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RESIDUAL FINITENESS OF COMMUTATIVE RINGS AND SCHEMES

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Introduction. This work grew out of a preliminary announcement (Notices of the Amer. Math. Soc. *18* (1971)). Here we modify the definition of residual finiteness given in [**2**]. This allows us, first of all, to consider a broader class of rings which are "essentially" residually finite and, secondly, to extend the notion to schemes. We then show that, for various topologies on the category of schemes, our notion of residual finiteness is local so that all relevant questions appear already at the ring level.

The first section contains the basic definitions and results of the theory. They are, as one would expect, very elementary. In the second section we deal with quasifinite homomorphisms; these are clearly finite homomorphisms on the level of the residual extensions, so the role they play is apparent. We then use Zariski Main Theorem in order to characterize "essentially" residually finite domains of finite type over $\mathbf{Z}$ or over a finite field (Theorem 2.3).

The third section centers around the behaviour of "essential" residual finiteness under completion and henselization. We show that the behaviour of residual finiteness in the sense of [**2**] is not as good and we produce counter-examples to a (false) statement in [**2**] — Corollary 5.3. In order that this statement remain true it suffices to use some sort of "unibranch" condition; in good cases, this condition is also necessary (Proposition 3.2 and Proposition 3.3).

In the fourth section we globalize most of the results for schemes. The last section is devoted to a brief report on residual finiteness at associated primes and at primes in the support of a module, a topic which seems to bear connection with torsion theories.

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1. Definitions and basic results. Let (comm) denote the category of commutative rings. If $A \in$ (comm), we let A_{red} stand for the reduced ring of A, i.e., $A_{\text{red}} = A/N$ where N is the nilradical of A. It is immediate that a homomorphism $\phi : A \to B$ canonically induces a homomorphism $\phi_{\text{red}} : A_{\text{red}} \to B_{\text{red}}$.

We know that the assignment $A \mapsto \text{spec } A$ extends to a functor (comm) $\to$ (top) where (top) stands for the category of topological spaces.

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If p : (comm) $\to$ (top) is a sub "mapping" (but not necessarily a subfunctor) of the functor spec, we say that a ring $A \in$ (comm) is *p-residually finite* if the residue field of A at every $P \in p(A)$ is a finite field. Clearly, this happens if and only if the integral domain A/P has a finite number of elements for every $P \in p(A)$, in which case $p(A) \subset \max A$, where $\max A$ is the maximal spectrum of A.

Examples. (1) p = spec. In this case we have the following result which completely characterizes a spec-residually finite ring:

PROPOSITION 1.1. *A ring $A \in$ (comm) is spec-residually finite if and only if A_{red} is a Von Neumann regular ring with finite residue field at every prime ideal.*

Proof. It is well-known that for $A \in$ (comm), the Krull dimension of A (heretofore denoted dim A) is zero if and only if A_{red} is Von Neumann regular [**1**, Chapitre II, § 4, Exercice 16 d]. The result now follows from the fact that A and A_{red} have homeomorphic spectra and the same residue field at corresponding primes.

Besides artinian local rings with finite residue field and rings with finitely many elements, other typical instances of spec-residually finite rings are:
(a) the ring of global sections of the sheaf of locally constant mappings $X \to k$ where X is a compact totally disconnected topological space and k is a finite field; a non-reduced non-noetherian example is given by
(b) Let A be a non-discrete valuation ring of rank one having finite residue field at its maximal ideal (e.g., the ring of a suitable inertial extension of a p-adic valuation of $\mathbf{Q}$). Choose non-zero elements $a, a_1, a_2, \ldots$ of A such that the sequence of values $v(a), v(a_1), v(a_2), \ldots$ decreases indefinitely. Then $A/(a)$ clearly satisfies all the requirements.

(2) p = max. The class of max-residually finite rings is perhaps too broad to allow any interesting characterization. For instance, any ring of finite type over a finite field is max-residually finite.

(3) p = esspec, where $\text{esspec}(A) = \text{spec}\, A - \min(A)$ with $\min(A)$ standing for the space of minimal prime ideals of A. Clearly, if A is esspec-residually finite then dim $A \leqq 1$. This is the type of residual finiteness which we will consider for the whole of this and the next three sections.

A homomorphism $\phi : A \to B$ is said to be *residually finite at $P \in$* spec B if the residual extension $k(\phi^{-1}(P)) \subset k(P)$ is finite; if ϕ is residually finite at every $P \in$ spec B outside min (B), we say that ϕ is an *ess-residually finite* homomorphism.

PROPOSITION 1.2. *Let A be an esspec-residually finite ring and let $\phi : A \to B$ be an ess-residually finite homomorphism. Suppose, moreover, that ϕ satisfies any one of the following conditions:*

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(i) ϕ *is integral (not necessarily injective);*

(ii) ${}^a\phi$: spec $B \rightarrow$ spec A *is injective;*

(iii) ϕ *is injective and* B *is a domain whose quotient field is an algebraic extension of the quotient field of* A.

