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ERRATUM TO: MIXTURE REPRESENTATIONS OF RESIDUAL LIFETIMES OF USED SYSTEMS

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Abstract

We have found a mistake in the proofs of Navarro (2008, Theorem 2.3(b) and 2.3(c)) due to misapplication of properties of hazard rate and likelihood ratio orders. In this paper we show with an example that the stated results do not hold. This example is interesting since it proves some unexpected properties for these orderings under the formation of coherent systems. The result stated in Navarro (2008, Theorem 2.3(a)) for the usual stochastic order is correct.

Keywords: Coherent system; signature; stochastic ordering

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1. Introduction

Let T be the lifetime of a coherent (or mixed) system with n components having independent and identically distributed (i.i.d.) lifetimes $X_1, \dots, X_n$. Let $(s_1, \dots, s_n)$ be the signature of the system. Then the reliability function of the residual lifetime of the system can be written (see [1]) as

$$\mathbb{P}(T - t > x \mid T > t) = \sum_{i=1}^{n} p_i(t)\mathbb{P}(X_{i:n} - t > x \mid X_{i:n} > t), \tag{1.1}$$

where $p_i(t) = s_i\mathbb{P}(X_{i:n} > t)/\overline{F}_T(t)$ and $X_{1:n}, \dots, X_{n:n}$ are the order statistics from $X_1, \dots, X_n$. The function $p_i(t)$ may be identified as the probability of $\{T = X_{i:n} \mid T > t\}$. Based on this representation, the following theorem was stated in [1] for the stochastic, hazard rate, and likelihood ratio orders represented, respectively, by $\le_{\text{st}}$, $\le_{\text{hr}}$ and $\le_{\text{lr}}$; see [3] for their definitions and basic properties.

Theorem 1.1. *Let $\boldsymbol{p}_1(t)$ and $\boldsymbol{p}_2(t)$ be the vectors of the coefficients in (1.1), for a fixed $t \ge 0$, of two mixed systems with i.i.d. component lifetimes $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$, distributed according to common continuous distributions F and G, respectively. Let T_1 and T_2 be their respective lifetimes. Then:*

(i) *if $F \le_{\text{hr}} G$ and $\boldsymbol{p}_1(t) \le_{\text{st}} \boldsymbol{p}_2(t)$, then $(T_1 - t \mid T_1 > t) \le_{\text{st}} (T_2 - t \mid T_2 > t)$;*

(ii) *if $F \le_{\text{hr}} G$ and $\boldsymbol{p}_1(t) \le_{\text{hr}} \boldsymbol{p}_2(t)$, then $(T_1 - t \mid T_1 > t) \le_{\text{hr}} (T_2 - t \mid T_2 > t)$;*

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(iii) *if F and G are absolutely continuous,* $F \leq_{\mathrm{lr}} G$ *and* $\boldsymbol{p}_1(t) \leq_{\mathrm{lr}} \boldsymbol{p}_2(t)$, *then* $(T_1 - t \mid T_1 > t) \leq_{\mathrm{lr}} (T_2 - t \mid T_2 > t)$.

We have realized that the proofs of Navarro (2008, Theorems 2.3(b) and 2.3(c)) are incorrect because of a misapplication of [3, Theorems 1.B.14 and 1.C.17]. However, the result stated in Navarro (2008, Theorem 2.3(a)) for the stochastic order is completely correct. Furthermore, the following example shows that the results stated in (ii) and (iii) do not hold for system's lifetimes (i.e. $t = 0$) even when both systems have the same structure. That is, the example proves that if two coherent systems have hazard rate (or likelihood ratio) ordered i.i.d. component lifetimes, then the systems are not necessarily hazard rate (likelihood ratio) ordered. Therefore, this counterexample is interesting since it shows these unexpected properties for coherent systems. Conditions for the preservation of these orders under the formation of coherent systems were given in [2].

