J. Austral. Math. Soc. 21 (Series A) (1976), 49–55.

FREE MODULES OVER SOME MODULAR GROUP RINGS

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(Received 29 October 1973)

Communicated by G. E. Wall

1. Introduction

Let K be a field and G a finite group with subgroup H. We say that (G,H) is a K-free pair if whenever M is a finitely generated KG-module whose restriction M_H is a free KH-module, then M is a free KG-module. In this paper pairs of groups with this property will be investigated.

If K has characteristic p and G is a cyclic p-group then (G,H) is a K-free pair provided H is a non-trivial subgroup of G. Several other examples of such pairs are given. One of the major results is that if K has characteristic 2 and G is the quaternion group of order 8 then (G,H) is K-free for any non-trivial subgroup H of G. Several conditions on the existence of such pairs are included in this paper.

Almost all of the results in this paper concern cases where the field K has characteristic $p(\neq 0)$ and G is a p-group. There exist examples of K-free pairs (G,H) where G is not a p-group. But the results are incomplete and are not included here.

Throughout this paper all modules will be assumed to be finitely generated. If G is a group $1(G)$ will denote the identity KG-module. If U is a subgroup of G and M is a KU-module let $M^G = KG \otimes_{KU} M$. If L is a KG-module, L_U denotes the restriction of L to a KU-module. For $x, y \in G$, $x^y = yxy^{-1}$ and $U^x = xUx^{-1}$. The radical of KG is indicated by rad KG and $\tilde{G} = \sum_{g \in G} g$.

2. Generalities

In this section K is a field and H is a subgroup of group G.

PROPOSITION 2.1. *Let T be a subgroup of G with $H \subseteq T \subseteq G$.*
(i) *If (G,T) and (T,H) are K-free pairs then (G,H) is a K-free pair.*
(ii) *If (G,H) is K-free then (G,T) is a K-free pair.*

PROOF. (i) Let M be a KG-module such that M_H is a free module. Then M_T is a free module since $(M_T)_H = M_H$. So M is a free module.

(ii) Let M be a KG-module such that M_T is free. Then M_H is a free module.
We shall need the following several times.

LEMMA 2.2. *Let K have characteristic $p > 0$. Let G be a p-group. Suppose M is a KG-module. Then KG is a direct summand of M if and only if $\tilde{G}M \neq (0)$.*

PROOF. It is well known that since G is a p-group $K\tilde{G}$ is the unique minimal ideal in KG. If $\tilde{G}M \neq (0)$ then there exists some $m \in M$ with $\tilde{G}m \neq 0$. So the annihilator of m in KG is the zero ideal. Hence the mapping $KG \to M$ by $\alpha \to \alpha m$ for $\alpha \in KG$ is a monomorphism. **Since KG is an injective left KG-module [see Curtis and Reiner (1962; page 321)] this homomorphism must split.**

PROPOSITION 2.3. *Let K have characteristic $p > 0$ and let G be a p-group. Suppose E is a finite extension of K. If (G, H) is a K-free pair, it is an E-free pair.*

PROOF. Let M be an EG-module such that M_H is a free EH-module. By restriction M is a finitely generated KG-module. Since $EH = E \otimes_K KH$, we have that M_H is free as a KH-module, hence it is free as a KG-module. So $\tilde{G}M \neq (0)$ and EG is a direct summand of M. By induction on the dimension of M we get that M is a free EG-module.

THEOREM 2.4. *Suppose (G, H) is a K-free pair. Then there exists no subgroup C of G with $C \neq \{1\}$,* and *$C^x \cap H = \{1\}$ for all $x \in G$.*

PROOF. Suppose there did exist such a subgroup. Then by the Mackey sub**group theorem [Curtis and Reiner (1962; page 324)]**

$$(1(C)^G)_H = \sum_x 1(C^x \cap H)^H$$

where x runs through a set of representaives of the $H-C$ double cosets. Since $C^x \cap H = \{1\}$ and $1(\{1\})^H = KH$, $(1(C)^G)_H$ is a free KH-module. But $1(C)^G$ is not a free KG-module.

