Correspondence

DEAR EDITOR,

Recently my attention has been drawn to the November issue of your *Gazette*. I would like to comment on the note 85.57 entitled 'Fibonacci goes hyperbolic' written by A. Robert Pargeter.

The results presented in the above note are already available in the literature [1]. In fact, I am a bit surprised that A. Robert Pargeter did not make any reference to my paper which was published some 6 years ago in the journal devoted exclusively to the topics of Fibonacci numbers, series and triangles. One can easily conclude that the main 'findings' presented in the note 85.57 are the same as those presented on pages 131 and 132 in [1].

Reference

1. Z. Trzaska, On Fibonacci Hyperbolic Trigonometry and Modified Numerical Triangles, *The Fibonacci Quarterly* **34** (No. 2, May 1996) pp. 129-138.

Z. TRZASKA

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DEAR EDITOR,

There is, I think, a direct proof of S. Simons' curious identity in Note 85.38 of the July 2001 *Gazette* which comes from evaluating in two ways the coefficient of t^q in the expansion of $f(x, t) = (x + t)^q (1 - t)^{-(q+1)}$. Since the coefficient of t^r in the binomial expansion of $(1 - t)^{-(q+1)}$ is $(q + r)!/(q!\,r!)$, the coefficient of t^q in the expansion of $f(x, t)$ is

$$\sum_{r=0}^{q} \frac{q!}{(q-r)!\,r!} x^r . \frac{(q+r)!}{q!\,r!}$$

– the right-hand side of the required identity. On the other hand, we can rewrite $f(x, t)$ as follows:

$$\begin{aligned} f(x, t) &= \frac{(-1)^q}{1-t}\left[1 - \frac{1+x}{1-t}\right]^q \\ &= \sum_{r=0}^{q} (-1)^{q+r} \frac{q!}{r!\,(q-r)!} \frac{(1+x)^r}{(1-t)^{r+1}}. \end{aligned}$$

Thus, by picking out the terms in t^q in the expansions of $(1 - t)^{-(r+1)}$, the coefficient of t^q in the expansion of $f(x, t)$ is the left-hand side of the required identity, namely

$$\sum_{r=0}^{q} (-1)^{q+r} \frac{q!}{r!\,(q-r)!} \frac{(q+r)!}{q!\,r!} (1+x)^r .$$

A similar calculation yields a direct proof of formulae (10) and (12) of G. Rządkowski's Note 85.36 in the same issue.

Thus, for (10) we use

$$(1 - t^2)^{-(n-k)} = \sum_{m=0}^{\infty} \binom{n-k+m-1}{m} t^{2k},$$

so the coefficient of t^{2k+1} in the expansion of $(1+t)^{2n}(1-t^2)^{-(n-k)}$ is $\sum_{m=0}^{k} \binom{2n}{2m+1}\binom{n-m-1}{k-m}$, the LHS of (10). But $(1+t)^{2n}(1-t^2)^{-(n-k)}$ $= (1-t)^{n+k}(1-t)^{-(n-k)}$ and on expanding the coefficient of t^{2k+1} is also

$$\sum_{r=0}^{2k+1} \binom{n+k}{r}\binom{n+k-r}{2k+1-r}$$

$$= \sum_{r=0}^{2k+1} \binom{n+k}{2k+1}\binom{2k+1}{r}, \text{ on regrouping the binomial coefficients,}$$

$$= \binom{n+k}{2k+1}.2^{2k+1}, \text{ as required.}$$

Yours sincerely,

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DEAR EDITOR,

Square-triangular numbers again

As shown by Peter A. Braza and Jingcheng Tong [1], the numbers that are both square and triangular are precisely the numbers $N_k = (p_k q_k)^2$ where the fractions p_k / q_k are the convergents to $\sqrt{2}$.

As $p_{k+1} = p_k + 2q_k$, $q_{k+1} = p_k + q_k$, these numbers satisfy the recurrence relation

$$N_{k+3} = 35N_{k+2} - 35N_{k+1} + N_k$$

with $N_1 = 1, N_2 = 36, N_3 = 1225$. Their generating function is

$$\sum_{k \geqslant 1} N_k q^k = \frac{q(1+q)}{1 - 35q + 35q^2 - q^3}.$$

Reference

1. Peter A. Braza and Jingcheng Tong, Square-triangular numbers, revisited, *Math. Gaz.* **85** (July 2001) pp. 270-273.

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DEAR EDITOR,

Readers may be interested to know that the on-line cumulative index of the *Mathematical Gazette* now extends from 1960 to 2001. It can be accessed from the MA web site: http://www.m-a.org.uk/. A search facility is also being developed.

BILL RICHARDSON

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DEAR EDITOR,

One of my calculus students offered the following 'evaluation' of $\lim_{x \to \infty} \frac{\ln(x^2 + e^x)}{\ln(x^4 + e^{2x})}$:

$$\lim_{x \to \infty} \frac{\ln(x^2 + e^x)}{\ln(x^4 + e^{2x})} = \lim_{x \to \infty} \frac{\ln x^2 + \ln e^x}{\ln x^4 + \ln e^{2x}} \qquad (!)$$

$$= \lim_{x \to \infty} \frac{2 \ln x + x}{4 \ln x + 2x} = \frac{1}{2} \text{ which is correct (!!).}$$

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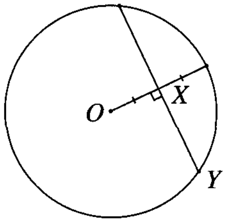
DEAR EDITOR,

A surprising ruler and compass construction

I am a high school maths teacher in Sydney, Australia. Before the summer holidays at the end of 2001, our Mathematics Department ran a competition. We were being cruel and invited students to use straight edge and compasses to draw a regular heptagon in a circle. A 13 year-old student, Daniel Pace, submitted a drawing. Upon inspection it had us scrambling to check the accuracy. We found it was less than 0.3% deviant from the real thing. This was the method used:–

1. Bisect the radius of the circle (at X).
2. Produce the perpendicular bisector until it intersects the circumference (at Y).
3. The distance XY is used to draw the (approximate) heptagon.



Taking the radius of the circle as 1, the 'real thing' has a side-length of 0.868, whereas the student's approximation has a side length 0.866 (working to 3 d.p.). I would be delighted to have this student receive some accolade for his elegant method; despite being aware that the construction was known to be impossible, he still went ahead and made the attempt.

Yours sincerely,

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Editor's Note: The construction described gives the side of the 'heptagon' as $\frac{1}{2}\sqrt{3}$ of the radius, implying that the angle subtended by one side at the centre is $\cos^{-1}\frac{5}{8}$. Thus the approximation becomes $\cos\frac{2}{7}\pi \approx \frac{5}{8}$. That this is feasible can be easily seen by considering the expression, easily derived from De Moivre's Theorem,

$$f(\theta) = \cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

and checking that $f(0.625) \approx 1$.

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