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BOUNDARY BEHAVIOUR OF SOLUTIONS OF THE NON-PARAMETRIC LEAST AREA PROBLEM

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Previous work concerning boundary regularity of solutions of the non-parametric least area problem leaves open the question of regularity of solutions at points where the mean curvature of the boundary of the domain vanishes. We here prove that the solutions may be discontinuous at such points, even when the given boundary data is smooth. We also give a sufficient condition which will ensure continuity at such points.

Suppose Σ is an open portion of the boundary $\partial\Omega$ of a C^2 domain $\Omega \subset \mathbb{R}^n$, and let H_Σ be the mean curvature of Σ relative to the inward pointing unit normal.

It is known ([*1*, *2*], [*4*]) that solutions of the non-parametric least area problem

$$\int_\Omega \sqrt{1+|Du|^2} + \int_{\partial\Omega} |u-\psi|\,dH^{n-1} \to \min , \quad u \in BV(\Omega) , \tag{0.1}$$

where ψ is a given Lipschitz function on $\partial\Omega$, are $C^2(\Omega) \cap W^{1,1}(\Omega)$ functions which attain the given boundary data ψ on Σ provided $H_\Sigma \geq 0$. It is also known ([*4*]) that if Σ is C^4 and $H_\Sigma < 0$ on Σ, then the trace of u on Σ is a locally Lipschitz function on Σ. In either of these two cases $\left(H_\Sigma \geq 0 \text{ on } \Sigma ,\ H_\Sigma < 0 \text{ on } \Sigma\right)$ it is known

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([1], [4, 5]) that u extends to be Hölder continuous on $\Omega \cup \Sigma$.

These results leave open the case when H_Σ changes sign on Σ and the case when $H_\Sigma \le 0$ on Σ with $H_\Sigma = 0$ at some point of Σ . We want to discuss these cases here. It is shown (Theorem 2) that, for the case $H_\Sigma \le 0$, u may have quite pathological trace on Σ and may exhibit discontinuities, even in case Σ is C^4 and ψ is constant on Σ . On the other hand in case H_Σ changes sign at a point $\xi \in \Sigma$ where $\partial_\Sigma H_\Sigma \ne 0$ (∂_Σ equals tangential gradient operator on Σ), u must be continuous at ξ . In fact, at such points ξ , it is shown (in Theorem 1) that the trace of u on Σ satisfies a Lipschitz condition, and u itself satisfies a Hölder condition.

Both the above theorems depend on construction of barriers, using a somewhat non-standard method of construction. For convenience the constructions here are carried out only in case $n = 2$, but the reader will see that only purely technical modifications are needed to give analogous results for arbitrary n .

1. Barriers

As mentioned above, we here assume $n = 2$, so that Σ is a C^2 arc $\subset \mathbb{R}^2$ and H_Σ denotes the curvature of Σ relative to the inward unit normal.

The existence of boundary barriers for solutions of the minimal surface equation (and solutions of the non-parametric least area problem) in case $H_\Sigma \ge 0$ on Σ has been discussed by many authors. (See for example [3], [6], [7].) Here we take a slightly different approach to the question, and obtain some new results.

To begin, we introduce convenient coordinate axes for $\mathbb{R}^3$. We suppose $0 \in \Sigma$ and that $(0, 1)$ is the inward pointing unit normal to Σ at 0 , and we introduce new coordinates (y_1, y_2, y_3) for $\mathbb{R}^3$ with $y_1 = x_1$, $y_2 = x_3$, and $y_3 = x_2$. Then since Σ is C^2 we have $\delta > 0$

so that the boundary cylinder $(\Sigma \cap \{x : |x_1| < \delta\}) \times \mathbb{R}$ can be expressed as $\left\{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_3 = \omega(y_1, y_2), |y_1| < \delta\right\}$, where ω is a C^2 function on $(-\delta, \delta) \times \mathbb{R}$ with $\partial\omega/\partial y_2 \equiv 0$ and $(\partial\omega/\partial y_1)(0) = 0$, and where

$$(1.1) \qquad |D\omega(y_1, y_2)| \le L|y_1| \quad \text{for} \quad |y_1| < \delta .$$

H_Σ, when considered as a function on $(\Sigma \cap \{x : |x_1| < \delta\}) \times \mathbb{R}$ which is constant on vertical lines, is given by

$$(1.2) \qquad \begin{aligned} H_\Sigma(x_1, x_2, x_3) &\equiv H_\Sigma(y_1, y_2, y_3) \\ &= -M(\omega)(y_1, y_2) . \end{aligned}$$

