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ANALYTIC DISCS IN THE POLYNOMIAL HULL OF A DISC FIBRATION OVER THE SPHERE

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It is shown that for each point p in the interior of the polynomial hull of a disc fibration X over the unit sphere $\partial\mathbb{B}^n$ there exists an H^∞ analytic disc with boundary in X and passing through p.

1. INTRODUCTION

Let $\mathbb{B}^n = \{z \in \mathbb{C}^n ; |z| < 1\}$ be the open unit ball centred at the point 0 in the n dimensional complex space $\mathbb{C}^n$. Let φ be a continuous function on the unit sphere $\partial\mathbb{B}^n$. In this short note we investigate the presence of analytic discs in the polynomial hull of the compact set

$$X = \left\{(z, w) \in \partial\mathbb{B}^n \times \mathbb{C} ; |w| \leqslant e^{-\varphi(z)}\right\}$$

fibred over $\partial\mathbb{B}^n$. Recall that the polynomial hull $\widehat{K}$ of a compact set $K \subseteq \mathbb{C}^m$ is defined as

$$\widehat{K} = \left\{z \in \mathbb{C}^m ; \left|p(z)\right| \leqslant \max_K |p| \text{ for every polynomial } p \text{ in } m \text{ variables}\right\}$$

and that by the maximum principle the image $F(\Delta)$ of every H^∞ holomorphic mapping $F : \Delta \to \mathbb{C}^m$ with boundary in K, that is, $F^*(e^{i\theta}) \in K$ for almost every θ, belongs to the polynomial hull $\widehat{K}$ of K. For a bounded holomorphic mapping F on Δ the notation F^* is used to denote its almost everywhere defined boundary values.

Let Φ be the maximal plurisubharmonic function on $\mathbb{B}^n$, continuous on $\overline{\mathbb{B}^n}$, such that $\Phi|_{\partial\mathbb{B}^n} = \varphi$, that is, Φ is the unique solution on $\mathbb{B}^n$ of the Dirichlet problem for the Monge-Ampère operator [5]

$$\left\{\begin{array}{l} \Phi \in \mathrm{PSH}(\mathbb{B}^n) \cap \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{B}^n) \\ (d\,d^c\,\Phi)^n = 0 \text{ on } \mathbb{B}^n \\ \Phi|_{\partial\mathbb{B}^n} = \varphi \text{ on } \partial\mathbb{B}^n. \end{array}\right. \tag{1.1}$$

It is a classical result [4, p.99] that the polynomial hull of the set X is

$$\widehat{X} = \left\{(z, w) \in \overline{\mathbb{B}^n} \times \mathbb{C} ; |w| \leqslant e^{-\Phi(z)}\right\}.$$

In this note we show that the interior of the polynomial hull $\widehat{X}$ contains a lot of analytic discs with boundaries in X. More precisely, we prove the following statement:

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Proposition 1.1. *For each point $(z_0, w_0) \in \operatorname{Int}(\widehat{X})$ there exists an H^∞ analytic disc $F : \Delta \to \mathbb{C}^n \times \mathbb{C}$ with boundary in X such that $F(0) = (z_0, w_0)$.*

Some remarks are in order. In the case $n = 1$, much more was proved in the series of papers [**1, 3, 9, 10**]. Using the graphs of analytic functions on $\Delta = \mathbb{B}^1$ with boundaries in X, a complete description of the polynomial hull of a fibration over the unit circle was given for geometrically much more complicated fibres, for example, in [**10**] it was only assumed that each fibre over the unit circle is a simply connected continuum.

In higher dimensions related results were proved by Whittlesey in [**11**] for disc fibrations over $\partial\mathbb{B}^2$ of the form

$$X = \Big\{(z,w) \in \partial\mathbb{B}^2 \times \mathbb{C};\, \big|w - \alpha(z)\big| \leqslant R(z)\Big\},$$

where α is a continuous complex valued function on $\partial\mathbb{B}^2$ and $R \in C^2(\partial\mathbb{B}^2)$ a positive real function such that $\big|\alpha(z)\big| \leqslant R(z)$, $z \in \partial\mathbb{B}^2$. Working with the assumption that $(\mathbb{B}^2 \times \mathbb{C}) \setminus \widehat{X}$ is a pseudoconvex domain, it was proved in [**11**] that the polynomial hull of X can be foliated by the graphs of analytic balls. On the other hand, there are examples of maximal plurisubharmonic functions on $\mathbb{B}^n$ for which for certain points $z \in \mathbb{B}^n$ there is no germ V of an analytic variety containing z and such that $\Phi|_V$ is harmonic, for example, Sibony's example [**2**, p.73]. Therefore, in view of these examples and our result, one can not, in general, expect to get a foliation of the whole $\widehat{X}$ with analytic discs. Namely, if (in the case $\alpha = 0$) there exists a nontrivial analytic disc $F = (f,g) : \Delta \to \widehat{X} \subseteq \overline{\mathbb{B}^n} \times \mathbb{C}$ such that its image $F(\Delta)$ touches $\partial\widehat{X} \cap (\mathbb{B}^n \times \mathbb{C})$, that is, if there exists $\xi_0 \in \Delta$ such that $F(\xi_0) \in \partial\widehat{X} \cap (\mathbb{B}^n \times \mathbb{C})$, then, by the maximum principle for the subharmonic function

$$\xi \longmapsto \big|g(\xi)\big|e^{\Phi(f(\xi))}$$

on Δ, we actually have $F(\Delta) \subseteq \partial\widehat{X} \cap (\mathbb{B}^n \times \mathbb{C})$. Hence $g(\xi) \neq 0$ and

$$\Phi\big(f(\xi)\big) = -\log\big|g(\xi)\big|$$

on Δ. Therefore $\Phi|_{f(\Delta)}$ is harmonic.

