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AN ERGODIC THEOREM FOR MULTIDIMENSIONAL SUPERADDITIVE PROCESSES

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1. Introduction. The ergodic theorem for multidimensional strongly subadditive processes relative to a semigroup $\mathscr{U}$ induced by a measure preserving point transformation on X was proved by R. T. Smythe [**18**]. His results have been generalized by M. A. Akçoğlu and U. Krengel [**4**] to the continuous parameter case. The definition of superadditivity they used is stronger than Smythe's but weaker than strong superadditivity. R. Emilion and B. Hachem [**10**] extended this result to strongly superadditive processes relative to a semigroup generated by a pair of commuting Markovian operators which are also L_∞-contractions. The basic tool in the proof is a technique which may be referred to as "reduction of dimension" and they used a version of it due to A. Brunel [**6**].

The purpose of this paper is to show that if $F = \{F_{(u,v)}\}_{u>0}$ is a bounded strongly superadditive process with respect to a two-dimensional strongly continuous Markovian semigroup of operators on L_1 then $u^{-2}F_{(u,u)}$ converges a.e. as $u \to \infty$. This result in the discrete case, can be obtained from R. Emilion and B. Hachem's result. However, we give a complete proof by a different method, namely by applying a version of reduction of dimension which is less complicated and more natural than that of A. Brunel's. This method has been introduced by N. Dunford and J. T. Schwartz [**9**] and further developed by T. R. Terrell [**19**] and M. A. Akçoğlu and A. del Junco [**2**]. We also prove the continuous parameter version of this ergodic theorem for strongly superadditive processes relative to a Markovian semigroup which are also L_∞-contractions. These results generalize the results in [**4**] (as $u \to \infty$) both in discrete and continuous parameter case as well as the results in [**5**].

Let $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ be the usual two dimensional real vector space, considered together with all its usual structure. In particular $\mathbf{R}^2$ is partially ordered in the usual way, i.e., for any $(u, v), (t, r) \in \mathbf{R}^2$, $(u, v) \leqq (t, r)$ if $u \leqq t$ and $v \leqq r$, $(u, v) < (t, r)$ if $(u, v) \leqq (t, r)$ and $(u, v) \neq (t, r)$. The positive cone of $\mathbf{R}^2$ is $\mathbf{R}^2_+$ and the interior of $\mathbf{R}^2_+$ is C. By $\mathbf{N}$ and $\mathbf{N}_+$ we will denote the set of nonnegative and positive integers respectively, and we have $\mathscr{N} = \mathbf{N}^2$ and $\mathscr{N}_+ = \mathbf{N}^2_+$. Let

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$$B = \{m2^{-k}:m, k \in \mathbf{N}_+\},$$

the set of positive binary numbers, then we will denote $K = B^2$. For any $k \in \mathbf{R}$, $\mathbf{k} = (k, k) \in \mathbf{R}^2$.

Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and $L_1 = L_1(X, \mathscr{F}, \mu)$ be the classical Banach space of real-valued integrable functions on X. L_1^+ will denote the positive cone of L_1. We shall not distinguish between the equivalence classes of functions and the individual functions. The relations below are often defined only modulo sets of measure zero; the words a.e. may or may not be omitted.

Consider a strongly continuous semigroup

$$\mathscr{U} = \{U_{(t,r)}\}_{(t,r) \in \mathbf{R}^2_+}$$

of positive L_1-contractions with $U_{\mathbf{0}} = I$, the identity operator on L_1. This means that

(1.1) $U_{(t,r)}$ is a linear operator on L_1 for each $(t, r) \in \mathbf{R}^2_+$

(1.2) $U_{(t,r)}L_1^+ \subset L_1^+$ and $\|U_{(t,r)}\|_1 \leqq 1$ for each $(t, r) \in \mathbf{R}^2_+$

(1.3) $U_{(t,r)}U_{(u,v)} = U_{(t+u,r+v)}$ for each $(t, r), (u, v) \in \mathbf{R}^2_+$

(1.4) $\lim_{(t,r)\to\mathbf{0}} \|U_{(t,r)}f - f\|_1 = 0$ for each $f \in L_1$.

$\mathscr{U}$ is called a Markovian semigroup if, in addition to (1.1)-(1.4), it satisfies

(1.5) $\displaystyle\int U_{(t,r)} f d\mu = \int f d\mu$

for each $f \in L_1$ and for each $(t, r) \in \mathbf{R}^2_+$.

A family of L_1-functions $F = \{F_{(u,v)}\}_{(u,v)\in C}$ is called a *$\mathscr{U}$-superadditive process* [**14**, **3**, **5**] if

(1.6) For each $(t, r) \in \mathbf{R}^2_+$ and $(u, v) \in C$ with $\mathbf{0} \leqq (t, r) < (u, v)$,

a) $F_{(u,v)} \geqq F_{(t,v)} + U_{(t,0)}F_{(u-t,v)}$ if $0 < t < u$

b) $F_{(u,v)} \geqq F_{(u,r)} + U_{(0,r)}F_{(u,v-r)}$ if $0 < r < v$.

If $\{-F_{(u,v)}\}_{(u,v)\in C}$ is $\mathscr{U}$-superadditive, then $\{F_{(u,v)}\}_{(u,v)\in C}$ is called *$\mathscr{U}$-subadditive*; and if both $\{-F_{(u,v)}\}_{(u,v)\in C}$ and $\{F_{(u,v)}\}_{(u,v)\in C}$ are $\mathscr{U}$-superadditive, then $\{F_{(u,v)}\}_{(u,v)\in C}$ is called *$\mathscr{U}$-additive* [**2**, **3**].

A family $F = \{F_{(u,v)}\}_{(u,v)\in C}$ of L_1-functions is called a *strongly $\mathscr{U}$-superadditive* [**18**] if it satisfies

(1.7) if $\mathbf{0} < (t, r) < (u, v)$

$$F_{(t,r)} \leqq F_{(u,v)} - U_{(t,0)}F_{(u-t,v)} - U_{(0,r)}F_{(u,v-r)} + U_{(t,r)}F_{(u-t,v-r)}.$$

Any strongly $\mathscr{U}$-superadditive process $F = \{F_{(u,v)}\}_{(u,v)\in C}$ which satisfies

(1.8) $\quad F_{(0,v)} = F_{(u,0)} \equiv 0, \quad u > 0, v > 0$

is necessarily a $\mathscr{U}$-superadditive process [**18**]. Below, when we mention a strongly $\mathscr{U}$-superadditive process we will mean a process satisfying (1.7) and (1.8).

A process $F = \{F_{(u,v)}\}_{(u,v)\in C}$ is called *bounded* if it satisfies

$$(1.9) \qquad \sup_{(u,v)>\mathbf{0}} \frac{1}{uv} \left|\left|F_{(u,v)}\right|\right|_1 = \gamma_F < \infty.$$

This constant γ_F is referred to as the "time constant" of the process F [**11**, **14**].

