the following:*
(a) There exist $\rho, \beta > 0$ such that $I(z) \geq \beta$ for all $z \in E$ with $\|z\| = \rho$.
(b) There exists $e \in E$ with $\|e\| \geq \beta$ such that $I(e) < 0$.

Proof. (a) It follows from (2.1) and the Sobolev embedding theorem that for any $\epsilon > 0$ there is a $c_\epsilon > 0$ such that

$$\int_\Omega F(x, u)\,dx + \int_\Omega G(x, v)\,dx \leq c\epsilon\|z\|^2 + c_\epsilon\|z\|^p$$

for all $z = (u, v) \in E$. This, jointly with (2.2) implies (a).

(b) By Fatou's Lemma, we have

$$\begin{aligned}\lim_{t\to\infty} \frac{I(t\varphi_1, t\varphi_1)}{t^2} &= \frac{1}{2}\int_\Omega (2 + \lambda + \mu)|\nabla\varphi_1|^2\,dx - \lim_{t\to\infty}\int_\Omega \frac{F(x, t\varphi_1) + G(x, t\varphi_1)}{t^2}\,dx \\ &\leq \frac{1}{2}\int_\Omega (2 + \lambda + \mu)|\nabla\varphi_1|^2\,dx - \int_\Omega \lim_{t\to\infty} \frac{F(x, t\varphi_1) + G(x, t\varphi_1)}{t^2\varphi_1^2}\varphi_1^2\,dx \\ &= \frac{1}{2}\int_\Omega (2 + \lambda + \mu)|\nabla\varphi_1|^2\,dx - \frac{1}{2}\int_\Omega (l + m)\varphi_1^2\,dx \\ &= \frac{1}{2}\left(2 + \lambda + \mu - \frac{l + m}{\lambda_1}\right)\int_\Omega |\nabla\varphi_1|^2\,dx < 0\end{aligned}$$

because of $\lambda_1 < A$. So $I(t\varphi_1, t\varphi_1) \to -\infty$ as $t \to \infty$ and part (b) is proved. □

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PROPOSITION 2.2. *If* $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ *is a non-trivial solution of (1.1), then we have* $\lambda_1 \leq \frac{m\lambda+\mu l-\sqrt{(m\lambda-\mu l)^2+4ml}}{2(\lambda\mu-1)}$.

Proof. Let $k = \frac{\mu l-m\lambda+\sqrt{(m\lambda-\mu l)^2+4ml}}{2m}$. It is apparent that $(u, v) = (u, k\tilde{v})$ is a non-trivial solution pair of the problem

$$\begin{cases} -\Delta u - \mu k \Delta \tilde{v} = g(x, k\tilde{v}), \quad -\Delta \tilde{v} - \frac{\lambda}{k}\Delta u = \frac{1}{k} f(x, u), & x \in \Omega, \\ u = \tilde{v} = 0, & x \in \partial\Omega, \end{cases}$$

that is

$$-\left(1 + \frac{\lambda}{k}\right) \Delta \left(u + \frac{1+\mu k}{1+\frac{\lambda}{k}}\tilde{v}\right) = g\,(x, k\tilde{v}) + \frac{1}{k} f(x, u).$$

By (H_3) and (H_4), we have

$$\begin{aligned} \left(1 + \frac{\lambda}{k}\right) \int_\Omega |\nabla \left(u + \frac{1+\mu k}{1+\frac{\lambda}{k}}\tilde{v}\right)|^2 \, dx &= \int_\Omega \left[g(x, k\tilde{v}) + \frac{1}{k} f(x, u)\right] \left(u + \frac{1+\mu k}{1+\frac{\lambda}{k}}\tilde{v}\right) dx \\ &\leq \int_\Omega \left[mk\tilde{v} + \frac{l}{k}u\right] \left(u + \frac{1+\mu k}{1+\frac{\lambda}{k}}\tilde{v}\right) dx \\ &= \frac{l}{k} \int_\Omega \left(u + \frac{mk^2}{l}\tilde{v}\right) \left(u + \frac{1+\mu k}{1+\frac{\lambda}{k}}\tilde{v}\right) dx. \end{aligned}$$

By the definition of k we know that $\frac{1+\mu k}{1+\frac{\lambda}{k}} = \frac{mk^2}{l}$, and hence

$$\lambda_1 \leq \frac{\frac{l}{k}}{1+\frac{\lambda}{k}} = \frac{l}{k+\lambda} = \frac{m\lambda + \mu l - \sqrt{(m\lambda - \mu l)^2 + 4ml}}{2(\lambda\mu - 1)}.$$

The proof is complete. □

PROPOSITION 2.3. *Under assumptions* $(H_1) - (H_4)$, *problem (1.1) possesses at least one non-trivial solution pair* $(u, v) \in E$.