Here M is the minimal surface operator defined by

$$M(w) = \sum_{i=1}^{2} D_i\left(D_i w/\sqrt{1+|Dw|^2}\right) .$$

(Notice that since $\partial\omega/\partial y_2 \equiv 0$, we can compute

$$M(\omega) = \left(\frac{\partial^2\omega}{\partial y_1^2}\right)\Big/\left(1+\left(\frac{\partial\omega}{\partial y_1}\right)^2\right)^{3/2} .)$$

With regard to the operator M, we need the following technical lemma, the proof of which is a simple algebraic computation. In this lemma we let the operator M_0 be given by

$$M_0(w) = \left(1+|Dw|^2\right)^{\frac{1}{2}}M(w) = \Delta w - \sum_{i,j=1}^{2} \frac{D_i w D_j w}{1+|Dw|^2} D_i D_j w .$$

LEMMA 1. *Let M_0 be the operator defined above and let ζ_1, ζ_2 be C^2 with $|D\zeta_1|, |D\zeta_2| \le 1$. Then*

$$|M_0(\zeta_1+\zeta_2)-M_0(\zeta_2)-\Delta\zeta_1| \le c\left(|D\zeta_1|\left|D^2\zeta_2\right|+\left|D^2\zeta_1\right|\right)\left(|D\zeta_1|+|D\zeta_2|\right) ,$$

where c is an absolute constant.

We are going to use this lemma to construct functions $\tilde{w}$ on conical domains in the (y_1, y_2) variables, such that $M(\tilde{w}) \ge 0$ and such that the graph $y_3 = \tilde{\omega}(y_1, y_2)$ transforms to a graph $x_3 = w(x_1, x_2)$ (under the transformation $y_1 = x_1$, $y_2 = x_3$, $y_3 = x_2$). Notice that then we have $M(w) = -M(\tilde{\omega}) \le 0$. In this way we construct solutions of $M(w) \le 0$; the reason for the indirect approach (*via* the coordinates y_1, y_2, y_3) is that the computations in the y coordinates are simpler, especially in view of the form we choose below for the function $\tilde{\omega}$.

We in fact look for a function $\tilde{\omega}$ of the form

$$\tilde{\omega}(y_1, y_2) = \omega(y_1, y_2) + \alpha y_2^{k+2}\phi(y_1/y_2) \ , \tag{1.3}$$

where k, α (constants) and ϕ are to be chosen. We first note that, writing $f(y_1, y_2) = y_2^{k+2}\phi(y_1/y_2)$, we have

$$\frac{\partial f}{\partial y_2}(y) = (k+2)y_2^{k+1}\phi(t) - y_2^{k+1}t\phi'(t) \ , \quad t = y_1/y_2$$

$$\frac{\partial^2 f}{\partial y_2^2}(y) = y_2^k\left[(k+2)(k+1)\phi(t)-(2k+2)t\phi'(t)+t^2\phi''(t)\right] \ ,$$

$$\frac{\partial^2 f}{\partial y_1 \partial y_2}(y) = y_2^k[(k+1)\phi'(t)-t\phi''(t)] \ ,$$

$$\frac{\partial f}{\partial y_1}(y) = y_2^{k+1}\phi'(t) \ ,$$

$$\frac{\partial^2 f}{\partial y_1^2}(y) = y_2^k\phi''(t) \ .$$

Thus in particular we deduce that

$$\Delta f(y) = y_2^k\left[\left(1+t^2\right)\phi''(t)-2(k+1)t\phi'(t)+(k+2)(k+1)\phi(t)\right] \ .$$

Combining these calculations, and using Lemma 1, we deduce (taking $\tilde{\omega}$ as in (1.3)) that, for $K|t| \le 1$,

$$\begin{aligned}\left|M_0(\tilde{\omega})-\left\{M_0(\omega)+\alpha y_2^k\left|\left(1+t^2\right)\phi''(t)-2(k+1)t\phi'(t)+(k+2)(k+1)\phi(t)\right|\right\}\right| \\ \le C\alpha y_2^k\left(y_2\left(|\phi(t)|+|\phi'(t)|\right)+|\phi''(t)|\right)\left(|y_1|+\alpha y_2^{k+1}\left(|\phi(t)|+|\phi'(t)|\right)\right) \ ,\end{aligned} \tag{1.4}$$

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with $C \geq 1$ depending only on k, Σ, provided $\alpha|Df| \leq 1$.