There is another way of defining superadditivity [**2**]: For any interval $I = [a, b]$ in $\mathbf{R}^2_+$ where

$$[a, b] = [a_1, b_1] \times [a_2, b_2]$$

with $a_i \leqq b_i$ and $a_i, b_i \in \mathbf{R}$, $i = 1, 2$, define

$$(1.10) \qquad \widetilde{F}_I = U_{(a_1,a_2)}F_{(b_1-a_1,b_2-a_2)}.$$

Notice that in this case we have, for any $(u, v) \in \mathbf{R}^2_+$,

$$\widetilde{F}_{(u,v)+I} = U_{(u,v)}\widetilde{F}_I.$$

Similarly to the above, a family of L_1-functions

$$F = \{F_{(u,v)}\}_{(u,v)\in\mathscr{N}_+} \quad \text{or} \quad F = \{F_{(u,v)}\}_{(u,v)\in K}$$

defined on $\mathscr{N}_+$ or K respectively are called (strongly) $\mathscr{U}$-superadditive if they satisfy (1.6) ((1.7)) for each (u, v) in $\mathscr{N}_+$ or K.

Notice that, for each $(u, v) \in C$, $F_{(u,v)}$ is a class of functions in L_1, not an actual function. That is why to be able to speak about a.e. convergence of $u^{-2}F_{\mathbf{u}}$ when $F_{(u,v)}$ denotes equivalence class of L_1-functions and (u, v) ranges in $\mathbf{R}^2_+$, we either have to select suitable representatives or let (u, v) range through a countable set only. For convenience we will take $\mathbf{Q}^2_+$ as this countable set where $\mathbf{Q}_+$ is the set of positive rational numbers, and we will say that

$$q - \lim_{(u,v)\to(\infty,\infty)} F_{(u,v)} \text{ exists a.e.}$$

when the limit exists a.e. as (u, v) approaches (∞, ∞) along pairs of positive rationals [**2**, **3**]. This will be equivalent to the existence of

$$\lim_{(u,v)\to(\infty,\infty)} F_{(u,v)}(x)$$

when we take limit along $(u, v) \in \mathbf{R}^2_+$ for suitable choice of representatives $F_{(u,v)}(x)$ for $F_{(u,v)}$'s [**2**, **3**]. Similarly we will write

$$q - \sup_{(u,v)>\mathbf{0}} F_{(u,v)}$$

if we take the supremum over pairs of positive rationals.

2. Preliminaries.

(2.1) *Existence of the time constant*. We will give the existence of γ_F in the continuous parameter case and the proof in the discrete case follows along the same lines [**18**, **8**].

THEOREM 2.2. [**18**] *Let* $F = \{F_{(u,v)}\}_{(u,v)\in C}$ *be a positive superadditive process with respect to a positive strongly continuous semigroup of Markovian operators* $\mathcal{U} = \{U_{(t,r)}\}_{(t,r)\in\mathbf{R}^2}$ *such that* $U_{\mathbf{0}} = I$. *Then*

$$\lim_{(u,v)\to(\infty,\infty)} \int \frac{F_{(u,v)}}{uv} d\mu = \gamma_F.$$

Proof. Let

$$g(u, v) = \int F_{(u,v)} d\mu.$$

Since

$$\gamma_F = \sup_{(u,v)>\mathbf{0}} \frac{1}{uv} g(u, v),$$

it is enough to show that

$$\gamma_F = \liminf_{(u,v)\to\mathbf{0}} \frac{1}{uv} g(u, v).$$

First, assume $\gamma_F = \gamma < \infty$. Given $\epsilon > 0$, one can pick $(u_0, v_0) \in C$ such that

$$\frac{1}{u_0 v_0} g(u_0, v_0) > \gamma - \epsilon.$$

Let $(u, v) \in C$. Without loss of generality, one can assume $(u, v) > (u_0, v_0)$. Then there exist integers $n, m \in \mathbf{N}^+$ such that

$$(u, v) = (nu_0 + \delta_1, mv_0 + \delta_2)$$

where $0 \leqq \delta_1 < u_0$ and $0 \leqq \delta_2 < v_0$. Then

$$g(u, v) \geqq g(nu_0, mv_0) + g(nu_0, \delta_2) + g(\delta_1, mv_0) + g(\delta_1, \delta_2).$$

Moreover, successive applications of (1.6) (a) and (b) gives that

$$g(nu_0, mv_0) \geqq nmg(u_0, v_0).$$

Hence we have

$$(2.3) \quad \frac{g(u, v)}{uv} \geqq \frac{nm}{uv} g(u_0, v_0) + \frac{1}{uv}[g(nu_0, \delta_2) + g(\delta_1, mv_0) + g(\delta_1, \delta_2)].$$

Now, for any fixed δ_1 and δ_2, the functions $g(\delta_1, s)$ and $g(t, \delta_2)$ are one-parameter superadditive functions of s and t respectively. It is a well-known result that [**8**, **12**] for a superadditive function $g(x)$ with

$$\sup_{x>0} \frac{1}{x} g(x) < \infty,$$

$\lim_{x \to \infty} \frac{1}{x} g(x)$ exists and is finite. Thus,

$$\lim_{n\to\infty} \frac{g(nu_0, \delta_2)}{nu_0} \quad \text{and} \quad \lim_{m\to\infty} \frac{g(\delta_1, mv_0)}{mv_0}$$

exist and are finite. Consequently

$$\liminf_{(u,v)\to(\infty,\infty)} \frac{1}{uv}[g(nu_0, \delta_2) + g(\delta_1, mv_0) + g(\delta_1, \delta_2)] = 0,$$

since $g(\delta_1, \delta_2) < \infty$. Therefore (2.3) implies that

$$\liminf_{(u,v)\to(\infty,\infty)} \frac{1}{uv} g(u, v) \geqq \gamma - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this completes the proof when $\gamma < \infty$. If $\gamma = \infty$, let $M > 0$ be an arbitrary large number. Pick $(u_0, v_0) \in C$ such that

$$\frac{1}{u_0 v_0} g(u_0, v_0) > M.$$

Similarly, as in the case $\gamma < \infty$, one obtains

$$\frac{g(u, v)}{uv} \geqq \frac{mn}{uv} g(u_0, v_0)$$

where m, n are as above. Then

$$\liminf_{(u,v)\to(\infty,\infty)} \frac{1}{uv} g(u, v) > M,$$

which completes the proof in case $\gamma = \infty$.

Note that the condition $\|U_{(t,r)}\|_\infty < 1$ for each $(t, r) \in \mathbf{R}^2_+$ is not necessary for the above theorem. Also one needs only superadditivity.

