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LINEAR ISOMETRIES OF SPACES OF ABSOLUTELY CONTINUOUS FUNCTIONS

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1. Let X be an arbitrary compact subset of the real line $\mathbf{R}$ which has at least two points. For each finite complex valued function f on X we denote by $V(f; X)$ (and call it the *weak variation of f on X*) the least upper bound of the numbers $\sum_i |f(b_i) - f(a_i)|$ where $\{[a_i, b_i]\}$ is any sequence of non-overlapping intervals whose end points belong to X. A function f is said to be of *bounded variation* (BV) *on* X if $V(f; X) < \infty$. A function f is said to be absolutely continuous (AC) on X, if given any $\epsilon > 0$ there exists an $\eta > 0$ such that for every sequence of non-overlapping intervals $\{[a_i, b_i]\}$ whose end points belong to X, the inequality

$$\sum_i (b_i - a_i) < \eta$$

implies that

$$\sum_i |f(b_i) - f(a_i)| < \epsilon$$

([**7**], p. 221, 223).

We denote by AC(X) the linear space of all absolutely continuous complex valued functions on X and define a norm on it by

$$(1) \quad \|f\| = \|f\|_\infty + V(f; X), \quad f \in \mathrm{AC}(X)$$

where $\|f\|_\infty$ is the usual uniform norm.

Now let a and b be the greatest lower bound and the least upper bound of X, respectively. Since X is compact, a and b belong to X and hence $[a, b]\backslash X$ is an open subset of the real line $\mathbf{R}$. Clearly then $[a, b]\backslash X$ is the union of a countable number of disjoint open intervals. In order to show that AC(X) is a Banach space we first prove the following lemma.

LEMMA 1.1. *Let* $f \in \mathrm{AC}(X)$. *Then there is a unique function* F *on* $[a, b]$ *such that*

(i) $F|_X \equiv f$

(ii) *F is linear on the closure of each component of* $[a, b]\backslash X$. *We have* $F \in \mathrm{AC}\,[a, b]$ *and*

(iii) $V(f; X) = V(F; [a, b])$.

Proof. The existence and uniqueness of a continuous function F on $[a, b]$ with properties (i) and (ii) is obvious. It is easy to see that (iii) holds.

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To show that $F \in \mathrm{AC}\,[a, b]$ it is enough to show that the real and imaginary parts of F belong to $\mathrm{AC}\,[a, b]$. From (iii) it follows that F is of BV on $[a, b]$ and hence Re F and Im F are of BV on $[a, b]$. Clearly, Re F and Im F are absolutely continuous and hence N-functions ([**7**], p. 224) on X as well as on each component of $[a, b]\backslash X$. Since $[a, b]\backslash X$ has only a countable number of components Re F and Im F are N-functions on $[a, b]$. The result now follows from ([**3**], p. 288, Theorem 18.25).

Let $S_X = \{G | G \in \mathrm{AC}\,[a, b]$ and G is linear on the closure of each component of $[a, b]\backslash X\}$.

$$\|G\| = \|G\|_\infty + \int_a^b |G'(t)|dt$$

where $\|G\|_\infty$ is the usual uniform norm. It is well known that $\mathrm{AC}\,[a, b]$ with this norm is a Banach space.

PROPOSITION 1.2. *S_X is a closed subspace of* $\mathrm{AC}\,[a, b]$ *and* $\mathrm{AC}(X)$ *with the norm given by* (1) *is a Banach space which is isometrically isomorphic to S_X.*

Proof. Clearly, S_X is a closed subspace of $\mathrm{AC}\,[a, b]$ and hence it is complete. Now define a map ψ_X: $\mathrm{AC}(X) \to S_X$ by $f \to F$ where F is the unique extension of f as defined in Lemma 1.1. Clearly, ψ_X is well defined and is an isomorphism of $\mathrm{AC}(X)$ onto S_X. Now,

$$\|f\| = \|f\|_\infty + V(f; X) = \|f\|_\infty + V(F; [a, b])$$
$$= \|F\|_\infty + \int_a^b |F'(t)|dt = \|F\| = \|\psi_X(f)\|.$$

Therefore, ψ_X is an isometry. This implies that $\mathrm{AC}(X)$ is complete. Thus $\mathrm{AC}(X)$ is a Banach space which is isometrically isomorphic to S_X.