We now assume that

(1.5) $$M_0(\omega) \geq \mu y_1 + R(y_1), \quad |R(y_1)| \leq Ly_1^2 \leq \frac{\mu}{2}|y_1|, \quad |y_1| < \delta,$$

for some positive constants μ, L, δ. L, δ will (without loss of generality) be taken to coincide with the constants L, δ of (1.1).

Under the assumption (1.5), we want to discuss the possibility of selecting ϕ so that (1.4) implies $M(\tilde{\omega}) \geq 0$. To do this, we take $k = 1$ in the above discussion, we let $\varepsilon \in (0, \frac{1}{2})$ be arbitrary, and let ϕ_0 be a $C^{1,1}$ function on the interval $[-\varepsilon^2, 1]$ with $\phi_0(-\varepsilon^2) = \phi_0(1) = 0$, $\phi_0'' = 1$ on $[-\varepsilon^2, \varepsilon]$, $|\phi_0''| \leq 2\varepsilon$ on $[\varepsilon, 1]$, $|\phi_0'| \leq 2\varepsilon$ on $[-\varepsilon^2, 1]$, $t\phi_0'(t) \leq 2\phi_0(t)$ on $[-\varepsilon^2, 1]$, and $0 < \phi_0 < 2\varepsilon$ on $(-\varepsilon^2, 1)$. (It is left to the reader to check that such a function ϕ_0 exists.) Then we let

(1.6) $$\phi(t) = K^{-2}\phi_0(Kt) \quad (K \geq 1 \text{ arbitrary}).$$

With such a choice of ϕ (and with $k = 1$) one readily checks that (1.4)-(1.6) imply, for $-\varepsilon^2 \leq Kt \leq 1$, that

(1.7) $$M_0(\tilde{\omega}) \geq \mu y_1 + \alpha y_2 E(t) - C\alpha K^{-1} y_2^2 (1+\varepsilon y_2)^2 - Ly_1^2,$$

where $E(t) \geq \frac{1}{2}$ for $-\varepsilon^2 \leq Kt < \varepsilon$ and $E(t) \geq -6\varepsilon$ for $\varepsilon \leq Kt \leq 1$, provided $\alpha|Df| \leq 1$. This last restriction is guaranteed if we take $8\alpha y_2^2 \varepsilon \leq 1$. We now restrict α, ε such that

(1.8) $$\varepsilon K \leq 1, \quad 9\mu\varepsilon^2/K < 9\alpha < \mu^*\varepsilon/(KC), \quad \mu^* = \min\{\mu, 1\}.$$

Notice that K is still arbitrary. Also, we can ensure that $\alpha f \leq \delta$, $\alpha|Df| \leq 1$ for $0 \leq y_2 \leq K$, $-\varepsilon^2 \leq Kt \leq 1$, $|y_1| \leq \delta$, by taking

(1.9) $$8\alpha\varepsilon K \leq \delta.$$

(Then $\delta \in (0, 1)$ is also arbitrary.)

Subject to these restrictions one now easily checks from (1.7) the following:

(i) for $-\varepsilon^2 \le Kt \le \varepsilon$ and $0 \le y_2 \le K$ and $|y_1| \le \delta$,

$$M_0(\tilde{\omega}) \ge \mu y_1 - Ly_1^2 + \alpha y_2\left[\tfrac{1}{2}-4C\alpha K^{-1}y_2^2\right]$$
$$\ge -2\,\frac{\mu\varepsilon^2}{K}\,y_2 + \frac{\alpha y_2}{4} \ge 0\ ;$$

(ii) for $\varepsilon \le Kt \le 1$ and $0 \le y_2 \le K$ and $|y_1| \le \delta$,

$$M_0(\tilde{\omega}) \ge \frac{\mu}{2}\,y_1 - 6C\alpha y_2\left[\varepsilon+K^{-1}y_2\right]$$
$$\ge y_2\left[\frac{\mu\varepsilon}{2K} - 9C\alpha\right] \ge 0\ .$$

Thus subject to the restrictions (1.8), (1.9) we conclude

(1.10) $\quad M(\tilde{\omega}) \ge 0$ for $-\varepsilon^2 \le Kt \le 1$, $0 < y_2 \le K\delta$, $|y_1| \le \delta$.

This will be used in the next section in the manner we have already briefly alluded to above.

Next we wish to consider the possibility that, instead of (1.6), there holds

(1.11) $$M(\omega) \le -\mu|y_1|^k\ , \quad |y_1| \le \delta\ ,$$

where k is a positive integer and $\mu > 0$.