(2.4) *Additive Processes.* A classical example of an additive process

$G = \{G_t\}_{t>0}$ or $G = \{G_{(u,v)}\}_{(u,v)\in C}$ is

(2.5) $$G_t = \int_0^t T_s f ds \quad \text{or}$$

$$G_{(u,v)} = \int_0^u \int_0^v U_{(s_1,s_2)} f ds_1 ds_2$$

for $f \in L_1$, where

$$\tau = \{T_t\}_{t\geqq 0} \quad \text{and} \quad \mathscr{U} = \{U_{(t,r)}\}_{(t,r)\in \mathbf{R}^2}$$

are one and two dimensional strongly continuous semigroups of (positive) L_1-contradictions with $T_0 = I$ and $U_{\mathbf{0}} = I$, and the integrals are defined as the L_1-limit of the corresponding Riemann sums. These kinds of additive processes have been studied quite extensively in the literature [**1**, **15**, **9**, **16**, **17**]. Concerning the a.e. convergence of $t^{-1}G_t$ or $u^{-2}G_{\mathbf{u}}$ as $t \to \infty$ or $u \to \infty$, the properties of the additive processes of the form (2.5) are shared with general τ-additive and $\mathscr{U}$-additive processes. For, suppose $G = \{G_n\}_{n\in\mathbf{N}}$ is a τ-additive process where $\tau = \{T^k\}_{k\in\mathbf{N}}$ for some (positive) L_1-contraction T. Then

$$G_n = \sum_{i=0}^{n-1} T^i G_1,$$

hence

$$n^{-1}G_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i G_1.$$

Similarly for $G = \{G_{(n,m)}\}_{(n,m)\in\mathscr{N}}$, we have

$$\frac{1}{n^2}G_{\mathbf{n}} = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} T^i S^j G_{\mathbf{1}}$$

where G is $\mathscr{U}$-additive with

$$\mathscr{U} = \{T^n S^n\}_{(n,m)\in\mathscr{N}}$$

where T and S are commuting positive L_1-contractions. A.e. existence of the limit

$$\frac{1}{n}\sum_{i=0}^{n-1} T^i f$$

is a classical result now [**13**, **9**] for T is an L_1-contraction which is also an L_∞-contraction. We also have the following Maximal Ergodic Theorem [**9**]:

THEOREM 2.6. *Let T be a positive linear L_1-contraction with $\|T\|_\infty \leqq 1$, and let G be a $\tau = \{T^k\}_{k \in \mathbf{N}}$-additive process. Then for each $\alpha > 0$*

$$\mu(E) \leqq \frac{2}{\alpha} \int_{e(\alpha)} |G_1| d\mu \tag{2.7}$$

where

$$e(\alpha) = \{x \in X: |G_1(x)| > \alpha\} \quad \text{and}$$
$$E = \left\{x \in X: \sup_{n \geqq 1} \left|\frac{1}{n} G_n(x)\right| > \alpha\right\}.$$

Furthermore, let G be positive and bounded and T be Markovian. Then

$$\int_{e(\alpha)} G_1 d\mu \leqq \int G_1 d\mu = \frac{1}{n} \int G_n d\mu \leqq \gamma_G.$$

Therefore

$$\mu(E) \leqq \frac{2}{\alpha} \gamma_G. \tag{2.8}$$

For a continuous parameter positive τ-additive process $G = \{G_t\}_{t>0}$ we have

$$G_n - T_n G_{r-n} \leqq G_r \leqq G_{n+1} + T_n G_{r-n} \tag{2.9}$$

where r is a rational number with $n < r < n + 1$ for some $n \in \mathbf{N}_+$. If we let

$$\omega = q - \sup_{0 < t \leqq 1} G_t,$$

then $\omega \in L_1^+$. Define

$$G'_n = \sum_{i=0}^{n-1} T_1^i \omega, \quad n \geqq 1.$$

So, $G' = \{G'_n\}_{n \in \mathbf{N}}$ is a positive additive process. Observing that $\omega \geqq G_p$ for any rational p with $0 < p \leqq 1$, we have

$$T_n G_{r-n} \leqq T_n \omega = G'_{n+1} - G'_n$$

for any rational r with $n < r < n + 1$. Thus (2.9) implies that

$$G_n - (G'_{n+1} - G'_n) \leqq G_r \leqq G_{n+1} + (G'_{n+1} - G'_n).$$

Since both $\{G_n\}_{n \in \mathbf{N}}$ and $\{G'_n)_{n \in \mathbf{N}}$ are (discrete) positive additive processes,

$$\lim_{n \to \infty} \frac{1}{n} G_n \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} G'_n$$

exist and are finite a.e. Consequently

$$q - \lim_{t\to\infty}(1/t)G_t \text{ exists a.e.}$$

Moreover, we also have the maximal ergodic theorem for continuous parameter additive processes. Before giving this theorem, for convenience, we will adopt the following notation: It is known that $\int_0^t T_s f ds$ is defined as the L_1-limit of the corresponding Riemann sums. For our purposes, we will take a particular type of Riemann sums given as $I_0^k f = 0$ and

$$I_t^k f = \frac{1}{2^k}\sum_{i=0}^{[t2^k]} T^i_{2^{-k}} f$$

for any integer $k \geqq 1$ and $t > 0$ and $f \in L_1$, where $[a]$ is the largest integer strictly less than a for any $a \in \mathbf{R}$.

THEOREM 2.10. *Let $G = \{G_t\}_{t>0}$ be a bounded, positive, τ-additive process, where $\tau = \{T_t\}_{t\geqq 0}$ is a strongly continuous Markovian semigroup. Then, for each $\alpha > 0$,*

$$\mu(E) \leqq \frac{2}{\alpha}\gamma_G,$$

where

$$E = \left\{x: q - \sup_{t>0}\frac{1}{t}G_t(x) > \alpha\right\}.$$

Proof. Let $t \in B$ and let k be a positive integer such that $2^k t \in \mathbf{N}_+$. Then

$$G_t = L_1 - \lim_{k\to\infty}\frac{1}{2^k}\sum_{i=0}^{2^k t-1} T^i_{2^{-k}} G_{2^{-k}}.$$

Let

$$f_k^* = \sup_{1\leqq m<\infty}\frac{1}{m}\sum_{i=0}^{m-1} T^i_{2^{-k}} G_{2^{-k}}.$$

