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ADJOINTS OF A GEOMETRY

BY
ALAN L. C. CHEUNG

To give a geometric interpretation to the inverted incidence relation between the flats of a geometry has for years been a tempting idea in combinatorial geometries [1]. If G is a geometric lattice, the inverted lattice G' is not necessarily geometric. The problem has been to determine whether there is some geometric lattice G^{Δ}, to be called an *adjoint* of G, into which G' may be embedded. The present note shows how any adjoint G^{Δ} of a geometric lattice G must be related to G-extensions in such a way that the natural correspondence between flats of G and principal G-extensions is preserved. It follows that an adjoint may fail to exist, even for a geometry with as few as eight points.

An appropriate definition of an adjoint is the following:

An *adjoint* of a geometric lattice G is a geometric lattice G^{Δ} *of the same rank* into which there is an embedding (i.e. a one-one order-preserving function)

$$e: G' \to G^{\Delta}$$

of the inverted lattice G', mapping the points of G' *onto* the points of G^{Δ}.

From these simple restrictions, that the ranks of G and G^{Δ} are equal, and that the embedding is onto the points of G^{Δ}, three apparently stronger properties follow:

(i) e is cover-preserving
(ii) e is rank-preserving
(iii) e is $\wedge$-preserving.

Throughout our discussion we'll identify, whenever convenient, elements of G or elements of G', with elements of G^{Δ}, and therefore identify the copoints of G with the points of G^{Δ}.

The G-extensions have been described [2] in several equivalent ways, namely, in terms of linear subclasses, modular filters, elementary quotients and elementary strong maps. Here we need consider only linear subclasses and modular filters.

A *linear subclass* of G is a set C of copoints of G with the following property: for any copoints x, y and z of G, if x, y and z cover $x \wedge y \wedge z$, then $x, y \in C \Rightarrow z \in C$.

A *modular filter* of G is a set $M \subseteq G$ with the following properties:

(i) $x \in M, y \geq x \Rightarrow y \in M$
(ii) if $x, y \in M$ is a modular pair, then $x \wedge y \in M$.

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The correspondence between linear subclasses and modular filters of G is given by the following [2]:

PROPOSITION. *Let C be a linear subclass of G. Then the set M consisting of all elements x such that every copoint $z \geq x$ is in C is a modular filter of G. Conversely, if M is a modular filter of G, then the set of copoints in M is a linear subclass of G.* $\square$

A G-extension (and the corresponding linear subclass, etc.) is said to be *principal* if the modular filter has a least element.

The G-extensions, or, without further mention, the linear subclasses of G ordered by inclusion, form a lattice. We denote this lattice by $E(G)$. We shall see how any adjoint of G is embeddable in this lattice of extensions.

PROPOSITION. *If G^Δ is an adjoint of G, the copoints of G which, as points of G^Δ, lie beneath a flat x of G^Δ form a linear subclass $\hat{x}$ of G. The embedding $x \to \hat{x}$ of $G^\Delta \to E(G)$ is $\wedge$-preserving.*

Proof. In a geometric lattice, a set D of points is said to be linearly closed if the points in any line determined by any two points in D is in D. It's obvious that any flat $x \in G^\Delta$, when considered as a set of points of G^Δ, is linearly closed, which is the same as saying that the set of copoints of G identified with x is a linear subclass of G.

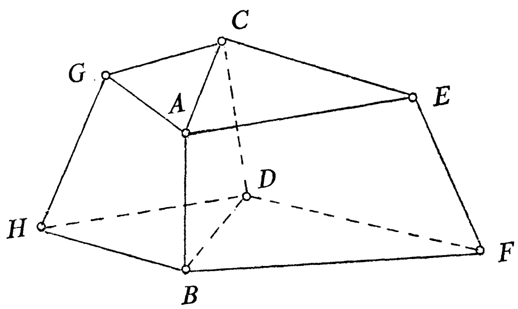
This embedding

$$G^\Delta \to E(G)$$

is $\wedge$-preserving because G^Δ and $E(G)$ are closure systems on the Boolean algebra of copoints of G, with $G^\Delta \subseteq E(G)$. $\square$

Note that the embedding $G^\Delta \to E(G)$ is not necessarily cover-preserving nor is it $\vee$-preserving. (The rank three geometry of six points in general position exhibits this phenomenon.)

Consider the 8-point geometry G:



where the only 4-point planes are

$$ABCD,\ ABEF,\ ABGH,\ CDEF \text{ and } CDGH.$$

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Consider the lines AB and CD. Let L be the linear subclass of G generated by the principal linear subclasses $\langle AB \rangle$ and $\langle CD \rangle$ and let M be the modular filter corresponding to L. Since $ABEF$, $ABGH$, $CDEF$ and $CDGH \in M$, EF is covered by $ABEF$ and $CDEF$, GH is covered by $ABGH$ and $CDGH$, so EF, $GH \in M$. Since EF, GH form a modular pair in G, so $\phi = EF \wedge GH \in M$. Hence L consists of all copoints of G, i.e. $L=1$ (of $E(G)$).

THEOREM. *The above* 8-*point geometry G has no adjoint.*

Proof. Suppose G has an adjoint G^{Δ}. Consider the flats x and y of G^{Δ} corresponding respectively to the lines AB and CD of G. Since $AB \vee_G CD = ABCD$, which covers AB and CD (in G), so x, y cover $x \wedge y$ in G^{Δ}. Since $x \vee y = 1$ in $E(G)$ and G^{Δ} is embedded in $E(G)$, we know that $x \vee y = 1$ in G^{Δ}. But in G^{Δ}, 1 does not cover x and y which contradicts the semimodularity of G^{Δ}. □

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DEPARTMENT OF PURE MATHEMATICS,
UNIVERSITY OF WATERLOO,
WATERLOO, ONTARIO, CANADA

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