2. By an isometry of a Banach space B_1 onto a Banach space B_2 we will mean a linear norm preserving map of B_1 onto B_2. The isometries of $\mathrm{AC}\,[0, 1]$ were investigated in [**1**] and in [**6**]. In this article, we show that the techniques of [**1**], in fact, can be employed to prove that if X and Y are compact subspaces of $\mathbf{R}$, then the existence of an isometry T of $\mathrm{AC}(X)$ onto $\mathrm{AC}(Y)$ implies that there exists an absolutely continuous homeomorphism τ of Y onto X. Moreover T can be described via τ.

Let V denote the closed unit ball of the space $L^\infty([a, b])$ provided with the weak-star topology and let W_X denote the compact space $X \times V$. Corresponding to each $f \in \mathrm{AC}(X)$ we define $\tilde{f} \in C\,(W_X)$ by

$$\tilde{f}(x, \alpha) = f(x) + \int_a^b F'(t)\bar{\alpha}(t)dt, \quad (x, \alpha) \in W_X$$

where F is the unique extension of f as defined in Lemma 1.1. It is easy to see that the following lemma holds.

LEMMA 2.1. *The mapping $f \to \tilde{f}$ establishes an isometry between* AC(X) *and the closed subspace $\tilde{S}_X$ of* C(W_X) *where* $\tilde{S}_X = \{\tilde{f} | f \in \text{AC}(X)\}$.

Next, for $(x, \alpha) \in W_X$ we define the continuous linear functional $L_{x,\alpha}$ on AC(X) by

$$L_{x,\alpha}(f) = \tilde{f}(x, \alpha), \quad f \in \text{AC}(X).$$

It follows from ([**2**], p. 441) that the extreme points of the unit ball U_X^* of AC*(X) constitute a subset of

$$\{\gamma L_{x,\alpha} | \quad \gamma \text{ is a complex number with } |\gamma| = 1, (x, \alpha) \in W_X\}.$$

Moreover, it is clear that if $L_{x,\alpha}$ is extreme in U_X^*, then α must be extreme in the unit ball of L^∞ $([a, b])$, i.e., $|\alpha| = 1$ almost everywhere on $[a, b]$ ([**4**], p. 138).

For a given x in X we denote by α_x the L^∞ function which takes the value 1 on $[a, x)$ (if $[a, x) \neq \emptyset$) and takes the value -1 on $(x, b]$ (if $(x, b] \neq \emptyset$). Let S be the set of all complex numbers with modulus one and having positive real part.

LEMMA 2.2. *For all $x \in X$ and $\gamma \in S$ the functional $L_{x,\gamma\alpha_x}$ is an extreme point of the unit ball in* AC*(X).

Proof. Given $x \in X$, define $H_x \in S_X$ by $H_x(x) = b - a$, $H_x' = \alpha_x$ a.e. on $[a, b]$. Now let $\gamma \in S$. There is a real number M such that

$$\gamma(b - a + Mi) = |b - a + Mi|.$$

Set

$$H_{x,\gamma} = \gamma(H_x + Mi) \quad \text{and} \quad h_{x,\gamma} = H_{x,\gamma}|_X.$$

Then $h_{x,\gamma} \in \text{AC}(X)$,

$$L_{x,\gamma\alpha_x}(h_{x,\gamma}) = h_{x,\gamma}(x) + \bar{\gamma} \int_a^b \gamma H_x'(t)\alpha_x(t)dt$$

$$= \|h_{x,\gamma}\|_\infty + \int_a^b |H_{x,\gamma}'(t)|dt = \|h_{x,\gamma}\|$$

and

$$|L_{t,\beta}(h_{x,\gamma})| < \|h_{x,\gamma}\| \quad \text{for } (t, \beta) \in W_X,\ (t, \beta) \neq (x, \gamma\alpha_x).$$

Now, a result of deLeeuw ([**5**], p. 61) shows that $L_{x,\gamma\alpha_x}$ is an extreme point of the unit ball in AC*(X).