Then for each $\epsilon > 0$, there exists $n_0 \in \mathbf{N}_+$ with

$$f_k^* \geqq \frac{1}{t}G_t - \epsilon \text{ a.e. for } k \geqq n_0.$$

Thus

$$\liminf_{k\to\infty} f_k^* \geqq \frac{1}{t}G_t \text{ a.e.}$$

Since $\{G_t\}_{t>0}$ is continuous in t on $(0, \infty)$ by additivity and boundedness, we have

$$\sup_{t\to 0} \frac{1}{t} G_t = q - \sup_{r>0} \frac{1}{r} G_r,$$

thus

$$\liminf_{k\to\infty} f_k^* \geqq q - \sup_{r>0} \frac{1}{r} G_r.$$

Let

$$f^* = q - \sup_{r>0} \frac{1}{r} G_r,$$

then

$$\liminf_{k\to\infty} f_k^*(x) \geqq f^*(x) \text{ a.e.}$$

Let $E_k = \{x: f_k^*(x) > \alpha\}$, then

$$E \subset \liminf_{k\to\infty} E_k.$$

Now, by Fatou's Lemma,

$$\mu(E) \leqq \liminf_{k\to\infty} \mu(E_k).$$

Therefore,

$$\frac{\alpha}{2} \mu(E) \leqq \liminf_{k} \frac{\alpha}{2} \mu(E_k)$$

$$\leqq \liminf_{k\to\infty} \int_{e(\alpha)} G_{2^k} d\mu$$

by Theorem 2.6, where $e(\alpha) = \{x: G_{2^k} > \alpha\}$. Since

$$\int G_{2^k} d\mu \leqq \gamma_G 2^k,$$

we have

$$\frac{\alpha}{2} \mu(E) \leqq \liminf_{k\to\infty} \frac{\gamma_G}{2^k} \leqq \gamma_G.$$

In the two dimensional case the a.e. existence of

$$\lim_{n\to\infty} \frac{1}{n^2} G_{\mathbf{n}}$$

is proved by A. Brunel [**6**]. Although it is not stated and proved in [**9**] all

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the necessary arguments for the proof of this result are included in [**9**] which are actually straightforward and self-contained. Brunel's proof, which is different from the arguments in [**9**], involves more complicated tools. That is why we will, nevertheless, sketch a proof for this theorem for general additive processes following the arguments in [**9**]. First we will need the following lemma which is stated and proved in [**9**], but we will state it here in terms of general additive processes.

LEMMA 2.11. *Let T and S be two Markovian operators which commute and* $\|T\|_\infty \leqq 1$ *and* $\|S\|_\infty \leqq 1$. *Let*

$$\mathscr{U} = \{T^n S^m\}_{(n,m)\in\mathscr{N}};$$

and let

$$G = \{G_{(n,m)}\}_{(n,m)\in\mathscr{N}}.$$

be a positive, bounded, $\mathscr{U}$*-additive process. If*

$$f^* = \sup_{n\geqq 1} (1/n^2)G_{\mathbf{n}},$$

then there exists a constant $K > 0$, *independent of G and* $\mathscr{U}$, *such that*

$$\mu(E) \leqq \frac{1}{K\cdot\alpha}\gamma_G \quad \text{for each } \alpha > 0,$$

where $E = \{x:f^*(x) > \alpha\}$.

THEOREM 2.12. *Let T, S and* $\mathscr{U}$ *be as in Lemma* 2.11. *If*

$$G = \{G_{(n,m)}\}_{(n,m)\in\mathscr{N}}.$$

is a bounded, $\mathscr{U}$*-additive process, then*

$$\lim_{n\to\infty} \frac{1}{n^2}G_{\mathbf{n}}$$

exists and is finite a.e.

Proof.

$$G_{\mathbf{n}} = \sum_{i,j=0}^{n-1} T^i S^j G_{\mathbf{1}}$$

for any $n \in \mathbf{N}_+$. The conclusion of the theorem is true if $G_{\mathbf{1}} \in L_p, p > 1$, and in this case $G_{\mathbf{n}} \in L_p$ for each $n \in \mathbf{N}_+$; $p > 1$ [**9**].

Now, first, without loss of generality one can assume that G is positive. Since $L_1 \cap L_p$ is dense in L_1 for each $p > 1$ in L_1-topology, given $\epsilon > 0$, there exists $g \in L_1 \cap L_p$ such that

$$\|G_{\mathbf{1}} - g\|_1 < \epsilon.$$

Also

$$\frac{1}{n^2}\sum_{i,j=0}^{n-1} T^iS^jG_1$$

$$= \frac{1}{n^2}\sum_{i,j=0}^{n-1} T^iS^j(G_1 - S) + \frac{1}{n^2}\sum_{i,j=0}^{n-1} T^iS^jg.$$

Consider the set

$$\left\{x: \limsup_{n\to\infty} \frac{1}{n^2}G_n(x) > \liminf_{n\to\infty} \frac{1}{n^2}G_n\right\}.$$

Then to prove the theorem it is enough to show that this set has measure zero. Since $G_1 = (G_1 - g) + g$ and since

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{i,j=0}^{n-1} T^iS^jg \text{ exists a.e.,}$$

it is enough to consider $G_1 - g$ instead of G_1 and it is enough to show $\mu(E) = 0$ where

$$E = \left\{x: \limsup_{n\to\infty} \frac{1}{n^2}\sum_{i,j=0}^{n-1} T^iS^j|G_1 - s|\right.$$

$$\left. - \liminf_{n\to\infty} \frac{1}{n^2}\sum_{i,j=0}^{n-1} T^iS^j|G_1 - s| > \alpha\right\},$$

for any $\alpha > 0$. Let

$$E' = \left\{x: \limsup_{n\to\infty} \frac{1}{n^2}\sum_{i,j=0}^{n-1} T^iS^j|G_1 - s| > \alpha\right\}.$$

Then $E \subseteq E'$ and hence $\mu(E) \leqq \mu(E')$. By Lemma 2.11

$$\mu(E') \leqq \frac{1}{k\alpha}\gamma',$$

where γ' is the time constant of the process given by

$$\left\{\frac{1}{n^2}\sum_{i,j=0}^{n-1} T^iS^j|G_1 - s|\right\}_{n\geqq 1}.$$

But $\gamma' = \|G_1 - g\|_1 < \epsilon$. Hence

$$\mu(E) \leqq \frac{\epsilon}{k\alpha},$$

giving the result desired.

Remark 2.13. The conditions $\|T\|_\infty \leqq 1$ and $\|S\|_\infty \leqq 1$ are necessary since if this condition is dropped, then a.e. convergence may not hold [7].

(2.14) *Reduction of Dimension.* In this section, given any two-dimensional strongly continuous semigroup

$$\mathcal{U} = \{U_{(t,r)}\}_{(t,r)\in \mathbf{R}^2_+}$$

of positive L_1-contractions and a bounded $\mathcal{U}$-additive process

$$G = \{G_{(u,v)}\}_{(u,v)\in C},$$

we will define a one-dimensional semigroup τ and a τ-additive process $H = \{H_t\}_{t>0}$ by using a technique introduced by N. Dunford and J. T. Schwartz **[9]** and further developed by T. R. Terrell **[19]** and M. A. Akçoğlu and A. del Junco **[2]**. Here we will only give the results and properties of this technique and omit the details, for it is given in **[19]** and **[2]** explicitly with proofs.