3. Some Examples

PROPOSITION 3.1. *Let K be a field of characteristic $p > 0$. Let G be cyclic of order p^a. If H is any non-trivial subgroup of G then (G, H) is a K-free pair.*

PROOF. Let $S = \langle x^p \rangle$ where x is a generator of G. If we show that (G, S) is a K-free pair an easy induction proves the proposition.

Let M be an indecomposible KG-module of K-dimension n. The Jordan canonical form of the matrix of x on M is

https://doi.org/10.1017/S1446788700016906 Published online by Cambridge University Press

$$A = \begin{bmatrix} 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & 1 & 1 \\ & & & & & & 1 \end{bmatrix}.$$

Relative to some basis for M this is the matrix for x. So x^p has matrix

$$A^p = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 & \cdots & & \\ & 1 & \cdots & & 0 & 1 & & \\ & & \cdot & & & \cdot & & \\ & & & \cdot & & & \cdot & \\ & & & & \cdot & & & \cdot \\ & & & & & & & 1 \\ & & & & & \cdot & & 0 \\ & & & & & & \cdot & \cdot \\ & & & & & & \cdot & \cdot \\ & & & & & & \cdot & \cdot \\ & & & & & & & 0 \\ & & & & & & & 1 \end{bmatrix}$$

where the non-zero entries occur along the diagonal and in the $(i, i+p)$ positions for $i = 1, \cdots, n-p$. The K-dimension of $M/(1 - x^p)M$ is p.

Suppose M is a free KS-module. Then M_S is isomorphic to the sum of t copies of KS. Thus $n = p^{a-1}t$ and t is the K-dimension of $M_S/(\text{rad } KS \cdot M_S)$. Since

$$\text{rad } KS = (1 - x^p)KS$$

we must have $t = p$. Therefore $n = p^a$ and $M \cong KG$.

We can develop more examples using the following.

THEOREM 3.2. *Let K be a field of characteristic $p > 0$ and G a p-group.*

https://doi.org/10.1017/S1446788700016906 Published online by Cambridge University Press

Suppose T, H are subgroups of G with $T \vartriangle G$ and $T \subseteq H \subseteq G$. If $(G/T, H/T)$ is a K-free pair so is (G, H).

PROOF. Let M be a KG-module such that M_H is a free KH-module. Let $\tilde{T} = \sum_{g \in T} g$. The set $L = \tilde{T}M$ is a submodule of M since $T \vartriangle G$. For all $g \in T$, $g\tilde{T} = \tilde{T}$. So we can regard L as a G/T-module. We claim that $L_{(H/T)}$ is a free module. This follows from the fact that M_H is a direct sum of copies of KH and $\tilde{T}(KH) \cong 1(T)^H$ while $1(T)^H \cong K(H/T)$ as $K(H/T)$-modules.

Hence L is a free $K(G/H)$-module. Let $x_1, \cdots, x_n$ be a compete set of coset representatives of T in G. If $X = \sum_{i=1}^{n} x_i$, by Lemma 2.2 there exists an element $l \in L$ with $Xl \neq 0$. But $l = \tilde{T}m$ for some $m \in M$. So $Xl = X\tilde{T}m = \tilde{G}m \neq 0$. Lemma 2.2 says that KG is a direct summand of M. An easy induction proves the theorem.

COROLLARY 3.3. *Let K have characteristic $p > 0$ and let G and S be p-groups. If H is a subgroup of G with (G, H) a K-free pair then $(G \times S, H \times S)$ is a K-free pair.*

PROOF. $G \times S/S \cong G$ so $((G \times S)/S, (H \times S)/S)$ is a K-free pair.