Now, let Y be another arbitrary compact subset of $\mathbf{R}$ with c and d as its greatest lower bound and least upper bound respectively. Let T be an isometry of AC(X) onto AC(Y). We denote by ψ_Y (analogous to ψ_X) the

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isometry of AC(Y) onto the closed subspace S_Y (analogous to S_X) of AC$[c, d]$. Then $\hat{T} = \psi_Y T \psi_X^{-1}$ is an isometry of S_X onto S_Y. Also the adjoint of T, namely T^*, is an isometry of AC$(Y)^*$ onto AC$(X)^*$ and hence maps the set of extreme points of U_Y^* onto the set of extreme points of U_X^*.

The function which is identically equal to 1 on a set Q will be denoted by 1 and it will be always clear from the context what Q is meant.

LEMMA 2.3. *$T(1)$ is a constant function on Y.*

Proof. Let $y \in Y$, $\gamma \in S$ and let α_y denote the function in $L^\infty([c, d])$ analogous to the function α_x in $L^\infty([a, b])$. The fact that $L_{y,\gamma\alpha_y}$ is an extreme point of U_Y^* implies that $T^* L_{y,\gamma\alpha_y}$ is a functional of the form $\delta L_{x,\beta}$, where $|\delta| = 1$ and $(x, \beta) \in W_X$. Set $g = T(1)$ and $G = \hat{T}(1)$. Then

$$\left| g(y) + \bar{\gamma} \int_c^d G'(t)\alpha_y(t)dt \right| = |L_{y,\gamma\alpha_y}(g)| = |T^* L_{y,\gamma\alpha_y}(1)| = 1.$$

Since γ is an arbitrary element of S, we must have either $g(y) = 0$ or $|g(y)| = 1$. Since $\|g\| = 1$, we have

$$|g(y)| = 1 \quad \text{and} \quad \int_c^d G'(t)\alpha_y(t)dt = 0$$

for each $y \in Y$. Therefore

$$0 = G(y) - G(c) - G(d) + G(y), \quad G(y) = \tfrac{1}{2}(G(c) + G(d))$$

for each $y \in Y$. Since $g = G|_Y$, g is a constant function on Y.

For $y \in Y$ and $\gamma \in S$, the functional $T^* L_{y,\gamma\alpha_y}$ must be of the form $\delta L_{x,\beta}$ where δ, x, β, as such, will depend on y and γ but it is easy to see that δ is constant for all $y \in Y$ and $\gamma \in S$ and $\delta = T(1)$. In what follows we suppose that $T(1) = 1$, for otherwise we may replace T by $T/T(1)$. Hence for $y \in Y$ and $\gamma \in S$, the functional $T^* L_{y,\gamma\alpha_y}$ will be of the form $L_{x,\beta}$ for some $x \in X$ and $\beta \in L^\infty([a, b])$ such that $|\beta| = 1$ a.e. on $[a, b]$. For each $y \in Y$ let $h_y \in$ AC(Y) be defined by $h_y = H_y|_Y$ where $H_y \in$ AC $[c, d]$ is defined as $H_y(y) = d - c$, $H_y' = \alpha_y$ a.e. on $[c, d]$. Let M be a real number such that

$$\gamma(d - c + Mi) = |d - c + Mi|.$$

Let $H_{y,\gamma} = \gamma(H_y + Mi)$ and let $h_{y,\gamma} = H_{y,\gamma}|_Y$.