Proof. Proposition 2.1 and Lemma 2.1 implies that there exists a $(PS)_c$−sequence $\{z_n\} \subset E$ for I, that is

$$I(z_n) \to c, \quad I'(z_n) \to 0, \tag{2.4}$$

where $c > 0$. To get a non-trivial solution, we only need to show that $\{z_n\}$ is bounded in E. For this purpose, we suppose, by contradiction, that $\|z_n\| \to \infty$ as $n \to \infty$ and let

$$t_n = \frac{2\sqrt{c}}{\|z_n\|}, \quad w_n = t_n z_n = \frac{2\sqrt{c}z_n}{\|z_n\|} = \left(\frac{2\sqrt{c}u_n}{\|z_n\|}, \frac{2\sqrt{c}v_n}{\|z_n\|}\right) \triangleq \left(w_n^1, w_n^2\right). \tag{2.5}$$

Obviously, $\{w_n\}$ is bounded in E. By extracting a sub-sequence, we may suppose that

$$w_n \rightharpoonup w \ \in E, \quad w_n \to w \ a.e. \text{ in } \Omega$$

as $n \to \infty$, where $w = (w_1, w_2)$.

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We claim that

$$w \not\equiv 0.$$

In fact, by $(H_2)-(H_4)$, we see that there exists $M > 0$ such that $|f(x,t)/t| \le M$, $|g(x,t)/t| \le M$ for all $x \in \Omega$ and $t \ge 0$. Supposing $w \equiv 0$, by Sobolev embedding theorem that, $w_n^1 \to 0$, $w_n^2 \to 0$ in $L^2(\Omega)$, as $n \to \infty$. Then it follows from (2.4) and (2.5) that

$$\begin{aligned} 4c &= \int_\Omega \left[\frac{f(x,u_n)}{u_n}|w_n^1|^2 + \frac{g(x,v_n)}{v_n}|w_n^2|^2\right] dx + o(1) \\ &\le M\int_\Omega \left[|w_n^1|^2 + |w_n^2|^2\right] dx + o(1) \to 0 \end{aligned}$$

as $n \to \infty$, which is impossible as $c > 0$. Hence, the claim is proved.

Set

$$p_n(x) = \begin{cases} \frac{f(x,u_n)}{u_n} & \text{if } u_n(x) > 0; \\ 0 & \text{if } u_n(x) \le 0, \end{cases} \qquad q_n(x) = \begin{cases} \frac{g(x,v_n)}{v_n} & \text{if } v_n(x) > 0; \\ 0 & \text{if } v_n(x) \le 0. \end{cases}$$

By $(H_2)-(H_4)$, we see that

$$0 \le p_n(x) \le l, \quad 0 \le q_n(x) \le m, \quad \forall x \in \Omega,$$

and there exist two functions $p(x),\ q(x) \in L^\infty(\Omega)$ such that

$$p_n \rightharpoonup p,\ q_n \rightharpoonup q \text{ in } L^2(\Omega)$$

as $n \to \infty$. It results to

$$p_n(x)w_n^1 \rightharpoonup p(x)\max\{w^1(x),0\},\ q_n(x)w_n^2 \rightharpoonup q(x)\max\{w^2(x),0\} \text{ in } L^2(\Omega)$$

as $n \to \infty$. Since $\{z_n\}$ is a $(PS)_c-$sequence of I, then from (2.3) we have $\forall(\varphi,\psi) \in E$, so that

$$\begin{aligned} o(1) = &\int_\Omega \left[\nabla w_n^1\nabla\psi + \nabla w_n^2\nabla\varphi + \lambda\nabla w_n^1\nabla\varphi + \mu\nabla w_n^2\nabla\psi\right] dx \\ &- \int_\Omega p_n(x)w_n^1\varphi\, dx - \int_\Omega q_n(x)w_n^2\psi\, dx. \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} &\int_\Omega \left[\nabla w^1\nabla\psi + \nabla w^2\nabla\varphi + \lambda\nabla w^1\nabla\varphi + \mu\nabla w^2\nabla\psi\right] dx - \int_\Omega p(x)\max\{w^1,0\}\varphi dx \\ &- \int_\Omega q(x)\max\{w^2,0\}\psi\, dx = 0. \end{aligned} \tag{2.6}$$

