INTEGRALS INVOLVING E-FUNCTIONS

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1. In this paper three integrals involving E-functions are evaluated in terms of E-functions. The formulae to be established are:

$$\int_0^\infty t^{2\gamma-1} W_{k,m}(t) W_{-k,m}(t) E(p\,;\,\alpha_r : q\,;\,\beta_s : zt^{-2n})\,dt$$
$$= (2\pi)^{\frac{1}{2}-n}(2n)^{2\gamma-\frac{1}{2}} E\{p+4n\,;\,\alpha_r : q+2n\,;\,\beta_s : z(2n)^{-2n}\}, \quad \text{.....................(1)}$$

where n is a positive integer, $|\arg z| < \pi$, $R(\gamma \pm m+\frac{1}{2}) > 0$, $\alpha_{p+\nu} = (2\gamma+\nu)/2n$ $(\nu = 1, 2, \ldots, 2n)$, $\alpha_{p+2n+i} = (\gamma+m-\frac{1}{2}+i)/n$, $\alpha_{p+3n+i} = (\gamma-m-\frac{1}{2}+i)/n$, $\beta_{q+i} = (\gamma+k+i)/n$, $\beta_{q+n+i} = (\gamma-k+i)\,/\,n$ $(i = 1, 2, \ldots, n)$.

$$\int_0^\infty t^{2\lambda-1} K_{2\mu}(t) K_{2\nu}(t) E(p\,;\,\alpha_r : q\,;\,\beta_s : zt^{-2n})\,dt$$
$$= n^{2\lambda-\frac{3}{2}}\pi^{\frac{3}{2}-n} 2^{-n-1} E(p+4n\,;\,\alpha_r : q+2n\,;\,\beta_s : zn^{-2n}), \quad \text{...........(2)}$$

where n is a positive integer, $|\arg z| < \pi$, $R(\lambda \pm \mu \pm \nu) > 0$, $\alpha_{p+i+1} = (\lambda+\mu+\nu+i)/n$, $\alpha_{p+n+i+1} = (\lambda-\mu+\nu+i)/n$, $\alpha_{p+2n+i+1} = (\lambda+\mu-\nu+i)/n$, $\alpha_{p+3n+i+1} = (\lambda-\mu-\nu+i)/n$ $(i = 0, 1, 2, \ldots, n-1)$, $\beta_{q+j+1} = (2\lambda+j)/2n$ $(j = 0, 1, 2, \ldots, 2n-1)$.

$$\int_0^{\frac{\pi}{2}} \cos 2\mu\theta (\cos\theta)^{2\lambda-2} E\{p\,;\,\alpha_r : q\,;\,\beta_s : z(\cos\theta)^{-2n}\}\,d\theta = \frac{1}{2}\sqrt{\frac{\pi}{n}}\, E(p+2n\,;\,\alpha_r : q+2n\,;\,\beta_s : z), \text{...(3)}$$

where n is a positive integer, $R(\lambda) > \frac{1}{2}$, $|\arg z| < \pi$, $\alpha_{p+i+1} = (2\lambda-1+i)/2n$ $(i = 0, 1, 2, \ldots, 2n-1)$, $\beta_{q+j+1} = (\lambda+\mu+j)/n$, $\beta_{q+n+j+1} = (\lambda-\mu+j)/n$ $(j = 0, 1, 2, \ldots, n-1)$.

2. The following results will be required.

If $|\arg z| < \pi$ [1, p. 374],

$$E(p\,;\,\alpha_r : q\,;\,\beta_s : z) = \frac{1}{2\pi i}\int \frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{s=1}^{q}\Gamma(\beta_s-\xi)}\, z^{\xi}\,d\xi, \quad \text{.....................(4)}$$

where the contour is of the Barnes type.

If $R(\gamma \pm m+\frac{1}{2}) > 0$ [1, Ex. 118, p. 397],

$$\int_0^\infty t^{2\gamma-1} W_{k,m}(t) W_{-k,m}(t)\,dt = \frac{\Gamma(2\gamma+1)\Gamma(\gamma+m+\frac{1}{2})\Gamma(\gamma-m+\frac{1}{2})}{2\Gamma(\gamma+k+1)\Gamma(\gamma-k+1)}. \quad \text{............(5)}$$

If $R(\lambda \pm \mu \pm \nu) > 0$,

$$\int_0^\infty t^{2\lambda-1} K_{2\mu}(t) K_{2\nu}(t)\,dt = 2^{2\lambda-3}\Gamma(\lambda+\mu+\nu)\Gamma(\lambda-\mu+\nu)\Gamma(\lambda+\mu-\nu)\Gamma(\lambda-\mu-\nu)[\Gamma(2\lambda)]^{-1}. \quad \text{......(6)}$$

This follows from an integral due to Titchmarsh [2].

If $R(\lambda) > \frac{1}{2}$ [1, Ex. 95, p. 340],

$$\int_0^{\frac{\pi}{2}} \cos 2\mu\theta (\cos\theta)^{2\lambda-2}\,d\theta = \frac{\pi\Gamma(2\lambda-1)}{2^{2\lambda-1}\Gamma(\lambda+\mu)\Gamma(\lambda-\mu)}. \quad \text{........................(7)}$$

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If m is a positive integer,

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{mz-\frac{1}{2}} \prod_{l=1}^{m} \Gamma\left(z+\frac{l-1}{m}\right). \qquad \text{(8)}$$

3. *Proof.* To prove the first formula, we substitute the contour integral (4) for the E-function in the integrand of (1) and change the order of integration; the integral then becomes

$$\frac{1}{2\pi i}\int \frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{s=1}^{q}\Gamma(\beta_s-\xi)} z^{\xi}\, d\xi \int_0^{\infty} t^{2\gamma-2n\xi-1} W_{k,m}(t) W_{-k,m}(t)\, dt.$$

If we now use (5) and (8), it takes the form

$$\frac{(2\pi)^{\frac{1}{2}-n}(2n)^{2\gamma-\frac{1}{2}}}{2\pi i}\int \frac{\Gamma(\xi)\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)\prod_{l=1}^{2n}\Gamma\{(2\gamma+l)/2n-\xi\}}{\prod_{s=1}^{q}\Gamma(\beta_s-\xi)\prod_{l=1}^{n}\Gamma\{(\gamma+k+l)/n-\xi\}}$$

$$\times \frac{\prod_{l=1}^{n}\Gamma\{(\gamma+m-\frac{1}{2}+l)/n-\xi\}\prod_{l=1}^{n}\Gamma\{(\gamma-m-\frac{1}{2}+l)/n-\xi\}}{\prod_{l=1}^{n}\Gamma\{(\gamma-k+l)/n-\xi\}} z^{\xi}(2n)^{-2n\xi}\, d\xi$$

and the result follows from (4).

Formulae (2) and (3) can also be derived in the same way by using (6) and (7) respectively.

REFERENCES

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