Then B *is esspec-residually finite.*

Proof. Since ϕ is an ess-residually finite homomorphism, it suffices to check that the contraction of a non-minimal prime of B is a non-minimal prime of A, which we easily do:

(i) If $P \in$ spec B is non-minimal, let Q be a proper generization of P. Then $\phi^{-1}(Q)$ is a proper generization of $\phi^{-1}(P)$ since the homomorphism ϕ is integral.

(ii) The contention is trivial here since the non-empty fibres of ϕ have only one point.

(iii) There is a standard argument in this case, namely, if $P \in$ spec B is non-zero pick $b \in P$, $b \neq 0$, and write an algebraic equation of minimal degree for b over A, say, $a_n b^n + \ldots + a_0 = 0$. Then $a_0 \neq 0$ and it is clear that $a_0 \in P \cap A$.

In the opposite direction, we have:

PROPOSITION 1.3. *Let* B *be an esspec-residually finite ring and let* $\phi : A \rightarrow B$ *be a homomorphism satisfying any of the following conditions:*

(i) ϕ *is flat;*

(ii) ker ϕ *is a nilideal (i.e.,* ${}^a\phi$ *is dominant) and* spec B *is irreducible;*

(iii) ϕ *is injective and integral.*

Then A *has finite residue field at every non-minimal prime* $\mathfrak{p}$ *such that the fiber of* ϕ *over* $\mathfrak{p}$ *is non-empty.*

Proof. It suffices to show that if $P \in$ spec B has a non-minimal contraction $\phi^{-1}(P)$ then the fiber of P contains a non-minimal prime of B. We proceed to do so:

(i) A flat homomorphism satisfies the going-down property [**5**, Chapter 2, (5.D)]. Therefore, given a proper generization q of $\phi^{-1}(P)$, there exists a proper generization of P lying over q. Thus, in this case, P itself is a non-minimal prime.

(ii) Since B (respectively A) is residually finite at a given prime if and only if its reduced ring is residually finite at the corresponding prime and since the canonical homeomorphism spec $B_{\text{red}} \simeq$ spec B preserves the fibres of ϕ, we may — by passing to ϕ_{red} — assume that B is a domain and ϕ is injective. The result is now clear since the generic point of spec B maps onto the generic point of spec A.

(iii) Let $q \in$ spec A be a minimal generization of $\phi^{-1}(P)$. Since ϕ is injective, there exists a minimal prime Q of B lying over q [**1**, Chapitre II, § 2, Proposition 16]. But ϕ is also integral; therefore, by the going-up property, there exists a prime M of B, a specialization of Q, lying over $\phi^{-1}(P)$. Such an M clearly answers our problem.

Remark. In all three conditions of Proposition 1.3 the kernel of ϕ was assumed to be "small"; more precisely, $\ker \phi \subset \bigcup p$ with p running through min (A). To see that such a hypothesis is essential it suffices to consider a surjection $A \to A/q$ with q a maximal non-minimal prime of A having infinite residue field. On the other hand the remaining hypotheses are not at all necessary; thus, if $\dim B = 1$ and if the prime ideals of A have finite height (say, $\dim A < \infty$), Proposition 1.3 goes through solely under the assumption that ϕ is injective (or even that $\ker \phi$ is a nilideal). For, by starting with a minimal prime ideal, one can work up a chain of primes in B of length one. It is essential, however, that $\dim B = 1$ as the injective epimorphism $A \to \mathrm{K} \times k$ shows, where A is a discrete valuation ring with infinite residue field k and quotient field K.

COROLLARY 1.4. *Let A be a domain with quotient field K, let $L|K$ be a finite field extension and let B be a subring of L containing A.*

(a) *If A is noetherian and esspec-residually finite then B is esspec-residually finite.*

(b) *If A is integrally closed and if B is integral over A then A is esspec-residually finite if and only if B is so.*

Proof. (a) Since A is esspec-residually finite, $\dim \mathrm{A} \leqq 1$. By a result of Krull-Akizuki [**1**, Chapitre IV, § 2, Proposition 5], for every non-zero prime P of B, B/P is a finite A-module and, *a fortiori*, a finite $A/P \cap A$-module. Therefore the injection $A \subset B$ is an ess-residually finite homomorphism and the conclusion follows from Proposition 1.2(iii).

(b) Under these assumptions, the fibers of the injection $A \subset B$ are all finite [**1**, Chapitre IV, § 2, no. 3, Corollaire 2(i)]. Therefore, by a standard argument, the injection $A \subset B$ is ess-residually finite and we conclude from Proposition 1.2(i) and from Proposition 1.3(iii).