For any $x \in (0, \infty)$ and $\beta \in \mathbf{R}$, let

$$(2.15)\quad \Phi_x(\beta) = \begin{cases} \dfrac{x}{2\sqrt{\beta}}\beta^{-3/2}e^{-x^2/4\beta} & \text{if } \beta > 0 \\ 0 & \text{if } \beta \leqq 0 \end{cases}$$

and for $(u, v) \in \mathbf{R}^2_+$, define

$$\Phi_x(u, v) = \Phi_x(u) \cdot \Phi_x(v).$$

Then, for each fixed $x \in (0, \infty)$, $\Phi_x:\mathbf{R}^2 \to \mathbf{R}$ is a nonnegative continuous function vanishing on $\mathbf{R}^2 \backslash \mathbf{R}^2_+$. Moreover,

$$\int_{\mathbf{R}^2} \Phi_x(u, v)dudv = 1 \quad \text{and}$$

$$\int_{\mathbf{R}^2} \Phi_x(t - u, r - v)\Phi_y(u, v)dudv = \Phi_{x+y}(t, r)$$

for each $(t, r) \in \mathbf{R}^2_+$ and $x, y \in (0, \infty)$.

Given any strongly continuous semigroup

$$\mathcal{U} = \{U_{(t,r)}\}_{(t,r)\in \mathbf{R}^2_+}$$

of positive L_1-contractions, if we define

$$(2.16)\quad L_xf = \int_{\mathbf{R}^2} \Phi_x(u, v)U_{(u,v)}fdudv,$$

for $x \in (0, \infty)$ and $f \in L_1$, then $\tau = \{L_x\}_{x>0}$ has the following properties:

(2.17) L_x is a positive linear contraction on L_1 for any $x > 0$.
(2.18) $L_x L_y = L_{x+y}$ for each $x, y \in (0, \infty)$.
(2.19) $\|L_x\|_\infty \leqq 1$ for each $x \in (0, \infty)$ if

$$\|U_{(t,r)}\|_\infty \leqq 1 \quad \text{for each } (t, r) \in \mathbf{R}^2_+.$$

(2.20) L_x is Markovian for each $x \in (0, \infty)$ if $\mathscr{U}$ is Markovian.
(2.21) $\{L_x\}_{x>0}$ is strongly continuous. If $\mathscr{U}$ is strongly continuous with $U_0 = I$, then $\{L_x\}$ is also continuous at $x = 0$ and $L_0 = I$.

Let $G = \{G_{(u,v)}\}_{(u,v)\in C}$ be a (positive) $\mathscr{U}$-additive process. For $x \in (0, \infty)$ define

$$(2.22) \quad h_x = \int_{\mathbf{R}^2} \Phi_x(u, v)\widetilde{G}(du, dv),$$

and also define a new process $H = \{H_a\}_{a>0}$ by

$$(2.23) \quad H_a = \int_0^a h_x dx, \quad a > 0.$$

Here we use the definition (1.10). Then this new process H is a (positive) τ-additive process. Also, the following lemma is known [**2**]:

LEMMA 2.24. *Given a positive, bounded $\mathscr{U}$-additive process*

$$G = \{G_{(u,v)}\}_{(u,v)\in C},$$

then there exists a constant $\delta > 0$, independent of G and $\mathscr{U}$, such that

$$(2.25) \quad \frac{\delta}{\alpha^2} G_\alpha \leqq \frac{1}{\sqrt{\alpha}} H_{\sqrt{\alpha}}$$

for each $\alpha > 0$, where $H = \{H_t\}_{t>0}$ and τ are as defined in (2.23) *and* (2.16) *above.*

3. Ergodic theorems. Before proving the main result, we will give two technical lemmas. The first one is originally due to R. T. Smythe [**18**], in the one-dimensional case, in the form we will give here, is given by M. A. Akçoğlu and L. Sucheston [**5**], and by R. Emilion and B. Hachem [**10**] in the two-dimensional case. But [**10**] is available only as an announcement and the lemma is stated in it without proof. For this reason we provide a proof. Before stating it, we find it convenient to give a notation: for any $(n, m) \in \mathscr{N}$ and $k, l \in \mathbf{N}_+$, let

$$\Theta_k F_{(n,m)} = F_{(n+k,m)},$$
$$\phi_l F_{(n,m)} = F_{(n,m+l)} \quad \text{and}$$
$$\tau_k F_{(n,m)} = (\Theta_k - U_{(k,0)})F_{(n,m)},$$
$$\sigma_l F_{(n,m)} = (\phi_l - U_{(0,l)})F_{(n,m)}.$$

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Notice that, in this case, (1.7) takes the form

$$(1.7') \qquad F_{(k,l)} \leqq \sigma_l \tau_k F_{(n,m)}.$$

LEMMA 3.1. [5, 10] *Let* $F = \{F_{(n,m)}\}_{(n,m)\in\mathcal{N}}$ *be a bounded, positive, strongly* $\mathcal{U}$*-superadditive process, where* $\mathcal{U} = \{T^n S^m\}$ *with* T *and* S *commuting Markovian operators such that* $\|T\|_\infty \leqq 1$ *and* $\|S\|_\infty \leqq 1$. *Then there exist positive* $\mathcal{U}$*-additive processes* G^m, $m = 1, 2, \ldots$, *such that*

$$(3.2) \qquad G^m_{\mathbf{n}} \geqq \left(1 - \frac{n}{m}\right)^2 F_{\mathbf{n}}$$

for each $m \geqq 1$ *and* $1 \leqq n < m$.

Proof. Let $\{\eta(m)\}_{m\geqq 1}$ be a sequence of elements in L_1^+ defined by

$$\eta(m) = \frac{1}{m^2}\sum_{i,j=1}^{m}[F_{(i,j)} - TF_{(i-1,j)} - SF_{(i,j-1)} + TSF_{(i-1,j-1)}].$$

Define a process G^m by

$$G^m_{\mathbf{n}} = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} T^k S^l \eta(m), \quad m \geqq 1.$$

This is a $\mathcal{U}$-additive process. It is known that [5], for $1 \leqq n < m$,

$$\sum_{k=0}^{n-1} T^k\left[\sum_{i=1}^{m}(F_{(i,j)} - TF_{(i-1,j)})\right]$$

$$= (I - T^n)\sum_{i=1}^{m-1} F_{(i-j)} + \sum_{i=0}^{n-1} T^i F_{(m,j)}$$

and

$$\sum_{l=0}^{n-1} S^l\left[\sum_{j=1}^{m}(F_{(i,j)} - SF_{(i,j-1)})\right]$$

$$= (I - S^n)\sum_{j=1}^{m-1} F_{(i,j)} + \sum_{j=0}^{n-1} S^j F_{(i,m)}.$$

Therefore, for $m \geqq 1$ and $1 \leqq n < m$,

$$m^2 G^m_{\mathbf{n}} = (I - T^n)\sum_{i=1}^{m-1}\left[(I - S^n)\sum_{j=1}^{m-1} F_{(i,j)} + \sum_{j=0}^{n-1} S^j F_{(i,m)}\right]$$

$$+ \sum_{i=0}^{n-1} T^i\left[(I - S^n)\sum_{j=1}^{m-1} F_{(m,j)} + \sum_{j=0}^{n-1} S^j F_{(m,m)}\right].$$