Example 1.1. Let us consider the system structure shown in Figure 1. Let us consider two coherent systems with this structure and i.i.d. component lifetimes X_1, X_2, X_3 and Y_1, Y_2, Y_3 with distribution functions F and G, respectively. Then the system lifetimes are

$$T_1 = \max(X_1, \min(X_2, X_3)), \qquad T_2 = \max(Y_1, \min(Y_2, Y_3)).$$

As the signature only depends on the structure, both systems have the same signature $\boldsymbol{s} = (0, \frac{2}{3}, \frac{1}{3})$. Now let us assume that

$$F(t) = (1 - \mathrm{e}^{-t})^2, \qquad G(t) = (1 - \mathrm{e}^{-t})^5 \quad \text{for } t \geq 0.$$

Then the respective probability density functions are

$$f(t) = 2\mathrm{e}^{-t}(1 - \mathrm{e}^{-t}), \qquad g(t) = 5\mathrm{e}^{-t}(1 - \mathrm{e}^{-t})^4 \quad \text{for } t \geq 0.$$

Therefore.

$$\frac{g(t)}{f(t)} = \frac{5\mathrm{e}^{-t}(1 - \mathrm{e}^{-t})^4}{2\mathrm{e}^{-t}(1 - \mathrm{e}^{-t})} = \frac{5}{2}(1 - \mathrm{e}^{-t})^3,$$

which is an increasing function for $t \geq 0$. Hence, $X_1 \leq_{\mathrm{lr}} Y_1$, where $\leq_{\mathrm{lr}}$ denotes the likelihood ratio (lr) order. As the likelihood ratio order implies the hazard rate (hr) order, we also have $X_1 \leq_{\mathrm{hr}} Y_1$. The component hazard rate functions can be seen in Figure 2 (the dashed line is the hr of X_1). However, the systems are not hazard rate ordered as can be see in Figure 3 (the dashed line is the hr of T_1). For example, if h_1 is the hazard rate of T_1 and h_2 is the hazard rate of T_2, then we have

$$h_1(1) \approx 0.699\,263\,2 > 0.111\,746\,5 \approx h_2(1), \qquad h_1(5) \approx 1.009\,475 < 1.016\,182 \approx h_2(5).$$

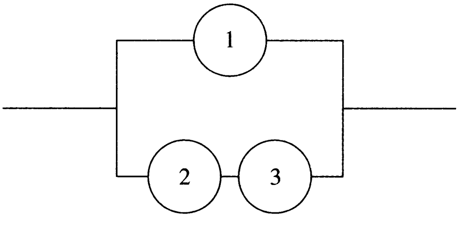

FIGURE 1: Structure of the systems studied in Example 1.1.

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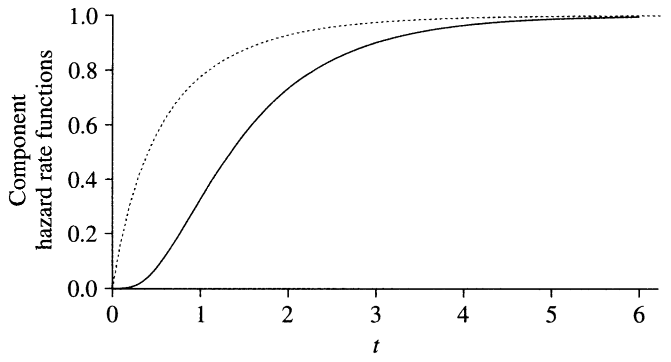

FIGURE 2: Component hazard rate functions for the systems studied in Example 1.1.

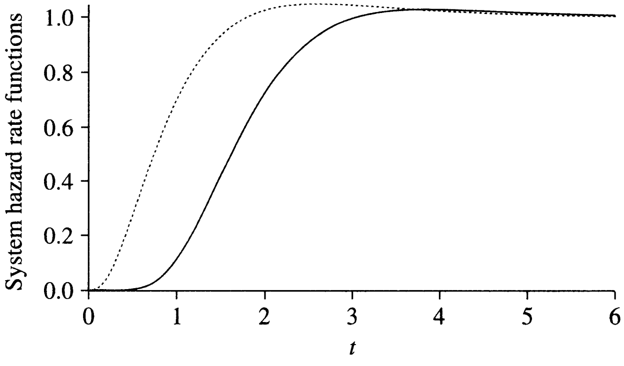

FIGURE 3: System hazard rate functions for the systems studied in Example 1.1.