COROLLARY 1.5. *Let $\phi : A \to B$ be an epimorphism. If A is esspec-residually finite, so is B. In particular, rings of fractions and affine open sets of esspec-residually finite rings are esspec-residually finite.*

Proof. It follows immediately from Proposition 1.2(ii) because the mapping ${}^a\phi$ corresponding to an epimorphism $\phi : A \to B$ is injective and admits only trivial residual extensions [**4**, Proposition 1.5].

COROLLARY 1.6. *If A is an esspec-residually finite Prüfer domain and if B is a subring of the quotient field of A containing A, then B is esspec-residually finite.*

Proof. Since B is A-torsion free and A is Prüfer, B is A-flat. It follows that the injection $A \subset B$ is an epimorphism [**7**, Compl. au texte, Corollaire 1].

We may remark that if A is a domain which is neither noetherian nor of arithmetic type, we do not know whether esspec-residual finiteness is preserved under a dominant birational homomorphism $A \to B$; the first to look at is the case of a non-noetherian integrally closed local domain A which is not a valuation ring.

2. Characterization of a class of esspec-residually finite rings. Let us recall a stronger concept than that of a residually finite homomorphism — this concept has partially been met along the first section.

A homomorphism $\phi : A \to B$ is said to be *quasifinite at a prime* $P \in$ spec B if

(1) ϕ is of finite type;

(2) P is isolated in its fiber (i.e., in the fiber of ${}^a\phi$ over $\phi^{-1}(P)$).

It can be shown that if condition (1) holds then condition (2) is equivalent to

(2′) $B_P/\phi^{-1}(P)B_P$ is a finite $k(\phi^{-1}(P))$-module.

From this one easily deduces that a homomorphism $\phi : A \to B$ of finite type is quasifinite (meaning quasi-finite at every prime of B) if and only if, for every $p \in$ spec A, the fiber $B \otimes_A k(p)$ is a finite $k(p)$-module.

The following result is certainly part of the mathematical folklore: its geometrical counterpart is given in [**3**, Chapitre II, Corollaire 7.4.4] and our proof follows closely the proof there.

PROPOSITION 2.1. *Let A and B be noetherian domains of dimension one and let $\phi : A \to B$ be a homomorphism of finite type.*

(a) *If B is not semilocal and if ϕ is quasifinite then ϕ is injective.*

(b) *If ϕ is injective then it is quasifinite.*

Proof. Clearly, any non-zero ideal I of a noetherian domain D of dimension one is contained in only finitely many primes of D. Indeed, let D' be the integral closure of D in its quotient field; D' is noetherian [**1**, Chapitre VII, § 2, Proposition 5], hence a Dedekind domain, therefore ID' is contained in only finitely many primes of D' and so I is contained in only finitely many primes of D. In other words, we have shown that the proper closed sets of spec D are the finite subsets of spec D not containing the generic point; in particular, every infinite subset of spec D is dense everywhere. Now, the proof of the Proposition proper:

(a) Suppose ϕ is not injective. Then ${}^a\phi$ (spec B) is not dense in spec A. By the preceding remarks, ${}^a\phi$(spec B) is a finite set. On the other hand, by assumption, the fibers of ${}^a\phi$ are finite (possibly empty) sets. It follows that spec B is a finite set, which contradicts the hypothesis.

(b) If ϕ is injective, the generic point of spec B maps onto the generic point of spec A. Therefore, if $p \in$ spec A is non-generic, $({}^a\phi)^{-1}(p)$ is a closed set not including the generic point of spec B, hence it is in fact a finite set by the above remarks. Since spec $B \otimes_A k(p)$ is homeomorphic to $({}^a\phi)^{-1}(p)$, we have shown that every $P \in$ spec B lying over p is isolated in its fiber. Thus, we are left with the case of a point of spec B lying over the generic point of spec A: suppose the fiber over the generic point of spec A had infinitely many points.

By the preceding remarks, this fiber would be an everywhere dense subset of spec B. Let now $P \in$ spec B be a point outside the generic fiber; by Zariski Main Theorem [**6**, Chapitre IV, Théorèm 1], there is an affine open neighbourhood U of $\psi^{-1}(P)$ in spec A' such that U is isomorphic to $({}^a\psi)^{-1}(U)$, where A' is the integral closure of A in B and ψ is the canonical injection $A' \subset B$. On the other hand, $({}^a\psi)^{-1}(U)$ contains a point Q of the generic fiber of ϕ; since A' is integral over A, Q belongs to the generic fiber of ψ as well. But $({}^a\psi)^{-1}(U)$ is irreducible and its generic point is the generic point of spec B; therefore, there would be at least two points of $({}^a\psi)^{-1}(U)$ lying over the generic point of U and this contradicts the above isomorphism. (N.B.—we have shown that the generic fiber of ϕ has finitely many points. Pushing the reasoning one step further, we see that the generic fiber is in fact reduced to the generic point of spec B; indeed, since ϕ is of finite type, a standard argument — implicit in our preliminary remarks on quasifinite homomorphisms — shows that $B \otimes_A K$ is a finite K-module, where K is the quotient field of A. It easily follows that the quotient field of B is a finite extension of K, therefore the generic point of spec B is the only point of spec B lying over the generic point of spec A (cf., e.g., the proof of Proposition 1.2(iii)), as desired).