COROLLARY 3.4. *Let K have characteristic p and let $G = A_m(p) = \langle x, y / x^{p^{m-1}} = y^p = 1, x^y = x^{1+p^{m-2}} \rangle$ where m is an integer $m \geqq 4$. Let $H = \langle x^{p^{m-2}}, y \rangle$. If T is any subgroup of G with $H \subseteq T \subseteq G$ then (G, T) is a K-free pair.*

PROOF. By Proposition 2.1 it is sufficient to show that (G, H) is a K-free pair. Let $S = \langle x^{p^{m-3}}, y \rangle \cong \langle x^{p^{m-3}} \rangle \times \langle y \rangle$. By Corollary 3.3 and Proposition 3.1 (S, H) is a K-free pair. Now $H \vartriangle G$ and G/H is cyclic. So $(G/H, S/H)$ is a K-free pair. Hence (G, S) is K-free. By Proposition 2.1, (G, H) is a K-free pair.

COROLLARY 3.5. *Let K be a field of characteristic $p > 0$ and let $G = B_m(p) = \langle x, y, z \mid x^{p^{m-2}} = y^p = z^p = 1, xy = yx, yz = zy, x^z = xy \rangle$ where $m \geqq 4$. Let $H = \langle x^{p^{m-2}}, y, z \rangle$. Then if T is any subgroup with $H \subseteq T \subseteq G$, (G, T) is a K-free pair.*

PROOF. We need only note that $\langle y \rangle \vartriangle G$ and $(G/\langle y \rangle, H/\langle y \rangle)$ is a K-free pair.

4. The Quaternion Group

THEOREM 4.1. *Let K be a field of characteristic 2. Let G be the quaternion group of order 8, i.e. $G = \langle x, y \mid x^4 = y^4 = 1, x^2 = y^2 = (xy)^2 \rangle$. If H is any non-trivial subgroup of G then (G, H) is a K-free pair.*

PROOF. Let $T = \langle x^2 \rangle$. Since all non-trivial subgroups of G contain T, it will be sufficient to prove that (G, T) is K-free.

Throughout this proof we suppose M is a KG-module such that M_T is a free

KT-module, but M is not free as a KG-module. It will be shown that this leads to a contradiction. Assume further that M has minimal K-dimension among such modules.

Let $L = (1 + y^2)M$. Then L is a submodule of M. Let $N = M/L$. Since the elements of T act trivially on L and on N, these modules may be regarded as $K\bar{G}$-modules where $\bar{G} = G/T$. We can write $\bar{G} = \langle \bar{x}, \bar{y} \rangle$ where $\bar{x} = xT$, $\bar{y} = yT$. Since $M_{\langle x \rangle}$, $M_{\langle y \rangle}$, $M_{\langle xy \rangle}$ are free modules, $L_{\langle \bar{x} \rangle}$, $L_{\langle \bar{y} \rangle}$, $L_{\langle \bar{x}\bar{y} \rangle}$, $N_{\langle \bar{x} \rangle}$, $N_{\langle \bar{y} \rangle}$, and $N_{\langle \bar{x}\bar{y} \rangle}$ are free modules.

We shall need the following

LEMMA 4.2. *Let $S = \langle y \rangle$. Let $m_1, \cdots, m_t \in M$ such that $\{m_i + (\text{rad}\, KS)M_S\}$ is a K-basis for $M_S/(\text{rad}\, KS)M_S$. Then $m_1, \cdots, m_t$ is a KS-basis for M_S.*

PROOF. Clearly the KS-dimension of M_S is t since M_S is a free module. Let $M' = \sum_{i=1}^{t} KSm_i$. Then

$$M_S = M' + (\text{rad}\, KS)M_S.$$

Nakayama's lemma [see Bass (1968; page 85)] says that $M_S = M'$. A simple dimension argument proves the lemma.