Therefore, w^1 and w^2 satisfy

$$\begin{cases} -\Delta w^1 - \mu\Delta w^2 = q(x)\max\{w^2,0\} \ge 0,\ x \in \Omega, \\ -\Delta w^2 - \lambda\Delta w^1 = p(x)\max\{w^1,0\} \ge 0,\ x \in \Omega. \end{cases} \tag{2.7}$$

Choosing $(\varphi_1, 0)$ as a test function in (2.6), we can get that

$$\int_\Omega \left[\nabla w^2 \nabla \varphi_1 + \lambda \nabla w^1 \nabla \varphi_1\right] dx = \int_\Omega p(x)\max\{w^1, 0\}\varphi_1 \, dx = l \int_{\Omega \cap \{x : w^1(x) > 0\}} w^1 \varphi_1 \, dx,$$

but

$$\int_\Omega \left[\nabla w^2 \nabla \varphi_1 + \lambda \nabla w^1 \nabla \varphi_1\right] dx = \int_\Omega \left[\lambda_1 w^2 \varphi_1 + \lambda \lambda_1 w^1 \varphi_1\right] dx,$$

thus we have

$$\int_{\Omega \cap \{x : w^1(x) > 0\}} (l - \lambda\lambda_1) w^1 \varphi_1 \, dx \le \int_{\Omega \cap \{x : w^2(x) > 0\}} \lambda_1 w^2 \varphi_1 \, dx. \tag{2.8}$$

Similarly, choosing $(0, \varphi_1)$ as a test function in (2.6), we can get

$$\int_{\Omega \cap \{x : w^2(x) > 0\}} (m - \mu\lambda_1) w^2 \varphi_1 \, dx \le \int_{\Omega \cap \{x : w^1(x) > 0\}} \lambda_1 w^1 \varphi_1 \, dx. \tag{2.9}$$

If $\Omega \cap \{x : w^2(x) > 0\} = \emptyset$, then from (2.7) we know that the maximum principle implies that $w^1 = -\mu w^2 \ge 0$ in Ω, but $w = (w_1, w_2) \not\equiv 0$, so we must have $\Omega \cap \{x : w^1(x) > 0\} \ne \emptyset$. Hence we can conclude from (2.8) that $l - \lambda\lambda_1 \le 0$, which contradicts $\lambda_1 < A$. Therefore $\Omega \cap \{x : w^2(x) > 0\} \ne \emptyset$. Similarly, we have $\Omega \cap \{x : w^1(x) > 0\} \ne \emptyset$. Thus, combining (2.8) and (2.9), we can get

$$(l - \lambda\lambda_1)(m - \mu\lambda_1) \le \lambda_1^2,$$

which is impossible since $\lambda_1 < A$.

Thus, we must have $\|z_n\| \le c < +\infty$ and the Proposition is proved. □

The proof for Theorem 1.1 will be completed by the following Proposition.

PROPOSITION 2.4. *If* $(H_1) - (H_4)$ *hold, then* I^∞ *is assumed.*

Proof. By Proposition 2.3, we know that I^∞ is well defined and finite. Now we show that I^∞ is assumed. Using (2.1) and Sobolev embedding theorem, we get

$$\|z\|^2 = \int_\Omega f(x, u)u \, dx + \int_\Omega g(x, v)v \, dx \le \epsilon c\|z\|^2 + c_\epsilon \|z\|^p.$$

When ϵ is small enough, we have

$$\|z\| \ge c > 0. \tag{2.10}$$

Suppose now $z_n = (u_n, v_n) \not\equiv 0$ is a minimising sequence of I^∞. By Proposition 2.3, we see that $\{z_n\}$ is uniformly bounded in E. So we may assume $z_n \to z = (u, v)$ in E and $I'(z) = 0$. Since (2.10) implies $z \ne (0, 0)$, it follows that $I^\infty = \lim_{n\to\infty} I(z_n) = I(z)$. Consequently, I^∞ is assumed by $z \in E \setminus \{0\}$. The proof is complete. □

Proof of Theorem 1.1. This is a direct consequence of Proposition 2.3 and 2.4. □

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