LEMMA 2.4. *Let $y \in Y$, $\gamma \in S$, $g \in$ AC(Y) and*

$$\|h_{y,\gamma} + g\| = \|h_{y,\gamma}\| + \|g\|.$$

Then $\|g\| = L_{y,\gamma\alpha_y}(g)$.

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Proof. It is easy to see that

$$\|h_{y,\gamma} + g\|_\infty = \|h_{y,\gamma}\|_\infty + \|g\|_\infty$$

and that

$$\int_c^d |H_{y,\gamma}'(t) + G'(t)|dt = \int_c^d (|H_{y,\gamma}'(t)| + |G'(t)|)dt$$

where $G = \psi_Y(g)$. It follows that $g(y) = \|g\|_\infty$ and that $G' \geqq 0$ a.e. on (c, y), $G' \leqq 0$ a.e. on (y, d). This proves the lemma.

LEMMA 2.5. *Let $y \in Y$, $\gamma \in S$ and $T^*L_{y,\gamma\alpha_y} = L_{x,\beta}$. Let $k = T^{-1}(h_{y,\gamma})$, $f \in \mathrm{AC}(X)$ and $\|k + f\| = \|k\| + \|f\|$. Then*

$$\|f\| = L_{x,\beta}(f).$$

Proof. We have

$$\|h_{y,\gamma}\| + \|T(f)\| = \|k\| + \|f\| = \|k + f\| = \|T(k + f)\| = \|h_{y,\gamma} + T(f)\|.$$

Therefore by Lemma 2.4

$$\|f\| = \|T(f)\| = L_{y,\gamma\alpha_y}(T(f)) = T^*L_{y,\gamma\alpha_y}(f) = L_{x,\beta}(f).$$

For each $y \in Y$ and each $\gamma \in S$ let $A_{y,\gamma}$ be the set of all $g \in \mathrm{AC}(Y)$ such that $L_{y,\gamma\alpha_y}(g) = \|g\|$. Then, since T^{-1} is an isometry, we have

$$T^{-1}(A_{y,\gamma}) = \{T^{-1}(g) | \quad g \in A_{y,\gamma}\} = \{f \in \mathrm{AC}(X) | \quad T^*L_{y,\gamma\alpha_y}(f) = \|f\|\}.$$

For each measurable set $B \subset R$ let $|B|$ be its Lebesgue measure.

LEMMA 2.6. *Let $y \in Y$, $\gamma \in S$ and $T^*L_{y,\gamma\alpha_y} = L_{x,\beta}$. If E is an open subset of X which contains x, then there exists an $h \in \mathrm{AC}(X)$ such that*

$$L_{x,\beta}(h) = \|h\| \quad \text{and} \quad \max_{t\in(X\setminus E)} |h(t)| < |h(x)|.$$

Proof. We first assume that x is an interior point of $[a, b]$. Then there exists an open interval (p, q) such that $x \in (p, q) \cap X \subset E$. We claim that $T^{-1}(A_{y,\gamma})$ contains an element f_1 such that $\psi_X(f_1)$ is not constant on $(p, x]$. To see this, one may take $f_1 = T^{-1}(h_{y,\gamma})$ if $\hat{T}^{-1}(H_{y,\gamma})$ is not constant on $(p, x]$. Otherwise let χ be the characteristic function of $(p, x]$, $F_1(t) = \int_a^t \chi(s)ds$ $(t \in [a, b])$ and $f_1 = F_1|_X$. Further define $\beta_1(s) = \beta(s)$ on $[a, b]\setminus(p, x]$, $\beta_1(s) = 1$ on $(p, x]$. Then

$$\begin{aligned}\|f_1\| + \|T^{-1}(h_{y,\gamma})\| &= \|f_1\|_\infty + \int_a^b |F_1'(s)|ds + L_{x,\beta}(T^{-1}(h_{y,\gamma}))\\ &= f_1(x) + \int_p^x \chi(s)ds + T^{-1}(h_{y,\gamma})(x) + \int_a^b (\hat{T}^{-1}(H_{y,\gamma}))'(s)\bar{\beta}(s)ds\\ &= (f_1 + T^{-1}(h_{y,\gamma}))(x) + \int_a^b (F_1' + (\hat{T}^{-1}(H_{y,\gamma}))')(s)\bar{\beta}_1(s)ds\\ &\leqq \|f_1 + T^{-1}(h_{y,\gamma})\|.\end{aligned}$$

