A BIPARTITE RAMSEY PROBLEM AND THE ZARANKIEWICZ NUMBERS

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1. Introduction. Beineke and Schwenk [**1**] have defined the bipartite Ramsey number $R(m, n)$, for integers $m, n (1 \leq m \leq n)$, to be the smallest integer p such that any 2-colouring of the edges of the complete bipartite graph $K_{p,p}$ forces the appearance of a monochromatic $K_{m,n}$. In [**1**] the following results are established:

$$R(1, n) = 2n - 1 \tag{1.1}$$

$$R(2, n) \leq 4n - 3 \tag{1.2}$$

with equality if there is a Hadamard matrix of order $2(n-1)$, n odd,

$$R(2, 4) = 13 \tag{1.3}$$

$$R(3, n) \leq 8n - 5 \tag{1.4}$$

$$R(3, n) \geq 8n - 7 \tag{1.5}$$

if there is a Hadamard matrix of order $4(n-1)$,

$$R(3, 3) = 17. \tag{1.6}$$

On the basis of this evidence, Beineke and Schwenk formulated the conjecture

$$R(m, n) = 2^m(n-1) + 1. \tag{1.7}$$

In the present note, we strengthen (1.4) to

$$R(3, n) \leq 8n - 7, \tag{1.8}$$

thus establishing equality in (1.5). In particular, we show that

$$R(3, 4) = 25 \tag{1.9}$$

$$R(3, 5) = 33, \tag{1.10}$$

thus solving two specific problems listed by Harary [**5**]. Further, we show that (1.7) is false in general by providing a number of counter-examples.

An extension of the Beineke–Schwenk problem, which has been mentioned by Hales and Jewett [**4**] and by Guy [**3**], is the determination of those ordered pairs (x, y) such that in any 2-colouring of the edges of $K_{x,y}$, some $K_{m,n}$, with the m vertices a subset of the x and the n vertices a subset of the y, is monochromatic. We write

$$(x, y) \rightarrow (m, n)$$

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to denote the truth of the latter statement and

$$(x, y) \not\rightarrow (m, n)$$

to denote its falsity.

Guy [3] reports that S. Niven has determined some of those pairs (x, y) for which $(x, y) \rightarrow (m, n)$ in the cases $(m, n) = (2, 2), (2, 3), (2, 4)$. In §4 of the present note, we shall investigate some properties of the symbol $\rightarrow$, determine precisely those (x, y) for which $(x, y) \rightarrow (m, n)$ in the cases $(m, n) = (2, 2)$, $(2, 3)$, $(2, 4)$, and we solve most of the corresponding problem for $(m, n) = (3, 3)$.

The corresponding extremal problem, a special case of which was first posed by Zarankiewicz [6], asks for the smallest integer $Z = Z(x, y;\ m, n)$ such that any Z-edge subgraph of $K_{x,y}$ contains $K_{m,n}$ with the m vertices a subset of the x and the n vertices a subset of the y. It seems appropriate to refer to the numbers $Z(x, y;\ m, n)$ as the *Zarankiewicz numbers.* Upper and lower bounds for these numbers have been given, and for small values of the parameters many exact values are known; see [3] for a comprehensive summary of results and a list of references.

The connection between the Ramsey problem and the extremal problem is obvious and is stated in the following proposition.

Proposition 1.1. *$Z(x, y; m, n) \leq [\frac{1}{2}xy]$ implies $(x, y) \rightarrow (m, n)$, where $[p]$ denotes the smallest integer not less than p.*

Hence any method which gives an upper bound for $Z(x, y; m, n)$ also yields information about those (x, y) for which $(x, y) \rightarrow (m, n)$, and in the special case $x = y$ gives an upper bound for $R(m, n)$. We shall pursue this approach in §3.

2. An upper bound for $R(3, n)$.

Theorem 2.1. $R(3, n) \leq 8n - 7$.

Proof. Let A, B denote the two $(8n-7)$-sets into which the vertex set of $K_{8n-7,8n-7}$ is naturally partitioned. Suppose that the edges of $K_{8n-7,8n-7}$ are coloured in two colours, red and green, say. We have to show that there is either

(i) a subgraph $K_{3,n}$ with the 3 vertices a subset of A and the n vertices a subset of B, with all edges the same colour, henceforth referred to as a monochromatic $K_{3,n}$, or

(ii) a subgraph $K_{3,n}$ with the 3 vertices a subset of B and the n vertices a subset of A, with all edges the same colour, henceforth referred to as a monochromatic $K_{n,3}$.

For a vertex set $\{v_1, v_2, \ldots, v_r\} \subseteq A$(resp. B), define $\mathrm{st}_R(v_1, v_2, \ldots, v_r) = \{u \in B(\text{resp. } A)\colon$ edges $uv_1, uv_2, \ldots, uv_r$ all red$\}$, $\mathrm{st}_G(v_1, v_2, \ldots, v_r) = \{u \in B(\text{resp. } A)\colon$ edges $uv_1, uv_2, \ldots, uv_r$ all green$\}$.

Case (i). No vertex of $K_{8n-7,8n-7}$ has as many as $4n-2$ incident edges of any one colour.