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Moreover, if

$$k_{n,m} = (I - L^n) \sum_{t=1}^{m-1} F_t + \sum_{t=0}^{n-1} L_t F_m,$$

where L is an operator on L_1 and F is a process, then

$$k_{n,m} = \sum_{t=1}^{n-1} F_t + \sum_{t=n}^{m-1} (F_t - S^n F_{t-n}) + \sum_{t=0}^{n-1} (S^t F_m - S^n F_{m-n+t}).$$

Hence

$$\begin{aligned} m^2 G_{\mathbf{n}}^m = & \sum_{i,j=1}^{n-1} F_{(i,j)} \\ & + \sum_{i=1}^{n-1} \sum_{j=n}^{m-1} \sigma_n F_{(i,j-n)} + \sum_{i=1}^{n-1} \sum_{j=0}^{n-1} S^j \sigma_{n-j} F_{(i,m+j-n)} \\ & + \sum_{i=n}^{m-1} \sum_{j=1}^{n-1} \tau_n F_{(i-n,j)} + \sum_{i,j=n}^{m-1} \tau_n \sigma_n F_{(i-n,j-n)} \\ & + \sum_{i=n}^{m-1} \sum_{j=0}^{n-1} S^j \tau_n \sigma_{n-j} F_{(i-n,m+j-n)} \\ & + \sum_{i=0}^{n-1} \sum_{j=1}^{n-1} T^i \tau_{n-i} F_{(m+i-n,j)} \\ & + \sum_{i=0}^{n-1} \sum_{j=n}^{m-1} T^i \tau_{n-i} \sigma_n F_{(m+i-n,j-n)} \\ & + \sum_{i,j=0}^{n-1} T^i S^j \tau_{n-i} \sigma_{m-j} F_{(m+i-n,m+j-n)} . \end{aligned}$$

Since $F_{(n,m)} \geqq 0$ for each $(n, m) \in \mathcal{N}_+$ and since T and S are positive operators, we have

$$\begin{aligned} m^2 G_{\mathbf{n}}^m & \geqq \sum_{i,j=n}^{m-1} \tau_n \sigma_n F_{(i-n,j-n)} \\ & \geqq \sum_{i,j=n}^{m-1} F_{\mathbf{n}} = (m - n)^2 F_{\mathbf{n}} \end{aligned}$$

by (1.7′), giving (3.2). For all values of n, m with $1 \leqq n < m$, this process $G^m = \{G^m_{\mathbf{n}}\}_{n \geqq 1}$ is positive since $\{F_{(n,m)}\}$ is positive. Then, by additivity, it is positive for each $n \in \mathbf{N}_+$.

LEMMA 3.3. *Let F and $\mathscr{U}$ be as in Lemma* 3.1. *If*

$$f^* = \limsup_{n \geqq 1} \frac{1}{n^2} F_{\mathbf{n}},$$

then there exists a constant $K > 0$ which is independent of F on $\mathscr{U}$ such that

$$\mu(E) \leqq \frac{1}{K\alpha} \gamma_F$$

for each $\alpha > 0$, where $E = \{x: f^*(x) > \alpha\}$.

Proof. By Lemma 3.1, for each $m \leqq 1$, there is a positive, bounded $\mathscr{U}$-additive process $\{G^m_n\}_{n \leqq 1}$ such that

$$G^m_{\mathbf{n}} \geqq \left(1 - \frac{n}{m}\right)^2 F_{\mathbf{n}} \quad \text{for } i \leqq n < m.$$

On the other hand, as is shown in the proof of Lemma 2.11 following [9], there exists a constant $d > 0$, independent of f and $\mathscr{U}$, and $u = u(n)$ such that, for any $f \in L_1$,

$$\frac{1}{n^2} \sum_{i,j=0}^{n-1} T^i S^j f \leqq \frac{1}{u^2 d} \int_0^u \int_0^u U'_{(t,r)} f dt dr$$

where

$$U'_{(t,r)} = e^{-(t+r)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^i r^j}{i! j!} T^i S^j$$

as in [9]. Also, by Lemma 2.24 there is a constant $\delta > 0$, independent of f and

$$\mathscr{U}' = \{U'_{(t,r)}\}_{(t,r) \in \mathbf{R}^2_+},$$

such that

$$\frac{1}{u^2} \int_0^u \int_0^u U'_{(t,r)} f dt dr \leqq \frac{1}{\delta\sqrt{u}} H_{\sqrt{u}}, \quad f \in L_1,$$

where $H = \{H_t\}_{t > 0}$ is a positive, bounded, τ-additive process, with $\tau = \{L_x\}_{x \geqq 0}$ and

$$L_x(\cdot) = \int_{\mathbf{R}^2} \Phi_x(u, v) U'_{(u,v)}(\cdot) du dv$$

as defined in Section (2.14). Thus, for each $m \geqq 1$ and $1 \leqq n < m$, there is a $u = u(n)$ and a constant $K > 0$, independent of F and T and S, such that

$$(3.4) \quad \left(1 - \frac{n}{m}\right)^2 F_n \leqq \frac{1}{Kt} H_t, \quad t = \sqrt{u},$$

where

$$H_t = \int_0^t \left[\int_{R^2} \Phi_x(u, v) U'_{(u,v)} du dv \right] \eta(m) dx.$$

Since $H = \{H_t\}_{t>0}$ is a positive additive process, for any $t > 0$ with $k - 1 < t < k$, $k \in \mathbf{N}_+$,

$$H_t = \sum_{i=0}^{k-2} \int_0^1 L_{i+x} \eta(m) dx + \int_0^{t-(k-1)} L_{(k-1)+x} \eta(m) dx.$$

Since

$$\int_0^{t-(k-1)} L_{(k-1)+x} \eta(m) dx \leqq \int_0^1 L_{(k-1)+x} \eta(m) dx,$$

we see that

$$H_t \leqq \sum_{i=0}^{k-1} L_1^i \, \overline{\eta}(m),$$

where

$$\overline{\eta}(m) = \int_0^1 L_x \eta(m) dx$$

is an L_1^+-function for each $m \geqq 1$. Now,

$$H_t \leqq H_k = \sum_{i=0}^{k-1} L_1^i \overline{\eta}(m) \quad \text{for } k - 1 < t < k, \ k \in \mathbf{N}_+.$$