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Lemma 2.5 now shows that $f_1 \in T^{-1}(A_{y,\gamma})$. Clearly, F_1 is not constant on $(p, x]$. Thus there exists an $f_1 \in T^{-1}(A_{y,\gamma})$ and a point $e_1 \in (p, x)$ such that

$$F_1(e_1) \neq f_1(x) = \|f_1\|_\infty.$$

Similarly, there is an $F_2 \in S_X$ and a point $e_2 \in (x, q)$ such that

$$F_2|_X \in T^{-1}(A_{y,\gamma})$$

and that

$$F_2(e_2) \neq F_2(x) = \|F_2\|_\infty.$$

Define functions H_1 and H_2 as follows: If $e_1, e_2 \in X$ then $H_1(t) = F_1(e_1)$ for $t \in [a_1, e_1]$, $H_1(t) = F_1(t)$ for $t \in (e_1, b]$, $H_2(t) = F_2(t)$ for $t \in [a, e_2)$, $H_2(t) = F_2(e_2)$ for $t \in [e_2, b]$. If $e_1 \notin X$ then e_1 must belong to one of the components of $[a, b]\backslash X$ which must be an open interval say $(a_1, b_1) \subset (a, b)$. Thus $e_1 \in (a_1, b_1)$. Clearly then at least one of the $f_1(a_1)$ and $f_1(b_1)$ must be different from $f_1(x)$. Say for definiteness $f_1(a_1) \neq f_1(x)$. Then define $H_1(t) = F_1(a_1)$ for $t \in [a, a_1]$, $H_1(t) = F_1(t)$ for $t \in (a_1, b]$.

If $e_2 \notin X$, the definition of H_2 can be modified similarly. It is easy to see that $H_j|_X \in T^{-1}(A_{y,\gamma})$ $(j = 1,2)$ and that the function $h = (H_1 + H_2 + 1)|_X$ has the required properties. A slight modification in the above arguments will prove the result in the case when $x = a$ or $x = b$.

LEMMA 2.7. *Let* $y \in Y$, $D = \{t \in [a, b] | \ (T^{-1}(H_y))'(t) = 0\}$. *Then* $|D| = 0$.

Proof. Let $T^*L_{y,\alpha_y} = L_{x,\delta}$. Suppose that $|D| > 0$. Then for some positive real number η at least one of the two sets $D \cap [a, x - \eta]$, $D \cap [x + \eta, b]$ has a nonzero measure. Choose such a set and denote it by B. By Lemma 2.6 there exists an $h \in T^{-1}(A_{y,1})$, and an $\epsilon > 0$, such that

$$\sup_{t \in B} |h(t)| < |h(x)| - \epsilon.$$

Next choose a measurable function α with $|\alpha| = 1$ on B, $\alpha = 0$ on $[a, b]\backslash B$, $\int_a^b \alpha(s)ds = 0$ and such that $\alpha\bar{\delta}$ has a nonzero imaginary part on some subset of B with positive measure. Now define

$$H = \psi_X(h), \quad F(t) = H(t) + \epsilon \int_a^t \alpha(s)ds \quad (a \leqq t \leqq b), \quad f = F|_X.$$

It is easy to see that $\|f\|_\infty = f(x)$, $F'(t) = H'(t)$ a.e. on $[a, b]\backslash B$, $|H'(t)| = H'(t)\bar{\delta}(t)$ a.e. on $[a, b]$. We may choose h so that $H'(t)$ is either zero or one on B from which it follows that $F'(t) \neq 0$ a.e. on B.