Denote by R the red-coloured subgraph of $K_{8n-7,8n-7}$ and assume, without loss of generality, that there are more red edges than green edges. Let $A = A_1 \cup A_2$ and $B =$

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$B_1 \cup B_2$ where A_1, B_1 consist of vertices of degree $4n-3$ in R and A_2, B_2 consist of vertices of degree $4n-4$ in R.

Label the vertices in A $u_1, u_2, \ldots, u_{8n-7}$, with $d_R(u_1) \geq d_R(u_2) \geq \ldots \geq d_R(u_{8n-7})$, and the vertices in B $v_1, v_2, \ldots, v_{8n-7}$, with $d_R(v_1) \geq d_R(v_2) \geq \ldots \geq d_R(v_{8n-7})$, where $d_R(u)$, $d_R(v)$ denote the degrees in R of the vertices u, v. Then $A_1' = \{u_1, u_2, \ldots, u_{4n-3}\} \subseteq A_1$, $B_1' = \{v_1, v_2, \ldots, v_{4n-3}\} \subseteq B_1$. We claim that there is a vertex u of A_1' such that

$$|\mathrm{st}_R(u) \cap B_1'| \geq n+1. \tag{2.1}$$

For otherwise, number of red edges between A_1' and $B_1' \leq n(4n-3)$. Therefore, number of red edges between A_1' and $B \setminus B_1' \geq (3n-3)(4n-3)$. Therefore, number of red edges between $A \setminus A_1'$ and $B \setminus B_1' \leq (n-1)(4n-3)$. Therefore, number of green edges between $A \setminus A_1'$ and $B \setminus B_1' \geq (12n-13)(n-1)$. Now we count the members of the set $S = \{u, v, v', v'' : u \in A \setminus A_1',\ v, v', v'' \in B \setminus B_1',\ uv, uv', uv'' \text{ all green}\}$. Since $|A \setminus A_1'| = 4n-4$, and since $|S|$ will be minimised when the vertices of $A \setminus A_1'$ all have green degree as nearly equal as possible, we find

$$|S| \geq \binom{\frac{(12n-13)(n-1)}{4n-4}}{3} . (4n-4),$$

where the generalised binomial coefficient $\binom{x}{3}$ is defined by $\binom{x}{3} = \frac{1}{6}x(x-1)(x-2)$ for all $x \in \mathbb{R}$. Hence

$$|S| > (n-1)\binom{4n-4}{3} \qquad (n \geq 3)$$

$$= (n-1) \text{ . number of such triples } v, v', v'',$$

and so there is a green $K_{n,3}$. So (2.1) is established.

Now (2.1) implies that the number of red paths of length 2 originating at u is at least $(n+1)(4n-4) + (3n-4)(4n-5) = 16n^2 - 31n + 16$ and, since $(2n-2)(8n-8) = 16n^2 - 32n + 16$, either there is a vertex $u' \in A$ such that $|\mathrm{st}_R(u, u')| \geq 2n$, or there are n vertices $u_{i_1}, u_{i_2}, \ldots, u_{i_n} \in A$ such that $|\mathrm{st}_R(u, u_{i_j})| \geq 2n-1 \quad (j = 1, 2, \ldots, n)$. In the first case, the number of red edges connecting members of $\mathrm{st}_R(u, u')$ to members of $A \setminus \{u, u'\}$ is at least $2n(4n-6) > (n-1)(8n-9)$ and so there is a $u'' \in A$ such that $|\mathrm{st}_R(u, u', u'')| \geq n$ and we have a red $K_{3,n}$.

In the second case, the number of red edges connecting members of $\mathrm{st}_R(u, u_{i_1})$ to members of $A \setminus \{u, u_{i_1}\}$ is at least

$$(2n-1)(4n-6) > (n-1)(8n-9) \qquad (n > 3)$$

so that the same conclusion holds provided $n > 3$. If $n = 3$, then either the number of red edges connecting members of $\mathrm{st}_R(u, u_{i_j})$ to members of $A \setminus \{u, u_{i_j}\}$ is greater than

$$(2n-1)(4n-6), \quad \text{for some } j,$$

in which case the same argument works again, or each of $u_{i_1}, u_{i_2}, \ldots, u_{i_l}$ is red adjacent to each of the 5 vertices of $\mathrm{st}_R(u) \cap (A \setminus A_1')$ which, of course, gives a red $K_{3,3}$ at once.

Case (ii). There is a vertex u_1, say of A, with at least $4n-2$ incident edges of the same colour, say red.

Denote the vertices of B red-joined to u_1 by $v_1, v_2, \ldots, v_{4n-2}, \ldots$. Define

$$S_i = \{w \in A \setminus u_1 : \text{edge } wv_i \text{ is red}\} \qquad (i = 1, 2, \ldots, 4n-2),$$
$$T_i = \{w \in A \setminus u_1 : \text{edge } wv_i \text{ is green}\} \qquad (i = 1, 2, \ldots, 4n-2).$$

Hence

$$|S_i| + |T_i| = 8n - 8 \qquad (i = 1, 2, \ldots, 4n-2). \tag{2.2}$$

Case (ii) (a). $|T_1| + |T_2| + \ldots + |T_{4n-2}| > 4(n-1)(4n-1)$.

