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A classification of groups with a centralizer condition

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Let G be a finite group. A nontrivial subgroup M of G is called a *CC-subgroup* if M contains the centralizer in G of each of its nonidentity elements. The purpose of this paper is to classify groups with a *CC*-subgroup of order divisible by 3. Simple groups satisfying that condition are completely determined.

1. Introduction

The purpose of this paper is to prove the following theorem which confirms a conjecture of Feit (*cf.* [9]).

THEOREM A. *Let* G *be a finite group and let* M *be a CC-subgroup of* G . *Assume that* $3 \mid |M|$. *Then one of the following statements is true:*

(i) $N_G(M) = M$;

(ii) $M \triangleleft G$ *and* G *is a Frobenius group;*

(iii) M *is a noncyclic elementary abelian* S_3*-subgroup of* G ;

(iv) M *is a cyclic subgroup of* G *of odd order.*

In [1] there is a complete classification for the third case. Case *(iv)* is dealt with in [11]. These results yield:

THEOREM B. *Let* G *be a finite simple group and let* M *be a*

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CC-subgroup of G *. Assume that* $3 \mid |M|$ *. Then* G *is isomorphic to one of the following groups:*

(a) $\mathrm{PSL}(3, 4)$;

(b) $\mathrm{PSL}(2, 2^n)$, $n \geq 2$,

(c) $\mathrm{PSL}(2, 3^n)$, $n \geq 2$;

(d) $\mathrm{PSL}(2, p^n)$, $p > 3$, $12 \nmid p^n+\varepsilon$ *for* $\varepsilon = 1$ *or* -1 .

Conversely, all groups mentioned satisfy the assumptions of Theorem B.

Several authors have studied such groups. In [2] and [5] there is a complete description of groups with CC subgroups of order 3 and 9 . The author [1] gave a complete description of groups with a CC 3-subgroup of G . Herzog [9] and [10] and Ferguson [3] and [4] classified groups with a CC subgroup under additional conditions on the group G . Suzuki [12] classified groups with a CC subgroup of even order.

Our notation is standard and taken mainly from [7].

2. Two recent results

We need two definitions and some recent (still unpublished) results of Glauberman, Fletcher, and Stewart.

DEFINITION 1. Denote the symmetric group of degree four by S^4 . We say that G is S^4-free if S^4 is not involved in G .

DEFINITION 2. Suppose $A \subseteq T \subseteq G$ are groups such that A is abelian, T is an S_2-subgroup of G and whenever $a \in A$, $g \in G$, and $a^g \in T$ then $a^g \in A$. In this situation, we say that A is a strongly closed abelian 2-group in T with respect to G .

Recently Glauberman proved:

THEOREM 1. *A non-abelian simple group* G *is* S^4*-free if and only if* G *has a nonidentity strongly closed abelian* 2*-group.*

Recently Fletcher and Stewart proved:

THEOREM 2. *Let* G *be a non-abelian simple group* G *. Assume that* *the following conditions are satisfied:*

(i) no element of G *has order* 6 *, and*

(ii) some nonidentity 2*-subgroup of* G *is normalized by an element of order* 3 .

Then centralizers of involutions in G *have normal* 2*-complements.*

The following is an immediate consequence of Theorem 2 and the general classification theorem in [8].

COROLLARY. *Suppose* G *is a non-abelian simple group satisfying the hypotheses of Theorem* 2. *Then* G *is isomorphic to either* $\mathrm{PSL}(2, q)$ *for some* q *, or* $\mathrm{PSL}(3, 4)$.

3. Proof of Theorem A

Let G be a minimal counter example. $N_G(M)$ is a Frobenius group with Frobenius kernel M . Hence M is nilpotent and $Z(M) \neq 1$. Therefore M is a Hall subgroup of G and a TI-set by [9, Theorem 2.1 and 2.3]. By [9, Corollary 2.2 (b)] G contains a normal simple subgroup G^* containing M and satisfying $N_{G^*}(M) \neq M$. By induction hypothesis $G = G^*$ is simple. If $2 \mid |M|$ then the general classification theorem in [12] implies that G has no CC subgroup M such that $N_G(M) \neq M$. Therefore $2 \nmid |M|$ and G has no element of order 6 . Let H be an arbitrary 2-subgroup of G . If $3 \mid |N_G(H)|$ then by the corollary G is isomorphic to either $\mathrm{PSL}(2, q)$, for some q , or $\mathrm{PSL}(3, 4)$, in contradiction to our assumption that M is neither cyclic of odd order nor elementary abelian. It is easy to show that G is S^4-free if and only if whenever H is a 2-subgroup of G , then S^3 is not involved in $N_G(H)/C_G(H)$. Therefore G is S^4-free and by Glauberman's Theorem and [6], G is again isomorphic to either $\mathrm{PSL}(3, 4)$ or $\mathrm{PSL}(2, q)$, for some q , a contradiction.

4. Proof of Theorem B

Let G be a counterexample. If M is of even order then G is

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isomorphic to $PSL(2, 2^n)$, $n \geq 2$, by [12], a contradiction. Assume that $2 \nmid |M|$ and hence G has no element of order 6 . It follows from the proof of Theorem A that G is isomorphic to either PSL(3, 4) or $PSL(2, q)$ for some q . It is easy to check that in the latter case q has to satisfy one of the conditions *(b)*, *(c)*, or *(d)*, a final contradiction.

REMARK. If M , of Theorem A, is either a nilpotent subgroup or is disjoint from its conjugates, then Theorem A holds true when, for *(i)*, we substitute

*(i)** G *is a Frobenius group with complement* M .

Proof. This is an immediate corollary of [7, Theorem 2.7.7] and [9, Theorem 2.1].

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