The following example shows that Proposition 2.1(b) is no longer true in general if B has proper zero-divisors: let $A = \mathbf{Z}_p$ stand for the localization of the ring of integers at a prime ideal (p) and let $B = A[X]/(p(pX - 1))$, where X is an indeterminate. It is clear that $B \simeq \mathbf{Z}/(p)[X] \times \mathbf{Q}$ and that $\dim B = 1$. The canonical homomorphism $\phi : A \rightarrow B$ has two fibers; the fiber over the generic point is reduced to the prime $\mathbf{Z}/(p)[X] \times (0)$, while the fiber over the non-generic point consists of the minimal prime $(0) \times \mathbf{Q}$ and all of its infinitely many specializations $(f) \times \mathbf{Q}$, where f is an irreducible monic polynomial in $\mathbf{Z}/(p)[X]$.

Incidentally, the same example as above shows that, in contrast to the case of a quasifinite homomorphism, the set of points where a homomorphism is residually finite is not necessarily open — here this set is spec $B - \{(0) \times \mathbf{Q}\}$, which is not open since $(0) \times \mathbf{Q}$ is not a closed point.

COROLLARY 2.2. *Let A and B be noetherian domains of dimension one and let $\phi : A \rightarrow B$ be an injective homomorphism of finite type. Then ϕ factors through an injective finite homomorphism $A \rightarrow C$ and an open immersion $C \rightarrow B$.*

Proof. This follows from Proposition 2.1(b) and one of the variations of Zariski Main Theorem.

As an easy consequence, we obtain the following structural result:

THEOREM 2.3. *Let B be a domain finitely generated over its prime ring. Then B is esspec-residually finite if and only if exactly one of the following conditions is satisfied:*

(a) spec *B is an open subset of a finite* $\mathbf{Z}$*-algebra;*

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(b) spec B *is an open subset of a finite* $k[X]$*-algebra, where* k *is a finite field and* X *is a transcendental.*

Proof. Sufficiency is clear from Proposition 1.2(i) and Corollary 1.5. For necessity, we note that if char $B \neq 0$ then B contains a finite field k, in which case, except for one exception — namely, when B is a finite field containing k — B also contains a transcendental element x over k; clearly, B is finitely generated over $k[X]$. In any case, Corollary 2.2 is immediately applicable.

Typical instances of case (b) above are $k[x, y]$ with $xy = 1$ and $y^2 = x^3$ respectively; in both examples the inclusion $k[x] \subset k[x, y]$ corresponds, of course, to a projection onto the x-axis parallel to the y-axis. However, the first projection is a birational homomorphism while the second one is a double covering of the line.

3. Residual finiteness under henselization and completion. A *pair* (A, I) consists of a ring A (comm) and an ideal I of A. Pairs form a category, a morphism $f : (A, I) \rightarrow (A', I')$ consisting of a homomorphism $f : A \rightarrow A'$ such that $f(I) \subset I'$. A pair (A, I) is called a *Hensel pair* if A satisfies the usual Hensel lemma for monic polynomials relative to I. To any pair (A, I) one can assign a Hensel pair $({}^hA, {}^hI)$ and a morphism $(A, I) \rightarrow ({}^hA, {}^hI)$, called "canonical", which are universal for morphisms from (A, I) to Hensel pairs. For the main properties of the pair $({}^hA, {}^hI)$, the *henselization* of (A, I), we refer to [**6**, Chapitre XI].

Also, given a pair (A, I), we can consider its (separated) completion $(\hat{A}, \hat{I})$ defined by

$$\hat{A} = \varprojlim A/I^n, \ \hat{I} = \varprojlim I/I^n.$$

For properties of the completion we will refer to [**5**, Chapter 9].

PROPOSITION 3.1. *Let* (A, I) *be a pair such that* $I \subset \operatorname{rad} A$ *and let* $(\hat{A}, \hat{I})$ *and* $({}^hA, {}^hI)$ *be as above.*

(i) *If* A *is noetherian then* A *is esspec-residually finite if and only if* $\hat{A}$ *is so;*

(ii) *If* A *is noetherian or local with* $I = \operatorname{rad} A$ *then* ${}^hA$ *is esspec-residually finite as soon as* A *is. The converse holds with no assumptions on* A.

Proof. (i) Since the canonical homomorphism $A \rightarrow \hat{A}$ is flat [**5**, Chapter 9, Corollary 1 of Theorem 55], we deduce one implication by using Proposition 1.3(i). Conversely, if A is esspec-residually finite then dim $A \leqq 1$, so dim $\hat{A} \leqq 1$ [**5**, Chapter 9 (24.*D*)(*i'*)]; therefore we need only to check the maximal ideals of $\hat{A}$. But the residual extension at a maximal ideal is trivial [**5**, Chapter 9, Corollary to Theorem 56], hence the conclusion.