The number of quadruples $\{w, T_i, T_j, T_k\}$ with $1 \leq i < j < k \leq 4n-2$, $w \in T_i \cap T_j \cap T_k$, is at least

$$(4n-3)\binom{2n}{3} + (4n-5)\binom{2n-1}{3} > (n-1) \,.\, \binom{4n-2}{3}$$
$$= (n-1) \,.\, \text{number of such triples } T_i, T_j, T_k.$$

Hence there exist $v, v', v'' \in B$ such that $|\mathrm{st}_G(v, v', v'')| \geq n$ and we have a green $K_{n,3}$.

Case (ii) (b). $|S_1| + |S_2| + \ldots + |S_{4n-2}| > 4(n-1)(4n-3)$.

We first show that, if any vertex of A belongs to as many as $2n$ of the sets S_i, then either a red $K_{3,n}$ or a green $K_{n,3}$ is present. For, if $u_2 \in S_1 \cap S_2 \cap \ldots \cap S_{2n}$ say, then, if there is to be no red $K_{3,n}$, each of the $8n-9$ vertices of $A \setminus \{u_1, u_2\}$ can belong to at most $n-1$ of the sets $S_i (i = 1, 2, \ldots, 2n)$. Hence the sets S_i $(i = 1, 2, \ldots, 2n)$ satisfy $\sum_{i=1}^{2n} |S_i| \leq 8n^2 - 15n + 9$. By (2.2), we have $\sum_{i=1}^{2n} |T_i| \geq 8n^2 - n - 9$, so that the number of quadruples $\{w, T_i, T_j, T_k\}$ with $1 \leq i < j < k \leq 2n$, $w \in T_i \cap T_j \cap T_k$, is at least

$$(8n-9)\binom{n+1}{3} > (n-1)\binom{2n}{3} = (n-1) \,.\, \text{number of such triples } T_i, T_j, T_k.$$

Hence a green $K_{n,3}$ is present.

On the other hand, if among the $4(n-1)(4n-3)+1$ (or more) pairs (u, S_i), $u \in S_i \subseteq A$, no u appears more than $2n-1$ times, then at least $4n-3$ of the vertices of A, say $u_2, u_3, \ldots, u_{4n-2}$, appear exactly $2n-1$ times in such pairs. Therefore some S_i, say S_1, contains at least $2n-1$ of the vertices $u_2, u_3, \ldots, u_{4n-2}$, say $u_2, u_3, \ldots, u_{2n}$. The remaining $2n-2$ appearances of each of $u_2, u_3, \ldots, u_{2n}$ are distributed among $4n-3$ of the sets S_i and so the number of triples

$$\{u_i, u_j, S_k\}$$

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$2 \leq i < j \leq 2n$, $u_i, u_j \in S_k$, $2 \leq k \leq 4n-2$, is at least

$$(2n-2)\binom{n}{2}+(2n-2)\binom{n-1}{2}>(n-2)\binom{2n-1}{2}=(n-2)\,.\text{ number of such pairs } u_i, u_j.$$

Hence some pair appears in S_1 and $n-1$ of the other S_i and, together with u_1, this yields a red $K_{3,n}$.

Case (ii) (c). $|T_1|+|T_2|+\ldots+|T_{4n-2}|=4(n-1)(4n-1)$, $|S_1|+|S_2|+\ldots+|S_{4n-2}|=4(n-1)(4n-3)$.

By the argument of Case (ii) (a), we deduce at once that the only case needing consideration is when $4n-4$ of the vertices of $A\setminus\{u_1\}$, say $u_2, u_3, \ldots, u_{4n-3}$, lie in $2n$ of the sets T_i and the remainder, $u_{4n-2}, \ldots, u_{8n-7}$, lie in $2n-1$ of the sets T_i. These incidences yield $(4n-4)(2n)$-subsets and $(4n-4)(2n-1)$-subsets of a $(4n-2)$-set and, if we can show that some triple occurs in n of these subsets, we will have established the existence of a green $K_{n,3}$.

If this is not the case, then a simple count of triples reveals that every triple belongs to exactly $n-1$ of the subsets. Consider a fixed element and suppose that this element belongs to x of the $(2n)$-subsets and y of the $(2n-1)$-subsets. Then, by counting triples containing this element, we obtain the equation

$$x\binom{2n-1}{2}+y\binom{2n-2}{2}=(n-1)\binom{4n-3}{2}$$

i.e.

$$(2n-1)x+(2n-3)y=(2n-2)(4n-3).$$

This equation has solutions $x=n$, $y=3n-2$, and $x=3n-3$, $y=n-1$ so that, if we suppose that p of the elements yield the first solution and q the second solution, we obtain

$$\begin{aligned} pn+q(3n-3)&=(4n-4)\,.\,2n\\ p(3n-2)+q(n-1)&=(4n-4)\,.\,(2n-1)\\ p+q&=4n-2 \end{aligned}$$

giving $p=2n-2$, $q=2n$.