Let $\delta_1(t) = F'(t)/|F'(t)|$ a.e. on B, $\delta_1(t) = \delta(t)$ on $[a, b]\backslash B$. We have

$$\|f\| + \|T^{-1}(h_y)\| = \|f\|_\infty + \int_a^b |F'(t)|dt + L_{x,\delta}(T^{-1}(h_y))$$

$$= f(x) + \int_{[a,b]\backslash B} |F'(t)|dt + \int_B |F'(t)|dt + (T^{-1}(h_y))(x)$$

$$+ \int_{[a,b]\backslash B} (\hat{T}^{-1}(H_y))'(t)\bar{\delta}(t)dt = (f + T^{-1}(h_y))(x)$$

$$+ \int_{[a,b]\backslash B} (F'(t) + (\hat{T}^{-1}(H_y))'(t))\bar{\delta}(t)dt + \int_B |F'(t)|dt$$

$$= (f + T^{-1}(h_y))(x) + \int_a^b (F + \hat{T}^{-1}(H_y))'(t)\bar{\delta}_1(t)dt$$

$$\leqq \|f + T^{-1}(h_y)\|.$$

Also, since $\alpha\bar{\delta}$ has a nonzero imaginary part on some subset of B with positive measure, $L_{x,\delta}(f) \neq \|f\|$ which contradicts Lemma 2.5.

LEMMA 2.8. *Let* $y \in Y$, $\gamma \in S$, $T^*L_{y,\alpha_y} = L_{x,\delta}$, $T^*L_{y,\gamma\alpha_y} = L_{v,\beta}$. *Then* $\beta = \gamma\delta$ a.e.

Proof. Let M be a real number such that

$$\gamma(d - c + Mi) = |d - c + Mi|,$$
$$F = \gamma(\hat{T}^{-1}(H_y) + Mi),$$
$$f = F|_X.$$

Then $H_{y,\gamma} = \gamma(H_y + Mi)$, $T(f) = h_{y,\gamma}$ and

$$\|f\| = \|h_{y,\gamma}\| = L_{y,\gamma\alpha_y}(h_{y,\gamma}) = T^*L_{y,\gamma\alpha_y}(f) = f(v) + \int_a^b F'(t)\bar{\beta}(t)dt.$$

Thus

$$\int_a^b |(\hat{T}^{-1}(H_y))'(t)|dt = \int_a^b |F'(t)|dt = \int_a^b F'(t)\bar{\beta}(t)dt$$

$$= \int_a^b \gamma(\hat{T}^{-1}(H_y))'(t)\bar{\beta}(t)dt$$

and hence

$$\gamma\bar{\beta}(t)(\hat{T}^{-1}(H_y))'(t) = |(\hat{T}^{-1}(H_y))'(t)| \quad \text{a.e. on } [a, b].$$

Taking $\gamma = 1$ we get

$$\bar{\delta}(t)(\hat{T}^{-1}(H_y))'(t) = |(\hat{T}^{-1}(H_y))'(t)| \quad \text{a.e. on } [a, b].$$

Therefore, by Lemma 2.7, $\beta = \gamma\delta$ a.e.

Let $y \in Y$, $T^*L_{y,\alpha_y} = L_{x,\delta}$, $(x, \delta) \in W_X$. We see from Lemma 2.8 that for each $\gamma \in S$ there is a $v \in X$ such that

(2) $\quad T^*L_{y,\gamma\alpha_y} = L_{v,\gamma\delta}.$

It is easy to see that there is a unique v fulfilling (2). Define a function φ on S setting $\varphi(\gamma) = v$, where v is obtained by (2).