(ii) The canonical homomorphism $A \rightarrow {}^hA$ is flat because $({}^hA, {}^hI)$ is a direct limit of étale neighbourhoods of (A, I) [**6**, Chapitre XI, § 2, Théorèm 2]. Therefore, as before, we have that A is esspec-residually as soon as ${}^hA$ is. Conversely, suppose that A is esspec-residually finite. As in the proof of (i), it suffices to show that the dimension of ${}^hA$ is no greater than one. Now dim ${}^hA$ = dim A if A is local and I = rad A [**6**, Chapitre VIII, 4, Exercice]; or else, if A is noetherian then A and ${}^hA$ have the same separated completion, hence there is a faithfully flat homomorphism ${}^hA \to \hat{A}$. Therefore, pretty generally, dim ${}^hA \leqq$ dim $\hat{A}$ and so, dim ${}^hA \leqq 1$.

This seems to be a good place to mention the relation between the present work and [**2**]. A ring A is residually finite in the sense of [**2**] — heretofore designated simply as residually finite — if A/I has a finite number of elements for every non-zero ideal I. For noetherian domains our notion of esspec-residual finiteness coincides with residual finiteness. Since esspec-residual finiteness disregards what happens at the residue field of a minimal prime ideal, Proposition 3.1 is, so to say, a mechanical checking procedure. Not so residual finiteness; in fact, Corollary 5.3 of [**2**] is false in general and counterexamples are easily obtained. Thus, let A be the local ring of the nodal curve $k[X, Y]/(Y^2 - X^2(X + 1))$ at the origin, where k is a finite field of char $\neq 2$. It is easy to check that A is residually finite while its completion (or henselization for that matter), as it is well-known, has two distinct minimal prime ideals, hence, *a fortiori*, cannot be residually finite.

Thus, things go wrong for residual finiteness as soon as the completion (or henselization) "branches off". This is, however, the only evil as we now see.

First recall that a local ring A is *unibranch* if A_{red} is a domain whose integral closure is a local ring; the *formal fibers* of A are the fibers of the canonical homomorphism $A \to \hat{A}$. If $A \to B$ is an arbitrary homomorphism and if p is a prime ideal of A then the fiber over p, B $\otimes_A k(p)$, is *geometrically normal* if $B \otimes_A k$ is a normal ring for every field extension $k|k(p)$.

PROPOSITION 3.2. *Let* (A, I) *be a pair such that* $I \subset$ rad A *and suppose that the formal fibers of each stalk* A_m *are geometrically normal. Consider the following conditions:*

(i) A *is residually finite*, spec (A/I) *is connected and for every* $m \in$ max A, A_m *is unibranch;*

(ii) $\hat{A}$ *is residually finite.*

Then (i) *implies* (ii). *If, moreover,* A *is local then* (ii) *implies* (i).

Proof. In order to show that (i) implies (ii) it suffices, by the results of Proposition 3.1, to see that $\hat{A}$ is a domain under the present conditions. Now, generally, if $M \in$ max $\hat{A}$ then $m = M \cap A \in$ max A and $(A_m)^\wedge \simeq (\hat{A}_M)^\wedge$ [**5**, Chapter 9 (24.D)] and note that A is noetherian since it is residually finite. Therefore, using the assumption that A_m is unibranch and has geometrically normal formal fibers, we have that $(A_m)^\wedge$ is a domain [**3**, Chapter IV, Corollary (7.6.3)]. It follows that $(\hat{A}_M)^\wedge$, hence $\hat{A}_M$, is a domain. Thus, we have shown that $\hat{A}$ is "stalkwise" a domain. Furthermore $A/I \simeq \hat{A}/\hat{I}$ and $(\hat{A}, \hat{I})$ is a Hensel pair; therefore, by the "lifting idempotents" form of Hensel lemma, spec $\hat{A}$ is connected. Finally, $\hat{A}$ is noetherian since A is; by a standard argument, $\hat{A}$ is a domain.

Conversely, if $\hat{A}$ is residually finite, it is necessarily noetherian as remarked above, hence A is noetherian and the canonical homomorphism $A \to \hat{A}$ is flat. By Proposition 1.3(i), A is esspec-residually finite. On the other hand, $I \subset \operatorname{rad} A$ by hypothesis, therefore $A \to \hat{A}$ is in fact faithfully flat [**5**, Chapter 9, Theorem 56], hence, in particular, injective. Consequently, A is a domain and, *a fortiori*, residually finite. So far for residual finiteness; if, moreover, A is local then it is necessarily unibranch [**3**, Ibid.].

There is an exact analog of Proposition 3.2 for the henselization; here, moreover, the condition on the formal fibers is no longer needed [**3**, Chapitre IV, Proposition (18.6.12)]. We close the section with a result that globalizes Theorem 5.2 of [**2**].