Hence there is an element e which lies in n of the $(2n)$-sets and $(3n-2)$ of the $(2n-1)$-sets. So the number of pairs (e,f) with e,f lying together in the same set is

$$n(2n-1)+(3n-2)(2n-2)=8n^2-8n+4>(4n-3)(2n-1)$$

so that there is a fixed element f which lies together with e in at least $2n$ of the sets. The number of triples (e,f,g) with e,f,g lying together in the same set is at least $2n(2n-3)>(4n-4)(n-1)$ so that there is a fixed g which lies together with e and f in at least n of the sets.

This completes the proof in all possible cases.

COROLLARY 2.2. *$R(3,n)=8n-7$ if there is a Hadamard matrix of order $4(n-1)$.*

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This follows at once from the theorem and from the result (1.5) of Beineke and Schwenk. In particular, the known Hadamard matrices of orders 12 and 16 establish

$$R(3,4)=25$$
$$R(3,5)=33.$$

3. Upper bounds for the Zarankiewicz numbers. Henceforth we assume that suffices are ordered, i.e. we say that $K_{a,b}$ is a subgraph of $K_{x,y}$ if and only if the a vertices are a subset of the x and the b vertices a subset of the y.

Our upper bound method for the Zarankiewicz numbers is based on the following lemma.

LEMMA 3.1. *Suppose that, in a subgraph of $K_{x,y}$, the number of copies of $K_{a,b}$ is at least α. Then*

(i) *the number of copies of $K_{a,c}$ $(b<c)$ in the subgraph is at least*

$$\min_{d_i} \sum_{i=1}^{\binom{x}{a}} \binom{d_i}{c}$$

where the d_i $\left(1 \leqslant i \leqslant \binom{x}{a}\right)$ are non-negative integers subject to

$$\sum_{i=1}^{\binom{x}{a}} \binom{d_i}{b} \geqslant \alpha.$$

(ii) *The number of copies of $K_{c,b}$ $(a<c)$ in the subgraph is at least*

$$\min_{d_i} \sum_{i=1}^{\binom{y}{b}} \binom{d_i}{c}$$

where the d_i $\left(1 \leqslant i \leqslant \binom{y}{b}\right)$ are non-negative integers subject to

$$\sum_{i=1}^{\binom{y}{b}} \binom{d_i}{a} \geqslant \alpha.$$

Proof. (i) Let A, B denote respectively the x-vertex set and the y-vertex set. Let $d_i \left(1 \leqslant i \leqslant \binom{x}{a}\right)$ denote the number of vertices of B joined by an edge to each of the vertices in the i^{th} a-subset of A. Then the number of copies of $K_{a,b}$ is

$$\sum_{i=1}^{\binom{x}{a}} \binom{d_i}{b} \geqslant \alpha.$$

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But then the number of copies of $K_{a,c}$ is

$$\sum_{i=1}^{\binom{x}{a}} \binom{d_i}{c}.$$

(ii) Similar.

In order to establish an upper bound for $Z(x, y; m, n)$ for particular values of the parameters, Lemma 3.1 may be applied several times to a p-edge subgraph of $K_{x,y}$ and for suitable successive choices of a, b and c, it may be possible to prove that the number of copies of $K_{m,n} \geq 1$, so establishing $Z(x, y; m, n) \leq p$.

We have been unable to determine, in the general case, the optimal sequence of choices of a, b, c in the lemma. However, in the special case $x = y$, best results appear to be obtained by counting successively subgraphs $K_{1,2}, K_{3,2}, K_{3,4}, \ldots, K_{m,m-1}, K_{m,n}$ if m is odd, or $K_{2,1}, K_{2,3}, K_{4,3}, \ldots, K_{m,m-1}, K_{m,n}$ if m is even.

We illustrate the method for $Z(48, 48; 4, 4)$ and, in so doing, provide a counter-example to the conjecture (1.7) of Beineke and Schwenk.

THEOREM 3.2. $Z(48, 48; 4, 4) \leq 1148$.

Proof. Consider a 1148-edge subgraph of $K_{48,48}$. In Lemma 3.1(i), take $a = b = 1$, $c = 2$. Then the number of copies of $K_{1,2} \geq \sum_{i=1}^{48} \binom{d_i}{2}$, subject to $\sum_{i=1}^{48} \binom{d_i}{1} \geq 1148$. Hence the number of copies of $K_{1,2} \geq 44\binom{24}{2} + 4\binom{23}{2} = 13\,156$.

Now in Lemma 3.1(ii) take $a = 1$, $b = 2$, $c = 3$. The number of copies of $K_{3,2} \geq \sum_{i=1}^{1128} \binom{d_i}{3}$, subject to $\sum_{i=1}^{1128} \binom{d_i}{1} \geq 13\,156$. Hence the number of copies of $K_{3,2} \geq 748\binom{12}{3} + 380\binom{11}{3} = 227\,260$.

Now, in Lemma 3.1(i), take $a = 3$, $b = 2$, $c = 4$. The number of copies of $K_{3,4} \geq \sum_{i=1}^{17296} \binom{d_i}{4}$, subject to $\sum_{i=1}^{17296} \binom{d_i}{2} \geq 227\,260$. Hence the number of copies of $K_{3,4} \geq 10\,860\binom{6}{4} + 6436\binom{5}{4} = 195\,080$.