We also observe that in the case $\alpha = 0$ the complement $(\mathbb{B}^n \times \mathbb{C}) \setminus \widehat{X}$ is pseudoconvex if and only if the function Φ is pluriharmonic on $\mathbb{B}^n$. In this case a foliation of $\widehat{X}$ by the graphs of analytic balls is obvious: since Φ is pluriharmonic on $\mathbb{B}^n$, there exists an analytic function $H : \mathbb{B}^n \to \mathbb{C}$ such that $\Phi = \operatorname{Re}(H)$. Then the graphs of the family of analytic functions on the ball $H_\xi(z) := \xi e^{H(z)}$, $\xi \in \overline{\Delta}$, form a foliation of $\widehat{X}$.

2. Proof of the proposition

The proof of the proposition uses Poletsky's characterisation of the maximal plurisubharmonic function with the given continuous boundary data. It was proved in [**6, 7**] that for $z \in \mathbb{B}^n$ (one may replace $\mathbb{B}^n$ by any smoothly bounded strongly pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$) the value $\Phi(z)$ of the solution Φ of the problem (1.1) is given by

$$\Phi(z) = \inf_f \frac{1}{2\pi} \int_0^{2\pi} \varphi\big(f^*(e^{i\theta})\big)\, d\theta, \tag{2.1}$$

where the infimum is taken over all holomorphic mappings of the unit disc $f : \Delta \to \mathbb{B}^n$ with $f(0) = z$ and whose boundary values f^* satisfy $f^*(e^{i\theta}) \in \partial\mathbb{B}^n$ for almost every θ.

Recall that

$$\widehat{X} = \left\{(z, w) \in \overline{\mathbb{B}^n} \times \mathbb{C}\,;\, |w| \leqslant e^{-\Phi(z)}\right\}$$

and let $(z_0, w_0) \in \operatorname{Int}(\widehat{X})$. Because each fibre

$$\widehat{X}_z = \left\{w \in \mathbb{C}\,;\, |w| \leqslant e^{-\Phi(z)}\right\}$$

is a disc in the complex plane with centre at 0, we have that $w_0 = \eta\, e^{-\Phi(z_0)}$ for some $\eta \in \Delta$. By using a rotation in $\mathbb{C}$, we may assume that

$$w_0 = t\, e^{-\Phi(z_0)} \in \mathbb{R} \tag{2.2}$$

for some $t \in [0, 1)$.

Let $\varepsilon > 0$ be so small that

$$t\, e^{-\varphi(z)} + |w_0|(1 - e^{-\varepsilon}) < e^{-\varphi(z)}$$

for every $z \in \partial\mathbb{B}^n$. By Poletsky's theorem (2.1) there exists a holomorphic mapping $f : \Delta \to \mathbb{B}^n$ such that $f(0) = z_0$, $f^*(e^{i\theta}) \in \partial\mathbb{B}^n$ for almost every θ and

$$\Phi(z_0) \leqslant \frac{1}{2\pi} \int_0^{2\pi} \varphi\big(f^*(e^{i\theta})\big)\, d\theta \leqslant \Phi(z_0) + \varepsilon. \tag{2.3}$$

Let $u(e^{i\theta}) := \varphi\big(f^*(e^{i\theta})\big)$ and let $P[u]$ denote its Poisson integral

$$P[u](\xi) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left(\frac{e^{i\theta} + \xi}{e^{i\theta} - \xi}\right) u(e^{i\theta})\, d\theta.$$

Since φ is a continuous function on $\partial\mathbb{B}^n$, u is a bounded measurable function on $\partial\Delta$ and hence $P[u]$ has the nontangential limit $u(e^{i\theta})$ for almost every θ, [**8**]. Let $H[u]$ be the harmonic conjugate of $P[u]$ on Δ such that $H[u](0) = 0$. Although the function $H[u]$ is not necessarily bounded on Δ, this is the case for the function

$$g(\xi) = t\, e^{-(P[u](\xi) + i\, H[u](\xi))}, \tag{2.4}$$

which is holomorphic on Δ. For this function we have $g(0) = t\, e^{-P[u](0)}$ and the value $P[u](0)$ is given as the integral average of u over $\partial\Delta$. Thus

$$P[u](0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta})\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \varphi\big(f^*(e^{i\theta})\big)\, d\theta.$$

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The inequalities (2.3) and assumption (2.2) imply

$$w_0\, e^{-\varepsilon} \leqslant g(0) \leqslant w_0$$

and so

$$\big|w_0 - g(0)\big| \leqslant |w_0|(1 - e^{-\varepsilon}).$$

Hence for the boundary values g^* we have

$$\Big|g^*(e^{i\theta}) + \big(w_0 - g(0)\big)\Big| \leqslant t\, e^{-u(e^{i\theta})} + |w_0|(1 - e^{-\varepsilon}) < e^{-\varphi(f^*(e^{i\theta}))}$$

for almost every θ and the holomorphic disc

$$\xi \in \Delta \longmapsto F(\xi) = \Big(f(\xi), g(\xi) + \big(w_0 - g(0)\big)\Big)$$

is such that $F^*(e^{i\theta}) \in X$ for almost every θ and $F(0) = (z_0, w_0)$.

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