In this case we take arbitrary $\varepsilon \in (0, \tfrac{1}{2})$ and take any $\phi \in C^{1,1}([-1, 1])$ such that $0 < \phi$ on $(-1, 1)$, $|\phi'| + |\phi| \le \varepsilon$ on $[-1, 1]$, $\phi(t) = \phi(-t)$, $t \in [-1, 1]$, $\phi(-1) = \phi(1) = \phi'(-1) = \phi'(1) = 0$,

(1.12) $\quad \phi'' \le -\tfrac{1}{4}$ on $(-\varepsilon, \varepsilon)$, $|\phi''| \le 2\varepsilon$ on $[-1, 1] \sim [-\varepsilon, \varepsilon]$.

Provided that $(k+2)\alpha y_2^{k+1}\varepsilon \le 1$ (which ensures that $\alpha|Df| \le 1$), we deduce from (1.4) and (1.11) the following:

(i) for $|t| \le \varepsilon$, $0 < y_2 \le 1$, $|y_1| \le \delta$,

$$M_0(\tilde{\omega}) \le -\mu|y_1|^k + \alpha y_2^k\left[-\tfrac{1}{4}+4(k+2)^2\varepsilon+2C\left(|y_1|+\alpha\right)\right]$$
$$\le 0\ ,$$

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provided $\delta + \alpha \le (16C)^{-1}$, $\varepsilon \le 1/\left(32(k+2)^2\right)$;

(ii) for $t \in [-1, 1] \sim [-\varepsilon, \varepsilon]$, $0 < y_2 \le 1$, $|y_1| \le \delta$,

$$\begin{aligned} M_0(\tilde{\omega}) &\le -\mu\varepsilon^k y_2^k + \alpha y_2^k\Big[\varepsilon\left(1+4(k+2)^2\right)+2C\left(|y_1|+\alpha\right)\Big] \\ &\le y_2^k\left|-\mu\varepsilon^k+\alpha\left(1+4(k+2)^2\right)\varepsilon+2C(\alpha+\delta)\right| \\ &\le 0 , \end{aligned}$$

provided $\alpha \le \mu\varepsilon^k/\left(1+4(k+2)^2\right)$, $\alpha + \delta \le \mu\varepsilon^k/(2C)$.

Thus in any case we have

(1.13) $M(\tilde{\omega}) \le 0$ for $|y_1| \le \delta$, $0 < y_2 \le \delta$, $|y_1|/y_2 \le 1$,

provided we take $\varepsilon = 1/\left(32(k+2)^2\right)$ and provided the constants α, δ are chosen such that

(1.14) $\alpha \le \mu^*\delta\varepsilon^k/\left(1+4(k+2)^2\right)$, $\delta \le \mu^*\varepsilon^k/(16C)$, $\mu^* = \min\{\mu, 1\}$.

Notice that subject to these restrictions we automatically have $\alpha|Df| \le 1$ and $\alpha f \le \delta$.

We should also mention that in each of the above cases we have chosen ϕ such that $t\phi'(t) \le 2\phi(t)$. This ensures that $\partial f/\partial y_2 > 0$ $\left(f\right.$ as in (1.3)$\left.\right)$ and hence that the graph $y_3 = \tilde{\omega}\left(y_1, y_2\right)$ does transform to a graph $x_3 = w\left(x_1, x_2\right)$ are required.

2. Main results

We now use the barrier results of §1 to prove the theorems mentioned in the introduction. First we make a precise statement of the theorem concerning continuity of u at points $\xi \in \Sigma$ when H_Σ satisfies the conditions

(2.1) $H_\Sigma(\xi) = 0$, $|\partial_\Sigma H_\Sigma| \ge \beta > 0$ on $\partial\Omega \cap B_\delta(\xi)$, $\Sigma \in C^{2,1}$,

where β, δ are positive constants and where $\partial_\Sigma H_\Sigma$ denotes the tangential derivative of H_Σ along Σ .

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THEOREM 1. *Suppose* (2.1) *holds, and let* u *be any* $C^2(\Omega) \cap W^{1,1}(\Omega)$ *solution of the non-parametric least area problem* (0.1). *Then*

(2.2) $$|u(x)-\psi(\xi)| \le c|x-\xi|^{1/3}, \quad |x-\xi| < \delta, \quad x \in \Omega,$$

and

(2.3) $$|\bar{u}(x)-\psi(\xi)| \le c|x-\xi|, \quad |x-\xi| < \delta, \quad x \in \Sigma,$$

where c *depends only on* Σ, $\sup_{B_\delta \cap \Omega} |u|$ *and* Lip ψ, *and where* $\bar{u}$ *denotes the trace of* u *on* Σ.

Proof. We may assume that $\xi = 0$, $\psi(0) = 0$, and that $(0, 1)$ is the inward pointing normal to $\partial\Omega$ at 0.