Finally, in Lemma 3.1(ii), take $a = 3$, $b = c = 4$. The number of copies of $K_{4,4} \geq \sum_{i=1}^{194580} \binom{d_i}{4}$, subject to $\sum_{i=1}^{194\,580} \binom{d_i}{3} \geq 195\,080$. Hence the number of copies of $K_{4,4} \geq 167\binom{4}{4} \geq 1$.

COROLLARY 3.3. $R(4, 4) \leq 48$.

Proof. $Z(48, 48; 4, 4) \leq 1148 < 1152 = [\frac{1}{2} . 48 . 48]$. The result is an immediate consequence of Proposition 1.1.

Note that, when $m=n=4$, conjecture (1.7) states that $R(4,4)=49$.

Similar arguments to that of the proof of Theorem 3.2 have been used to construct the following table of upper bounds for the bipartite Ramsey numbers $R(m, n)$. A table of values of $f(m, n)=2^m(n-1)+1$ is included for purposes of comparison.

m \ n	4	5	6	7
4	48	65	82	98
5		115	149	182
6			257	328
7				566

Upper bounds for $R(m, n)$

m \ n	4	5	6	7
4	49	65	81	97
5		129	161	193
6			321	385
7				769

$f(m, n)=2^m(n-1)+1$

4. The pairs (x, y) **for which** $(x, y)\to(m, n)$. Given a pair of integers (m, n), $1\le m\le n$, we define the *critical set* for (m, n), denoted by $C_{m,n}$ to be the smallest set

$$\{(x_1, y_1), (x_2, y_2), \ldots, (x_p, y_p)\}$$

with the property that $(x, y)\to(m, n)\Leftrightarrow$ there exists i $(1\le i\le p)$ such that $x\ge x_i$, $y\ge y_i$.

It is clear that, for each pair (m, n), the critical set $C_{m,n}$ is well-defined, and its determination is equivalent to the determination of precisely those (x, y) for which $(x, y)\to(m, n)$.

Trivially, we have $C_{1,n}=\{(1, 2n-1)\}$ for all $n\ge 1$. We now record, in a series of lemmas, some properties of the symbol $\to$ and of the sets $C_{m,n}$, which will enable us to determine $C_{2,2}$, $C_{2,3}$, $C_{2,4}$ and to come close to determining $C_{3,3}$.

LEMMA 4.1

(i) $(x, y)\to(m, n)\Rightarrow\begin{cases}(x', y')\to(m, n) & \text{if } x'\ge x \text{ and } y'\ge y\\ (x, y)\to(m', n') & \text{if } m'\le m \text{ and } n'\le n\end{cases}$

(ii) $(x, y)\not\to(m, n)\Rightarrow\begin{cases}(x', y')\not\to(m, n) & \text{if } x'\le x \text{ and } y'\le y\\ (x, y)\not\to(m', n') & \text{if } m'\ge m \text{ and } n'\ge n.\end{cases}$

Proof. These are immediate consequences of the meaning of the symbol $\to$.

LEMMA 4.2. $(x, y)\in C_{m,m}\Leftrightarrow(y, x)\in C_{m,m}$.

Proof. This is an immediate consequence of the definition of $C_{m,m}$.

For the next lemma, we require some of the terminology of design theory. A *t-(b, v, r, k, λ)-design* is a collection of b k-subsets of a v-set such that every element of the v-set belongs to r of the k-subsets and such that every t-subset of the v-set is contained in exactly λ of the k-subsets. Clearly, the collection of b $(v-k)$-subsets of the v-set, which are the complements of the original k-subsets, forms a t-$(b, v, b-r, v-k, \lambda')$-design, with $\lambda'=\lambda\binom{v-k}{t}\Big/\binom{k}{t}$, called the complement of the original design. If, in the case $k=\frac{1}{2}v$, the design is isomorphic to its complement, then the design is called *self-complementary*.

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LEMMA 4.3. (i) *If there exists an* $m-\left(y, x, \frac{y}{2}, \frac{x}{2}, n-1\right)$*-design, then* $(x, y) \not\rightarrow (m, n)$.

(ii) *If there exists an* $(n-1)-\left(x, y, \frac{1}{2}x, \frac{1}{2}y, x\binom{\frac{1}{2}y}{n-1}\Big/\binom{y}{n-1}\right)$*-design with the properties* (a) *no m blocks have n points in common,* (b) *if* $m>2$ *the design is self-complementary, then* $(x, y) \not\rightarrow (m, n)$.

Proof. (i) We have to colour the edges of $K_{x,y}$ using two colours in such a way that no $K_{m,n}$ is monochromatic. Let M be a $y \times x$ incidence matrix of the design. Label the rows of M with the vertices of the y-set and the columns of M with the vertices of the x-set, and let M be the $y \times x$ adjacency matrix for the subgraph of colour 1. Suppose that some subgraph $K_{m,n}$ has all of its edges colour 1. Then some n rows of M have ones in m common positions, i.e. some n blocks of the design contain the same m elements—a contradiction.