This proves that the statement in Theorem 1.1(b) does not hold. As they are not hr ordered, then they are not lr ordered and, hence, Theorem 1.1(c) does not hold.

However, we would like to note that the result stated in Theorem 1.1(a) is completely correct. Note that from (1.1), we have

$$\mathbb{P}(T_1 - t > x \mid T_1 > t) = \sum_{i=1}^{n} p_i(t)\mathbb{P}(X_{i:n} - t > x \mid X_{i:n} > t)$$

and

$$\mathbb{P}(T_2 - t > x \mid T_2 > t) = \sum_{i=1}^{n} q_i(t)\mathbb{P}(Y_{i:n} - t > x \mid Y_{i:n} > t),$$

where $p_i(t) = \mathbb{P}(T_1 = X_{i:n} \mid T_1 > t)$ and $q_i(t) = \mathbb{P}(T_2 = Y_{i:n} \mid T_2 > t)$, for $i = 1, \dots, n$. Moreover, if we assume that $X_1 \leq_{\text{hr}} Y_1$, then we have

$$(X_{i:n} - t \mid X_{i:n} > t) \leq_{\text{st}} (Y_{i:n} - t \mid Y_{i:n} > t) \quad \text{for all } t \geq 0,$$

where $\leq_{\text{st}}$ denotes the (usual) stochastic (st) order, that is,

$$\mathbb{P}(X_{i:n} - t > x \mid X_{i:n} > t) \leq \mathbb{P}(Y_{i:n} - t > x \mid Y_{i:n} > t) \quad \text{for all } t, x \geq 0.$$

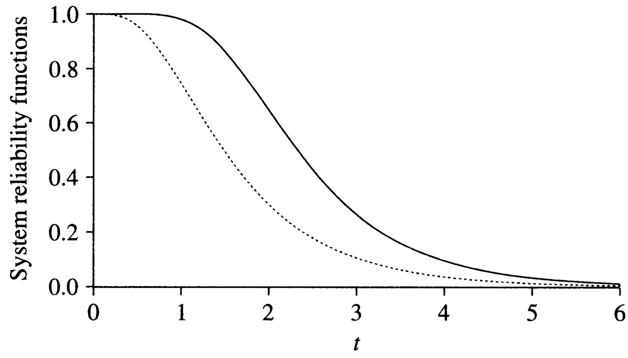

FIGURE 4: System reliability functions for the systems studied in Example 1.1.

Therefore,

$$\begin{aligned}\mathbb{P}(T_1 - t > x \mid T_1 > t) &= \sum_{i=1}^{n} p_i(t)\mathbb{P}(X_{i:n} - t > x \mid X_{i:n} > t)\\ &\le \sum_{i=1}^{n} p_i(t)\mathbb{P}(Y_{i:n} - t > x \mid Y_{i:n} > t)\\ &\le \sum_{i=1}^{n} q_i(t)\mathbb{P}(Y_{i:n} - t > x \mid Y_{i:n} > t)\\ &= \mathbb{P}(T_2 - t > x \mid T_2 > t) \quad \text{for all } t, x \ge 0,\end{aligned}$$

where the last inequality is obtained from [3, Theorem 1.A.6, p. 7]. The reliability functions of the systems considered in Example 1.1 can be seen in Figure 4 (the dashed line is the reliability function of T_1). Of course, as $X_1 \le_{\mathrm{st}} Y_1$, then $T_1 \le_{\mathrm{st}} T_2$, that is, the systems are st ordered when they are new ($t = 0$) but the used systems with age $t > 0$ are not necessarily st ordered. Furthermore, the ordering can be reversed when $t \to \infty$. For example, for $t = 5$, we have

$$(T_1 - 5 \mid T_1 > 5) >_{\mathrm{st}} (T_2 - 5 \mid T_2 > 5).$$

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