PROPOSITION 3.3. *Let* (A, I) *be a pair such that* $I \subset \operatorname{rad} A$. *Suppose that* A *is an infinite residually finite ring. Then there exists a bijection between the irreducible components of* $\hat{A}$ *(respectively* ${}^hA$*) and the isomorphism classes of pairs* B, θ *with* B *a residually finite ring and* $\theta : A \to B$ *an injective homomorphism that factors through the canonical homomorphism* $A \to \hat{A}$ *(respectively* $A \to {}^hA$*) and a surjective homomorphism* $\hat{A} \to B$ *(respectively* ${}^hA \to B$*).*

Proof. Let P be a minimal prime ideal of $\hat{A}$ (respectively ${}^hA$). Since the canonical homomorphism $A \to \hat{A}$ (respectively $A \to {}^hA$) is flat, $P \cap A$ is a minimal prime ideal of A (this is essentially the going-down property). Therefore $P \cap A = (0)$ because, as remarked before, a residually finite ring is necessarily a domain. It follows that the composition $A \to \hat{A} \to \hat{A}/P$ (respectively $A \to {}^hA \to {}^hA/P$) is injective and also that $\hat{A}/P$ (respectively ${}^hA/P$) is residually finite. Conversely, let B, θ be a pair as in the statement of the proposition; since A is infinite by hypothesis, so is B, hence $B \simeq \hat{A}/P$ (respectively $B \simeq {}^hA/P$) for some prime ideal P of $\hat{A}$ (respectively ${}^hA$) and $P \cap A = (0)$. On the other hand, $\dim \hat{A} \leqq 1$ (respectively, $\dim {}^hA \leqq 1$), hence P is necessarily minimal.

4. Globalization. Most of the definitions and results of the preceding sections can be easily extended for schemes (conceivably, for locally ringed spaces as well). In order to indicate some of them, we recall the concept of covering families.

Let C be a class of morphisms of schemes. A *surjective covering family in* C — here designated simply as *covering family in* C or C-*covering family* — is a family $\{X_i \to Y\}_{i \in I}$ of morphisms in C such that for every $y \in Y$ there exists an $i \in I$ such that the fiber over y of the morphism $X_i \to Y$ is non-empty. A property of schemes is said to be *local for* C if whenever $X_i \to Y$ is a covering family in C, then Y has the property if and only if each X_i has the property. In particular, by letting C be the class of open immersions, we retrieve the usual notion of a local property for a scheme.

We now mimic the definition given at the beginning of Section 1. Let (Sch) be the category of schemes and let P : (Sch) $\to$ (top) be a submapping of the forgetful functor Spec — by definition, Spec assigns to a scheme its underlying topological space. A scheme S is *P-residually finite* if the residue field at each point $s \in P(S)$ is finite. The main examples here again are $P = \text{Spec}$ and $P = \text{ESSpec}$, where $\text{ESSpec}(S) = \text{Spec}(S) - \text{Min}(S)$ and $\text{Min}(S)$ is the space of generic points of the irreducible components of S.

If S is a scheme and U an open subscheme of S, $\mathcal{O}_{U,x} \simeq \mathcal{O}_{S,x}$ for any $x \in U$. Therefore Spec-residual finiteness is a local property for the Zariski topology. In a similar way, we have:

LEMMA 4.1. *ESSpec-residual finiteness is a local property for the Zariski topology.*

Proof. It suffices to show that if $U \subset S$ is an open immersion and if $x \in U$ then x is the generic point of an irreducible component of U if and only if it is the generic point of an irreducible component of S. But this is an easy consequence of [**1**, Chapitre II § 4, Proposition 7].

The above lemma is basic for all the following results. Firstly, the analog of Corollary 1.4:

PROPOSITION 4.2. *Let $f : X \to Y$ be a dominant morphism of integral schemes (i.e., irreducible and reduced ones). Let $K(X)$ (respectively $K(Y)$) be the function field of X (respectively Y) and suppose that the extension $K(X)|K(Y)$ is finite.*

(a) *If Y is locally noetherian and ESSpec-residually finite then X is ESSpec-residually finite;*

(b) *If Y is a normal scheme and if, moreover, the morphism f is integral, then Y is ESSpec-residually finite if and only if X is ESSpec-residually finite.*

Proof. (a) Let $x \in \text{ESSpec}(X)$. Pick affine open neighbourhoods U and V of x and $f(x)$ respectively such that $f(U) \subset V$. Consider the homomorphism $A = \Gamma(V, \mathcal{O}_Y) \to \Gamma(U, \mathcal{O}_X) = B$ corresponding to the restriction $f\,|U : U \to V$; the latter is dominant, hence $A \to B$ is injective. Since the quotient field of A (respectively of B) is isomorphic to $K(Y)$ (respectively to $K(X)$), A is esspec-residually finite (Lemma 4.1) and noetherian (Y is locally noetherian), we deduce, via Corollary 1.4(a), that B is esspec-residually finite. Since $x \in U$ and x is an arbitrarily given non-generic point of X, we get the desired conclusion.