If all other edges are given colour 2, then the fact that the complementary design has the same parameters as the original implies that no $K_{m,n}$ has all of its edges colour 2.

(ii) As in (i), we use an $x \times y$ incidence matrix for the design as an $x \times y$ adjacency matrix for the subgraph of colour 1. Then no subgraph $K_{m,n}$ can have all of its edges colour 1, otherwise some m blocks of the design have n elements in common—a contradiction.

The same is true in colour 2 if the design is self-complementary. When $m=2$, the same is true in colour 2 whether or not the design is self-complementary, since if 2 blocks of the complement have intersection size $\geq n$ then the corresponding 2 blocks of the original also have intersection size $\geq n$.

LEMMA 4.4. (i) $(x, y) \in C_{m,n}$ *implies* $x \geq 2m-1, y \geq 2n-1$.

(ii)(a) $\left(2m, 2(n-1)\binom{2m-1}{m}\right) \not\rightarrow (m, n)$.

(b) $\left(2m-1, 2(n-1)\binom{2m-1}{m}+1\right) \in C_{m,n}$.

(iii)(a) $\left(2(m-1)\binom{2n-1}{n}, 2n\right) \not\rightarrow (m, n)$.

(b) $\left(2(m-1)\binom{2n-1}{n}+1, 2n-1\right) \in C_{m,n}$.

Proof. (i) If $x \leq 2m-2$ then, for any y, the edges of $K_{x,y}$ can be coloured in 2 colours so that no one of the x vertices has more than $m-1$ incident edges of each colour. Similarly if $y \leq 2n-2$.

(ii)(a) There is an $m-\left(2(n-1)\binom{2m-1}{m}, 2m, (n-1)\binom{2m-1}{m}, m, n-1\right)$-design consisting of every m-set of the $2m$-set repeated $n-1$ times. Result follows by Lemma 4.3(i).

(b) Let $s=2(n-1)\binom{2m-1}{m}+1$ and consider any 2-colouring of the edges of

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$K_{2m-1,s}$. Each of the s vertices defines a subset of the $(2m-1)$-set of size m or greater, each member of which is joined to it in the same colour. In $2(n-1)\binom{2m-1}{m}+1$ such subsets, some m-subset appears $2n-1$ times and hence n times in the same colour, and this yields a monochromatic $K_{m,n}$. Hence $\left(2m-1, 2(n-1)\binom{2m-1}{m}+1\right) \to (m, n)$. However, $\left(2m-2, 2(n-1)\binom{2m-1}{m}+1\right) \not\to (m, n)$ by (i) and $\left(2m-1, 2(n-1)\binom{2m-1}{m}\right) \not\to (m, n)$ by (ii)(a) and Lemma 4.1(ii). Hence $\left(2m-1, 2(n-1)\binom{2m-1}{m}+1\right) \in C_{m,n}$.

(iii)(a) There is a self-complementary $(n-1)-\left(2(m-1)\binom{2n-1}{n},\ 2n, (m-1)\binom{2n-1}{n},\ n, (m-1)(n+1)\right)$-design consisting of all the n-subsets of a $2n$-set repeated $m-1$ times and in this design no m blocks have n points in common. The result follows by Lemma 4.3(ii).

(b) This follows by an argument analogous to that of (ii)(b) above.

Lemma 4.5. *$C_{m,n} = \{x_1, y_1), (x_2, y_2), \ldots, (x_p, y_p)\}$ with $2m-1 = x_1 < x_2 < \ldots < x_p = 2(m-1)\binom{2n-1}{n}+1$ if and only if (i) $(x_i, y_i) \to (m, n)$ $(i = 1, 2, \ldots, p)$, and (ii) $(x_{i+1}-1, y_i-1) \not\to (m, n)$ $(i = 1, 2, \ldots, p-1)$.*

Proof. If (x, y) is such that $x \geqslant x_i$, $y \geqslant y_i$ for some i $(1 \leqslant i \leqslant p)$, then $(x, y) \to (m, n)$, by Lemma 4.1(i).

Suppose $(x, y) \to (m, n)$. We have to show $x \geqslant x_i$, $y \geqslant y_i$ for some i $(1 \leqslant i \leqslant p)$. We can assume $x_1 \leqslant x \leqslant x_p$ and so there is an i $(1 \leqslant i \leqslant p-1)$ such that $x_i \leqslant x < x_{i+1}$. It suffices to show $y \geqslant y_i$. Suppose, on the contrary, that $y < y_i$. Then, since $(x_{i+1}-1, y_i-1) \not\to (m, n)$, we have $(x, y_i-1) \not\to (m, n)$, since $x \leqslant x_{i+1}-1$, and so $(x, y) \not\to (m, n)$ since $y \leqslant y_i - 1$. This is a contradiction and the result follows.