Thus

$$\frac{H_t}{t} \leqq \left(\frac{k}{k-1}\right) \frac{H_k}{k} \quad \text{for } k - 1 < t < k.$$

It is known [5] that for any positive bounded superadditive process $\{H_k\}_{k \geqq 1}$ there exists an L_1^+-function Δ such that

$$H_k \leqq \sum_{i=0}^{k-1} L_1^i \Delta, \quad k \geqq 1.$$

Thus,

$$\frac{H_t}{t} \leqq \left(\frac{k}{k-1}\right) \left[\frac{1}{k} \sum_{i=0}^{k-1} L_1^i \Delta \right].$$

Moreover

$$\int \Delta d\mu = \gamma_H$$

[5]. Now, if

$$h^* = \limsup_{k\to\infty} \frac{1}{t} \sum_{i=0}^{k-1} L_1^i \Delta,$$

then

$$\limsup_{k\to\infty} \frac{H_k}{k} \leqq h^*,$$

and consequently,

$$\limsup_{k\to\infty} (1/t)H_t \leqq h^*.$$

Therefore, by (3.4),

$$\limsup_{n\to\infty} \frac{1}{n^2} F_{\mathbf{n}} \leqq \frac{1}{K} h^*.$$

Thus

$$f^* \leqq \frac{1}{K} h^*,$$

and hence

$$\mu(\{x:f^*(x) > \alpha\}) \leqq \mu(\{x:h^*(x) > \alpha K\}).$$

Then,

$$\mu(E) \leqq \frac{1}{\alpha K} \gamma_H.$$

Since

$$\int \Delta d\mu = \gamma_F = \lim_{m\to\infty} \int \eta(m) d\mu,$$

we have $\gamma_H = \gamma_F$ and hence

$$\mu(E) \leqq \frac{1}{\alpha K} \gamma_F.$$

THEOREM 3.5. *Let* $F = \{F_{(n,m)}\}_{(n,m)\in\mathcal{N}}$. *be a bounded strongly* $\mathcal{U}$*-superadditive process with*

$$\mathcal{U} = \{U_{(n,m)}\}_{(n,m)\in\mathcal{N}},$$

where each $U_{(n,m)}$ *is a Markovian operator on* L_1 *such that*

$$\|U_{(n,m)}\|_\infty \leqq 1, \quad (n, m) \in \mathcal{N}.$$

Then

$$\lim_{n\to\infty} \frac{1}{n^2} F_{\mathbf{n}} \textit{ exists a.e.}$$

Proof. Given $\epsilon > 0$, find $n_0 \in \mathbf{N}_+$ such that

$$n_0^{-2} \int F_{\mathbf{n}_0} d\mu > \gamma_F - \epsilon.$$

Then form a process

$$G_{\mathbf{n}} = \sum_{i,j=0}^{n-1} U_{(in_0, jn_0)} F_{\mathbf{n}_0}.$$

Then $G = \{G_{\mathbf{n}}\}_{n\geqq 1}$ is a bounded $\mathcal{U}'$-additive process, where

$$\mathcal{U}' = \{U_{(kn_0, ln_0)}\}_{(k,l)\in\mathbf{N}}.$$

Also

$$\gamma_G = \lim_{n\to\infty} \frac{1}{n^2} \int G_{\mathbf{n}} d\mu$$
$$= \lim_{n\to\infty} \left[\frac{1}{n^2} \int \sum_{i,j=0}^{n-1} U_{(in_0, jn_0)} F_{\mathbf{n}_0} d\mu\right]$$

and hence $\gamma_G > \gamma_F - \epsilon$. Notice that, by superadditivity, $F_{\mathbf{n}} \geqq G_{\mathbf{n}}$ on the points $\mathbf{n} = k\mathbf{n}_0$, $k = 0, 1, 2, \ldots$. Thus $F' = \{F'_{\mathbf{n}}\}$, where $F'_{\mathbf{n}} = F_{\mathbf{n}} - G_{\mathbf{n}}$ is a positive, bounded, strongly $\mathcal{U}$-superadditive process for $n = kn_0$, $k = 0, 1, 2, \ldots$. Moreover,

$$\gamma_{F'} = \gamma_F - \gamma_G < \epsilon.$$

Since

$$\lim_{n\to\infty} \frac{1}{n^2} G_{\mathbf{n}} \text{ exists a.e.}$$

by Theorem 2.12, and since

$$\mu(E) \leqq \frac{1}{K\alpha}\gamma_{F'} = \frac{\epsilon}{K\alpha},$$

for each $\alpha > 0$, where

$$E = \{x: \limsup_{k\geqq 0} \frac{1}{(kn_0)^2}(F_{\mathbf{kn}_0} - G_{\mathbf{kn}_0})(x) > \alpha\},$$

we see that

$$\lim_{n\to\infty} \frac{1}{n^2} F_{\mathbf{n}} \text{ exists a.e.}$$

Next, we will prove the ergodic theorem for continuous strongly superadditive processes.

THEOREM 3.6. *Let $F = \{F_{(u,v)}\}_{(u,v)\in C}$ be a bounded strongly $\mathscr{U}$-superadditive process where*

$$\mathscr{U} = \{U_{(t,r)}\}_{(t,r)\in \mathbf{R}^2_+}$$

is a strongly continuous Markovian semigroup on L_1 such that

$$\|U_{(t,r)}\|_\infty \leqq 1 \quad \text{for each } (t, r) \in \mathbf{R}^2_+$$

and $U_{\mathbf{0}} = I$. Assume that

$$\Omega = \sup \{ |U_{(t,r)}F_{(u,v)}| : t, r, u, v \in \mathbf{Q}_+,$$
$$t + u \leqq 1, r + v \leqq 1\} \in L_1.$$

Then $q - \lim_{u\to\infty} \frac{1}{u^2} F_{\mathbf{u}}$ *exists* a.e.