(b) Since the morphism $f : X \to Y$ is integral, we can cover Y by affine open sets U_α such that $f^{-1}(U_\alpha)$ is affine (actually, any affine covering will do) and such that the induced homomorphism

$$A_\alpha = \Gamma(U_\alpha, \mathcal{O}_Y) \to \Gamma(f^{-1}(U_\alpha), \mathcal{O}_X) = B_\alpha$$

is integral. The A_α's are integrally closed since Y is normal; therefore, a use of Corollary 1.4(b) and Lemma 4.1 finishes the proof.

PROPOSITION 4.3. *ESSpec-residual finiteness is a local property for the dominant monomorphisms whose source is an irreducible scheme.*

Proof. Let $\{f_i : X_i \to Y\}_{i \in I}$ be a covering family in the class of the dominant monomorphisms with irreducible source. Suppose that Y is ESSpec-residually finite and let $i \in I$ be an arbitrary index. Take affine open coverings $X = \cup U_j$ and $Y = \cup V_j$ such that $f_i(U_j) \subset V_j$; the restriction $f_i|U_j : U_j \to V_j$ is clearly a monomorphism as composite of the open immersion $U_j \to f_i^{-1}(V_j)$ and the fiber morphism $f_i^{-1}(V_j) \to V_j$. Therefore, the corresponding homomorphism $\Gamma(V_j, 0_Y) \to \Gamma(U_j, 0_{X_i})$ is an epimorphism; it then follows from Lemma 4.1 and Corollary 1.5 that X_i is ESSpec-residually finite. Conversely, let $y \in \mathrm{ESSpec}(Y)$ and let $i \in I$ be such that the fiber of f_i over y is non-empty; pick $x \in X_i$ in this fiber and let $U \subset X_i$ and $V \subset Y$ be affine open neighbourhoods of x and y, respectively. Since dominancy is a local property on both Y and X_i (X_i is irreducible !), the restriction $f_i|U : U \to V$ is dominant, which means that the kernel of the homomorphism $\Gamma(V, 0_Y) \to \Gamma(U, 0_{X_i})$ is a nil-ideal. Since U is irreducible too, we can apply Proposition 1.3(ii) and Lemma 4.1 to obtain the desired conclusion.

PROPOSITION 4.4. *ESSpec-residual finiteness is a local property for the class of flat monomorphisms.*

Proof. By examining the proof of Proposition 4.3, one sees that it suffices to show that flatness localizes well and then use Proposition 1.3(i). But this is indeed so because if $U \subset X$ and $V \subset Y$ are affine open neighbourhoods of x and $f(x)$ respectively, then the restriction $f|U : U \to V$ is flat as composite of the open immersion $U \to f^{-1}(V)$ and the fiber morphism $f^{-1}(V) \to V$.

Recall that a morphism $f : X \to Y$ of schemes is *locally quasifinite* if, for every $x \in X$, there exist affine open neighbourhoods U and V of x and $f(x)$ respectively such that $f(U) \subset V$ and such that the induced homomorphism $\Gamma(V, 0_Y) \to \Gamma(U, 0_X)$ is quasifinite.

PROPOSITION 4.5. *ESSpec-residual finiteness is a local property for the class of flat locally quasifinite morphisms (respectively dominant locally quasifinite morphisms whose source is an irreducible scheme). In particular, ESSpec-residual finiteness is a local property for the étale morphisms (respectively the unramified dominant morphisms whose source is an irreducible scheme).*

Proof. From the proofs of Proposition 4.3 and Proposition 4.4 we see that it is enough to show that if $f : X \to Y$ is a locally quasifinite morphism and if Y is ESSpec-residually finite, then X is ESSpec-residually finite. This is in turn a consequence of Zariski Main Theorem; indeed, if $x \in X$, there exist open neighbourhoods U of x and V of $f(x)$ such that $f(U) \subset V$ and such that the restriction morphism $f|U : U \to V$ factors through an open immersion $U \to V'$ and a finite morphism $V' \to V$. By Lemma 4.1, U is ESSpec-residually finite as soon as V' is. On the other hand, V' is ESSpec-residually finite as soon as V is: indeed, the question is purely local in which case it follows from Proposition 1.2(i).