Lemma 4.6. (i) *If $(2x+1, 2y+1) \not\to (2, n)$, then there is a collection $S_1, S_2, \ldots, S_{x+1}$ of $(x+1)$ y-subsets of a $(2y+1)$-set such that $|S_i \cap S_j| \leqslant n-2$ for any i, j $(1 \leqslant i < j \leqslant x+1)$.*

(ii) *If $(2x+1, 2y+1) \not\to (3, n)$, then there is a collection $S_1, S_2, \ldots, S_{x+1}$ of $(x+1)$ y-subsets of a $(2y+1)$-set such that, for any i, j, k $(1 \leqslant i < j < k \leqslant x+1)$*

$$|S_i \cap S_j| + |S_i \cap S_k| + |S_j \cap S_k| - |S_i \cap S_j \cap S_k| \leqslant n + y - 2.$$

Proof. (i) Suppose that the edges of $K_{2x+1,2y+1}$ have been coloured using 2 colours so that no $K_{2,n}$ is monochromatic. Adjacency in one colour to the $(2x+1)$ vertices yields at least $x+1$ subsets, each of size at most y, of the $(2y+1)$-vertex set. Denote these subsets by $T_1, T_2, \ldots, T_{x+1}$. Then for any i, j $(1 \leqslant i < j \leqslant x+1)$, we have

$$|T_i \cap T_j| \leqslant |T_i| + |T_j| + n - 2y - 2. \tag{4.1}$$

For otherwise, denoting the complement of T_k in the $(2y+1)$-vertex set by T'_k, we find

$$\begin{aligned} |T'_i \cap T'_j| &= 2y+1-|T_i \cup T_j| \\ &= 2y+1-\{|T_i|+|T_j|-|T_i \cap T_j|\} \\ &> 2y+1+n-2y-2=n-1 \end{aligned} \tag{4.2}$$

and so a monochromatic $K_{2,n}$ is present. Now to each T_i we adjoin arbitrary vertices of the $(2y+1)$-vertex set (if necessary) to form a y-set. Such adjunction preserves inequality (4.1) and leads to y-sets $S_1, S_2, \ldots, S_{x+1}$ with the stated property.

(ii) The proof is similar, using the relation

$$\begin{aligned} |T'_i \cap T'_j \cap T'_k| &= 2y+1-|T_i \cup T_j \cup T_k| \\ &= 2y+1-\{|T_i|+|T_j|+|T_k|-|T_i \cap T_j|-|T_i \cap T_k| \\ &\quad -|T_j \cap T_k|+|T_i \cap T_j \cap T_k|\} \end{aligned}$$

in place of (4.2).

We are now in a position to determine $C_{2,2}$, $C_{2,3}$ and $C_{2,4}$ completely.

THEOREM 4.7. $C_{2,2}=\{(3,7), (5,5), (7,3)\}$.

Proof. $(3,7), (7,3) \in C_{2,2}$ by Lemma 4.4, which also shows that $(4,6), (6,4) \not\rightarrow (2,2)$. An easy application of Lemma 3.1 shows that $Z(5,5;\ 2,2) \leq 13$, so that $(5,5) \rightarrow (2,2)$. The result follows from Lemma 4.5.

THEOREM 4.8. $C_{2,3}=\{(3,13), (5,11), (7,9), (15,7), (21,5)\}$.

Proof. Lemma 4.4 gives $(3,13), (21,5) \in C_{2,3}$ and $(4,12), (20,6) \not\rightarrow (2,3)$. By Lemma 4.5, it remains to show $(6,10), (14,8) \not\rightarrow (2,3)$ and $(5.11), (7,9), (15,7) \rightarrow (2,3)$. The existence of a $2-(10,6,5,3,2)$-design [**2**], together with Lemma 4.3(i), establishes $(6,10) \not\rightarrow (2,3)$.

Also there exists a $3-(14,8,7,4,1)$-design [**2**]. This is a Hadamard 3-design and, being the extension of a symmetric 2-design, any two blocks intersect in at most 2 points. Therefore, by Lemma 4.3(ii), $(14,8) \not\rightarrow (2,3)$.

Simple applications of Lemma 3.1 can be made to yield $Z(5,11;2,3) \leq 28$ and $Z(15,7;2,3) \leq 53$, so that $(5,11), (15,7) \rightarrow (2,3)$.

Finally, to show that $(7,9) \rightarrow (2,3)$, we need a slightly different argument based on Lemma 4.6(i). Suppose $(7,9) \not\rightarrow (2,3)$. Then there is a set of four 4-subsets S_1, S_2, S_3, S_4 of a 9-set, no 2 of which intersect in more than 1 point. So the size of the set $X=\{x, S_i, S_j\}$ with $1 \leq i < j \leq 4$, $x \in S_i \cap S_j$, is at most $\binom{4}{2}=6$. But the number of pairs $\{x, S_i\}$ with $1 \leq i \leq 4$, $x \in S_i$, is 16 and, since there are only 9 choices available for x, it follows that $|X| \geq 7$, a contradiction.

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THEOREM 4.9. $C_{2,4}=\{(3,19), (5,15), (9,13), (23,11), (37,9), (71,7)\}$.