LEMMA 2.9. *Let $y \in Y$ and let φ be as above. Then φ is constant.*

Proof. We show first that φ is continuous. Let $\gamma \in S$ and let E be an open neighbourhood of $\varphi(\gamma)$ in X. By Lemma 2.6 there exists an h in $\mathrm{AC}(X)$ such that $T^*L_{y,\gamma\alpha_y}(h) = \|h\|$ and

$$\sup_{t \in X \setminus E} |h(t)| < |h(\varphi(\gamma))| - \epsilon \quad \text{for some } \epsilon > 0.$$

Then

$$\|h\| = (T(h))(y) + \bar{\gamma} \int_c^d (\hat{T}(H))'(t)\bar{\alpha}_y(t)dt.$$

So it is clear that for z sufficiently close to γ, $\varphi(z) \in E$. Thus the mapping φ is continuous and $\varphi(S)$ is connected.

Now we prove that $\varphi(S)$ is a singleton. We proceed as follows: Suppose that $\varphi(S)$ has more points than one. Let $T^*L_{y,\alpha_y} = L_{x,\delta}$. We may choose a function $p \in \mathrm{AC}(X)$, an interval $I \subset \varphi(S)$ and a point $z \in S$ such that $p = 0$ on I, $p(\varphi(z)) \neq 0$ and

$$\int_a^b (\psi_X(p))'(t)\bar{\delta}(t)dt = 0.$$

If $\varphi(\gamma) \in I$, then

$$L_{y,\gamma\alpha_y}(T(p)) = L_{\varphi(\gamma),\gamma\delta}(p) = p(\varphi(\gamma)) = 0.$$

Since there are infinitely many such numbers γ, we have

$$L_{y,\gamma\alpha_y}(T(p)) = 0 \quad \text{for each } \gamma \in S.$$

However,

$$L_{y,z\alpha_y}(T(p)) = L_{\varphi(z),z\delta}(p) = p(\varphi(z)) \neq 0$$

which is a contradiction.

We now define a mapping τ: Y into X setting $\tau(y)$ to be the value of the function φ defined above. Thus

$$T^*L_{y,\alpha_y} = L_{\tau(y),\delta}.$$

Consideration of T^{-1} will show that τ is onto and one-to-one. It will then follow from the following theorem that τ is an absolutely continuous homeomorphism from Y onto X.

THEOREM 2.10. *Let T be an isometry of* AC(X) *onto* AC(Y) *with* $T(1) = 1$. *Let f_0 be the identity mapping of X onto itself and let* $\tau = T(f_0)$. *Then for each* $f \in$ AC(X) *and each* $y \in Y$

$$(T(f))(y) = f(\tau(y)).$$

Proof. Let $y \in Y$. We first suppose that $g \in \mathrm{AC}(Y)$ with $g(y) = 0$. Then for all $\gamma \in S$

$$\int_c^d G'(t)\bar{\alpha}_y(t)dt = \gamma L_{y,\gamma\alpha_y}(g) = \gamma T^* L_{y,\gamma\alpha_y}(T^{-1}(g)) = \gamma L_{\tau(y),\gamma\delta}(T^{-1}(g))$$
$$= \gamma(T^{-1}(g))(\tau(y)) + \int_a^b (\hat{T}^{-1}(G))'(t)\bar{\delta}(t)dt.$$

Therefore $(T^{-1}(g))(\tau(y)) = 0$.

For arbitrary $g \in \mathrm{AC}(Y)$, define g_1 by

$$g_1(t) = g(t) - g(y), \quad t \in Y.$$

Then

$$0 = (T^{-1}(g_1))(\tau(y)) = (T^{-1}(g))(\tau(y)) - g(y)(T^{-1}(1))(\tau(y))$$
$$= (T^{-1}(g))(\tau(y)) - g(y).$$

Replacing g by $T(f)$, we have for $y \in Y$ and $f \in \mathrm{AC}(X)$,

$$(T(f))(y) = f(\tau(y)).$$

If f_0 is the identity mapping of X onto itself, we have

$$\tau(y) = (T(f_0))(y)$$

and the theorem is proved.

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