Proof. Let $n \in \mathbf{N}_+$ and r be a rational number with $n < r < n + 1$. By superadditivity,

$$F_{\mathbf{r}} \geqq F_{\mathbf{n}} + U_{(n,0)}F_{(r-n,n)} + U_{(0,n)}F_{(n,r-n)} + U_{(n,n)}F_{\mathbf{r}-\mathbf{n}}$$
$$\geqq F_{\mathbf{n}} + \sum_{j=0}^{n-1} U_{(n,j)}F_{(r-n,1)} + \sum_{i=0}^{n-1} U_{(i,n)}F_{(1,r-n)} + U_{\mathbf{n}}F_{\mathbf{r}-\mathbf{n}}.$$

Therefore, if

$$Y_{r,n} = \sum_{i=0}^{n-1} U_{(i,n)}|F_{(1,r-n)}|$$
$$+ \sum_{j=0}^{n-1} U_{(n,j)}|F_{(r-n,1)}| + U_{\mathbf{n}}F_{\mathbf{r}-\mathbf{n}},$$

then $F_{\mathbf{r}} \geqq F_{\mathbf{n}} - Y_{r,n}$. Similarly,

$$F_{\mathbf{n}+\mathbf{1}} \geqq F_{\mathbf{r}} + U_{(r,0)}F_{(n+1-r,r)} + U_{(0,r)}F_{(r,n+1-r)} + U_{\mathbf{r}}F_{\mathbf{n}+\mathbf{1}-\mathbf{r}}$$
$$\geqq F_{\mathbf{r}} + \sum_{i=0}^{n-1} U_{(i,n)}[U_{(0,r-n)}F_{(1,n+1-r)}]$$
$$+ \sum_{j=0}^{n-1} U_{(n,j)}[U_{(r-n,0)}F_{(n+1-r,1)}]$$

$$+ U_{\mathbf{n}}[U_{(r-n,0)}F_{(n+1-r,r-n)}$$
$$+ U_{(0,r-n)}F_{(r-n,n+1-r)}$$
$$+ U_{\mathbf{r}-\mathbf{n}}F_{\mathbf{n}+\mathbf{1}-\mathbf{r}}].$$

Therefore, if

$$Z_{r,n} = \sum_{i=0}^{n-1} U_{(i,n)}|U_{(0,r-n)}F_{(1,n+1-r)}|$$
$$+ \sum_{j=0}^{n-1} U_{(n,j)}|U_{(r-n,0)}F_{(n+1-r,1)}|$$
$$+ U_{\mathbf{n}}[\,|U_{(r-n,0)}F_{(n+1-r,r-n)}|$$
$$+ |U_{(0,r-n)}F_{(r-n,n+1-r)}|$$
$$+ |U_{\mathbf{r}-\mathbf{n}}F_{\mathbf{n}+\mathbf{1}-\mathbf{r}}|\,],$$

then $F_{\mathbf{r}} \leqq F_{\mathbf{n}+\mathbf{1}} + Z_{r,n}$. Hence we have

$$F_{\mathbf{n}} - Y_{r,n} \leqq F_{\mathbf{r}} \leqq F_{\mathbf{n}+\mathbf{1}} + Z_{r,n}.$$

Since $Y_{r,n} \geqq 0$ and $Z_{r,n} \geqq 0$ and since

$$\lim_{n\to\infty} \frac{1}{n^2}F_{\mathbf{n}} \text{ exists a.e.}$$

by Theorem 3.5, to prove the theorem it is enough to show that

$$(3.7) \quad \frac{1}{n^2}Y_{r,n} \to 0 \text{ a.e. as } n \to \infty, \quad \text{and}$$

$$(3.8) \quad \frac{1}{n^2} Z_{r,n} \to 0 \text{ a.e. as } n \to \infty.$$

If we show that

$$(3.9) \quad 0 \leqq Y_{r,n} \leqq G_{\mathbf{n}+\mathbf{1}} - G_{\mathbf{n}} \quad \text{and}$$
$$(3.10) \quad 0 \leqq Z_{r,n} \leqq 3(G_{\mathbf{n}+\mathbf{1}} - G_{\mathbf{n}})$$

for some bounded additive process $\{G_{(m,n)}\}$, then (3.7) and (3.8) will follow from (3.9) and (3.10) respectively. Since

$$\lim_{n\to\infty} \frac{1}{n^2} G_{\mathbf{n}} \text{ exists a.e.}$$

by Theorem 2.12. For, let

$$G_{(n,m)} = \sum_{i=0}^{n-1} \sum_{i=0}^{m-1} U_{(i,j)}\Omega.$$

Obviously $G = \{G_{(n,m)}\}_{(n,m)\in\mathcal{N}}$ is a positive, bounded additive process. Then

$$Y_{r,n} \leqq \sum_{j=0}^{n-1} U_{(n,j)}\Omega + \sum_{i=0}^{n-1} U_{(i,n)}\Omega + U_{\mathbf{n}}\Omega = G_{\mathbf{n+1}} - G_{\mathbf{n}}$$

and

$$Z_{r,n} \leqq \sum_{j=0}^{n-1} U_{(n,j)}\Omega + \sum_{i=0}^{n-1} U_{(i,n)}\Omega + 3U_{\mathbf{n}}\Omega = 3(G_{\mathbf{n+1}} - G_{\mathbf{n}})$$

giving (3.9) and (3.10) respectively.

Further remarks. The results we have obtained are valid if $\mathcal{U}$ is a strongly continuous semigroup of positive L_1-contractions, not necessarily continuous at the origin. If $\mathcal{U}$ is such a semigroup, then for any $\alpha \in (0, \infty)$, $\{U_{(\alpha,t,\alpha r)}\}$ is a one-dimensional strongly continuous semigroup of L_1-contractions, $(t, r) \in C$. Then there exists a unique partition $\{\mathcal{C}, \mathcal{D}\}$ of X into its initially conservative and dissipative parts $\mathcal{C}$ and $\mathcal{D}$ respectively [**1**, **2**] such that

(i) $\chi U_{(t,r)}f = 0$ for any $f \in L_1$ and $(t, r) \in C$.

(ii) The restriction of $\mathcal{U}$ to $L_1(\mathcal{E})$ is a strongly continuous semigroup of positive $L_1(\mathcal{C})$-contractions which is also continuous at the origin where

$$L_1(\mathcal{C}) = \{f \in L_1\text{:support of } f \subseteq \mathcal{C}\}.$$

If $F = \{F_{(u,v)}\}_{(u,v)\in C}$ is a bounded $\mathcal{U}$-additive process and $\{\mathcal{C}, \mathcal{D}\}$ is the partition of X as above, then

$$\chi_{\mathcal{D}}F_{(u,v)} = 0$$

for each $(u, v) \in C$ [**2**], that is $F_{(u,v)} \in L_1(\mathcal{C})$. Thus we are allowed to restrict $\mathcal{U}$ to $\mathcal{C}$, and hence consider it as a semigroup which is strongly continuous at the origin also. Moreover, for the results of our work, there is no loss of generality in assuming that $U'_{\mathbf{0}} = I$, the identity operator, where $U'_{\mathbf{0}}$ belongs to the restriction of $\mathcal{U}$ to $\mathcal{C}$ [**2**, **3**]. Hence one obtains the conclusions of Theorem 3.5 and Theorem 3.6 on $\mathcal{C}$. It is obvious that in general there is no convergence on $\mathcal{D}$.

For notational convenience, all the proofs were given in the two-dimensional case. However, as seen from the method of proof that the extension of it to the n-dimensional case, $n \geqq 2$ is straightforward.

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