Proof. Lemma 4.4 gives $(3,19), (71,7)\to(2,4)$ and $(4,18), (70,8)\not\to(2,4)$. By Lemma 4.5, it remains to show $(8,14), (22,12), (36,10)\not\to(2,4)$ and $(5,15), (9,13), (23,11), (37,9)\to(2,4)$. The existence of a $2-(14, 8, 7, 4, 3)$-design **[2]**, together with Lemma 4.3(i), establishes $(8,14)\not\to(2,4)$. There exists a $3-(22, 12, 11, 6, 2)$-design **[2]** which is a Hadamard design and so the extension of a symmetric 2-design. Hence any 2 blocks intersect in at most 3 points and, by Lemma 4.3(ii), $(22,12)\not\to(2,4)$. Also there exists a $3-(36, 10, 18, 5, 3)$-design in which no 2 blocks have 4 points in common (see Appendix for details), so that Lemma 4.3(ii) implies $(36,10)\not\to(2,4)$.

On the other hand, an application of Lemma 3.1 can be used to show $Z(5,15;2,4)\leq 53$ and so $(5,15)\to(2,4)$. The three remaining results are each proved by application of Lemma 4.6(i).

Suppose $(9,13)\not\to(2,4)$. Then, by Lemma 4.6(i), there is a collection of 5 6-subsets $S_1,\ldots,S_5$ of a 13-set, no 2 intersecting in more than 2 points. So the size of the set

$$X=\{x, S_i, S_j\}$$

$1\leq i<j\leq 5$, $x\in S_i\cap S_j$, is at most $2\binom{5}{2}=20$. But the number of pairs

$$\{x, S_i\}$$

$1\leq i\leq 5$, $x\in S_i$, is 30 and, since there are only 13 choices available for x, we deduce that $|X|\geq 4\binom{3}{2}+9\binom{2}{2}=21$—a contradiction.

Suppose $(23,11)\not\to(2,4)$. Then, by Lemma 4.6(i), there is a collection of 12 5-subsets $S_1, S_2,\ldots,S_{12}$ of an 11-set, no 2 intersecting in more than 2 points. So the size of the set

$$X=\{x, S_i, S_j\}$$

$1\leq i<j\leq 12$, $x\in S_i\cap S_j$, is at most $2\binom{12}{2}=132$. But the number of pairs

$$\{x, S_i\}$$

$1\leq i\leq 12$, $x\in S_i$, is 60 and, since there are only 11 choices available for x, we deduce that $|X|\geq 5\binom{6}{2}+6\binom{5}{2}=135$—a contradiction.

Finally, suppose $(37,9)\not\to(2,4)$. Then, by Lemma 4.6(i), there is a collection of 19 4-subsets $S_1, S_2,\ldots,S_{19}$ of a 9-set, no 2 intersecting in more than 2 points. Clearly, some element is in at least 9 of these subsets and, of these 9, some further fixed element is in at least 4. To make up these 4 4-subsets, we require 4 mutually-disjoint 2-subsets of a 7-set—impossible.

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CONJECTURE 4.10. $C_{3,3} = \{(5, 41), (7, 29), (9, 23), (13, 17), (17, 13), (23, 9), (29, 7), (41, 5)\}$.

Partial proof. Lemma 4.4 gives $(5, 41), (41, 5) \in C_{3,3}$ and $(6, 40), (40, 6) \not\rightarrow (3, 3)$. The symmetric nature of $C_{3,3}$ is a consequence of Lemma 4.2. By Lemma 4.5, it remains to show $(8, 28), (12, 22), (16, 16) \not\rightarrow (3, 3)$ and $(7, 29), (9, 23), (13, 17) \rightarrow (3, 3)$. The existence of a $3-(28, 8, 14, 4, 2)$-design and a $3-(22, 12, 11, 6, 2)$-design [**2**], together with Lemma 5.3(i), establishes $(8, 28) \not\rightarrow (3, 3)$ and $(12, 22) \not\rightarrow (3, 3)$. Beineke and Schwenk [**1**] have shown $(16, 16) \not\rightarrow (3, 3)$.

On the other hand, an application of Lemma 3.1 gives $Z(7, 29; 3, 3) \leq 102$ and so $(7, 29) \rightarrow (3, 3)$. To show $(9, 23) \rightarrow (3, 3)$, we require a complicated application of Lemma 4.6(ii), the details of which we omit.

The only part of the conjecture which we have been unable to prove is the assertion $(13, 17) \rightarrow (3, 3)$.

Three further conjectures worthy of mention are the following:

CONJECTURE 4.11
(i) $(2n+1, 4n-3) \rightarrow (2, n)$
(ii) $(4n+1, 8n-7) \rightarrow (3, n)$
(iii) $C_{2,5} = \{3, 25), (5, 21), (7, 19), (11, 17), (31, 15), (83, 13), (133, 11), (253, 9)\}$.

Finally, we might ask if it is ever possible for an even number to appear as a member of a pair in a critical set, in any case for a critical set of the form $C_{2,n}$.

Appendix. We list the blocks of a $3-(36, 10, 18, 5, 3)$-design, on the point set $\{0, 1, \ldots, 9\}$, in which no 2 blocks have 4 common points.

12 345	12 570	13 690	23 467	24 580	34 680
12 368	12 890	14 568	23 490	24 689	35 790
12 379	13 489	14 590	23 589	25 678	36 789
12 478	13 567	14 679	23 560	26 790	45 670
12 460	13 470	15 789	23 780	34 569	47 890
12 569	13 580	16 780	24 579	34 578	56 890

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