Let $b_1, \cdots, b_t$ be a $K\langle \bar{y} \rangle$-basis for N. If $b_{t+i} = (1 + y)b_i$ then $b_1, \cdots, b_{2t}$ is a K-basis for N. Let $a_1, \cdots, a_t$ be a set of coset representatives of $b_1, \cdots, b_t$, respectively' in M. That is, for each i, $a_i \to b_i$ under the quotient map $M \to N = M/L$. Since this quotient map induces an isomorphism

$$M_S/(\text{rad}\, KS \cdot M_S) \cong N_{\bar{S}}/(\text{rad}\, K\bar{S} \cdot N_{\bar{S}}),$$

the elements $a_1, \cdots, a_t$ are a KS-basis for M_S. For each $i = 1, \cdots, t$, let $a_{t+i} = (1 + y)a_i$, $a_{2t+i} = (1 + y^2)a_i$ and $a_{3t+i} = (1 + y + y^2 + y^3)a_i$. Then $a_1, \cdots, a_{4t}$ is a K-basis for M.

LEMMA 4.3. *There exists no $K\bar{G}$-free submodules of N.*

PROOF. Write $N = N_1 \oplus \cdots \oplus N_S$ where each N_i is indecomposable. Suppose one of these, say N_1, is a free $K\bar{G}$-module. We can assume without loss of generality that b_1, b_2 are a $K\langle \bar{y} \rangle$ basis for N_1. Since one of these must be a $K\bar{G}$-basis for N_1, we lose nothing by assuming that $N_1 = K\bar{G} \cdot b_1$ and $b_2 = xb_1$. But then

$$\bar{G}a_1 = (1 + y + y^2 + y^3)a_1 + (1 + y + y^2 + y^3)a_2 \neq 0.$$

So M has a KG-free direct summand, by Lemma 2.2. This contradicts the minimality of the K-dimension of M.

Write $N = N_1 \oplus N_2 \oplus \cdots \oplus N_S$ where each N_i is indecomposable. Each N_i is free as a $K\langle \bar{x} \rangle$-module and as a $K\langle \bar{y} \rangle$-module but not as a $K\bar{G}$-module. Basev (1961) and Heller and Reiner (1961) [see Conlon (1964)] have given a complete list of representations of $\bar{G}$. The above requirements on each N_i dictate that each

N_i is a $C_n(\pi)$, in Conlon's notation. That is, there exists a basis for N such that, relative to this basis, x and y have matrices

$$y \leftrightarrow \begin{bmatrix} I & O \\ I & I \end{bmatrix}, \qquad x \leftrightarrow \begin{bmatrix} I & \\ A & I \end{bmatrix}$$

where $I = I_t$ is the $t \times t$ identity matrix and A is non-singular. In fact if the field K is large enough we can assume that A is triangular.

If these matrices are given relative to the basis $b_1, \cdots, b_{2t}$ for N ($b_{t+i} = (1+y)b_i$), then as before we can construct a K-basis $a_1, \cdots, a_{4t}$ for M. With respect to this basis x and y have matrices

$$y \leftrightarrow \begin{bmatrix} I & & & \\ I & I & & \\ & I & I & \\ & & I & I \end{bmatrix}$$

$$x \leftrightarrow \begin{bmatrix} I & & & \\ A & I & & \\ B & C & I & \\ D & E & A & I \end{bmatrix},$$

where B, C, D, E are to be determined. Now $x^2 = y^2$. This implies that $AC = I$ and $E = ABA^{-1}$. Furthermore $xy = y^3x$. By computing the matrices for this element it is easily seen that $I + A = C$ and $I + A + B = E$. Hence $I + A + A^2 = O$, and the minimum polynomial for A has at most two distinct roots.

Let F be an extension of K which contains the roots p, p^2 of the polynomial $1 + x + x^2$. In F, A is similar to the matrix

$$A' = \begin{bmatrix} pI_r & O \\ O & p^2 I_s \end{bmatrix}.$$

For convenience assume $A = A'$. But then

$I + A = A^2 = B + E = B + ABA^{-1}$. If

$$B = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix},$$

$$A^2 = B + ABA^{-1} = \begin{bmatrix} O & p^2X \\ pY & O \end{bmatrix}$$

which is impossible. This contradiction proves the theorem.

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