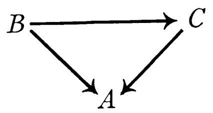
As in the ring case, we can characterize ESSpec-residually finite schemes that are of "good" type over a "good" ground ring. More precisely:

THEOREM 4.6. *Let R be an esspec-residually finite noetherian domain of dimension one and let $f : X \to \text{spec } R$ be a dominant morphism locally of finite type, where X is an integral scheme. Then X is ESSpec-residually finite if and only if f factors locally through an open immersion $U \to V$ and a finite morphism $V \to \text{spec } R$.*

Proof. If f factors in the stated way, it is clear that X is ESSpec-residually finite according to Proposition 1.2(i) and Lemma 4.1. Conversely, let X be ESSpec-residually finite; if $U \subset X$ is an affine open set, then $\Gamma(U, 0_X)$ is an esspec-residually finite domain endowed with an injective homomorphism $R \to \Gamma(U, 0_X)$ of finite type. If dim $\Gamma(U, 0_X) = 0$, this homomorphism is clearly quasifinite; otherwise, we can apply Proposition 2.1(b) to get the same conclusion. Therefore, f is a locally quasifinite morphism and an application of Zariski Main Theorem closes the proof.

5. The functor Ass$(-(M))$ and the functor Supp$(M \otimes_A -)$.

Let $A \in$ (comm) and let M be an A-module. Recall that a prime p $\in$ spec A is said to be *associated to* M if there exists an $x \in M$ such that p is minimal among the prime ideals containing the annihilator of x in A. The set of associated primes to M is denoted $\text{Ass}_A(M)$. If A (comm), we let $(\to A)$ denote the category of homomorphisms in (comm) whose target is A; a morphism in $(\to A)$ is a homomorphism $B \to C$ in (comm) such that the following diagram is commutative:



If A is besides noetherian, we denote $N(\to A)$ the full subcategory of $(\to A)$ whose objects are the morphisms with noetherian source. This is a remarkable category for one has:

PROPOSITION 5.1. *Let $A \in$ (comm) be noetherian and let $N(\to A)$ be as above. Let M be an A-module. For each $\phi : B \to A$ in $N(\to A)$ we set*

$$\text{Ass}_-(-_*(M))(\phi : B \to A) = \text{Ass}_B(\phi_*(M))$$

where $\phi_*(M)$ is the direct image of M under ϕ ("restriction of scalars"). Then $\text{Ass}_-(-_*(M))$ *extends to a subfunctor of the restriction of* spec *to* $N(\to A)$.

Proof. Since $\text{Ass}_-(-_*(M))$ is at any rate a submapping of spec, it suffices to check that a morphism $f : B \to C$ in $N(\to A)$ gives rise to a map in the opposite way $\text{Ass}_-(-_*(M))(f) : \text{Ass}_C((\phi_C)_*(M)) \to \text{Ass}_B((\phi_B)_*(M))$ induced by ${}^af : \text{spec } C \to \text{spec } B$. Now, since B and C are noetherian, ${}^af$ induces a map

$$\text{Ass}_C((\phi_C)_*(M)) \to \text{Ass}_B((f)_*(\phi_C)_*(M))$$

[**5**, Prop. (9.A)]. On the other hand, $(\phi_B)_*(M) = (\phi_C \circ f)_*(M) = (f_*)(\phi_C)_*(M)$ by the very definitions, hence the desired result.

COROLLARY 5.2. *Let* $A \in$ (comm) *be noetherian and let* M *be an* A*-module. Then* $\mathrm{Ass}_-(-_*(M))$ — *residual finiteness is a local property for the residually finite morphisms in* $N(\to A)$.

Proof. It readily follows from Proposition 5.1 and the fact that, for $f : B \to C$ in $N(\to A)$, one has $({}^a f)(\mathrm{Ass}_C((\phi_C)_*(M))) = \mathrm{Ass}_B((\phi_B)_*(M))$.

Recall that if $A \in$ (comm) and if M is an A-module, one defines $\mathrm{Supp}_A(M) = \{p \in \mathrm{spec}\, A | M_p = M \otimes_A A_p \neq 0\}$. Let $(A\to)$ denote the category of commutative A — algebras; then

$$\mathrm{Supp}_B(M \otimes_A B) \subset ({}^a f)^{-1}\mathrm{Supp}(_A(M))$$

for $f : A \to B$ in $(A\to)$ [**1**, Chapitre II, § 4, Proposition 19], so $\mathrm{Supp}_-(M \otimes_A -)$ is a functor on $(A\to)$.

PROPOSITION 5.3. $\mathrm{Supp}_-(M \otimes_A -)$ — *residual finiteness is a local property for the residually finite homomorphisms in* $(A\to)$, *provided* M *is of finite type over* A.

Proof. Let (C_i) be a covering family with source B in $(A\to)$. If B is Supp $(M \otimes_A -)$ — residually finite then so is C_i for every i because of functoriality. Conversely, if $p \in \mathrm{Supp}(M \otimes_A B)$, let $P \in \mathrm{spec}(C_i)$ (for some i) be such that it lies over p; since M is of finite type over A, $M \otimes_A B$ is of finite type over B, therefore, P belongs to Supp $((M \otimes_A B) \otimes_B C_i)$ [**1**, Ibid.]. Since $(M \otimes_A B) \otimes_B C_i \simeq M \otimes_A C_i$, the proof is finished.

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