For a suitable portion Σ of $\partial\Omega$ the condition (2.1) then guarantees that (1.5) holds for suitable μ, δ, and hence by the discussion of §1 we know that for any preassigned $K \ge 1$ there is a neighbourhood U of $\Sigma \cap \{x : |x_1| < \delta\}$ with $U \cap \partial\Omega = \Sigma \cap \{x : |x_1| < \delta\}$ and a $C^0(\overline{U \cap \Omega}) \cap C^2(U \cap \Omega)$ function w satisfying $M(w) \le 0$, $w(0) = 0$, $w|_{\partial U \cap \Omega} \ge K\delta$, and $w(x) \ge K|x_1|$, $x \in \Sigma \cap U$. Thus taking K such that $K\delta \ge 2\max\{\sup|\psi|, \text{Lip }\psi\}$ we can assert that $w \ge \psi$ on $\Sigma \cap U$ and $w \ge u$ on $\partial U \cap \Omega$. (Here we assume δ is chosen small enough to ensure that (1.1) holds with $L\delta < 1$.) Hence by standard barrier results for solutions to (0.1), we conclude that $u \le w$ everywhere in $U \cap \Omega$. The results (2.2), (2.3) now follow because, by construction of w, we have $w(x) \le c|x|^{1/3}$ for $x \in U \cap \Omega$ and $w(x) \le c|x|$ for $x \in \Sigma \cap U$, with c depending only on β, δ, and K.

We want to conclude this paper by presenting, for each integer $k \ge 1$, examples to show that u may have no limit at $\xi \in \partial\Omega$ even when $\Sigma = \partial\Omega$ is $C^{k+1,1}$ and when the curvature $H_\Sigma(x) \ge -|x|^k$, $x \in \Sigma \cap B_\delta(\xi)$, for some $\delta > 0$. Indeed we give an example to show that u may have no limit at ξ even when the function ψ of (0.1) is constant in a neighbourhood of ξ.

To do this we take any two bounded domains Ω_0 and $\Omega_1 \subset \mathbb{R}^2$ with $0 \in \partial\Omega_0 \cap \partial\Omega_1$, $\Sigma_j = \partial\Omega_j \cap B_{2\delta}(0)$ connected, $\Omega_0 \subset \Omega_1$,

$\Omega_1 \sim \Omega_0 \subset B_{\delta/2}(0)$, and such that the following conditions hold, in which H_j denotes the curvature of Σ_j relative to the inward unit normal η_j of Σ_j :

$$(2.4) \qquad \begin{cases} H_j \le 0 \text{ on } \Sigma_j , \quad \eta_j(0) = (0, 1) , \\ |\eta_j(x)-\eta_j(0)| < \tfrac{1}{2} , \quad x \in \Sigma_j , \quad j = 0, 1 ; \end{cases}$$

$$(2.5) \qquad -|x_1|^k \le H_0(x) \le -\tfrac{1}{2}|x_1|^k , \quad x \in \Sigma_0 , \quad |x_1| < \delta ;$$

and

$$(2.6) \qquad \begin{cases} H_1 \equiv 0 \text{ on } V_q \text{ for some neighbourhood } V_q \text{ of } \xi_q ; \\ \zeta_q \in \Sigma_0 \cap \Sigma_1 , \end{cases}$$

where $\{\xi_q\}$, $\{\zeta_q\}$ are sequences of points in $\mathbb{R}^2$ with $\lim \xi_q = \lim \zeta_q = 0$. (It is left to the reader to check that such domains Ω_0 and Ω_1 can be constructed in such a way that both $\partial\Omega_0$ and $\partial\Omega_1$ are $C^{k+1,1}$. We emphasize that, apart from the above restriction, Ω_0 and Ω_1 are quite arbitrary.)

Now let U be an arbitrary neighbourhood of $\{x \in \Sigma_0 : |x_1| < \delta\}$ with $U \cap \partial\Omega_0 = U \cap \Sigma_0 = \{x \in \partial\Omega_0 : |x_1| < \delta\}$. Evidently, since $H_0 < 0$ on $U \cap (\Sigma_0 \sim \{0\})$, we can find a solution u_0 of the problems (0.1) with Ω_0 in place of Ω and ψ_0 in place of ψ , where $\psi_0 \equiv 1$ on $\Sigma_0 \cap \{x : |x_1| < \delta/2\}$ and

$$\limsup_{\substack{\xi \to x \\ \xi \in U \cap \Omega_0}} u_0(\xi) < -1 \quad \text{for all } x \in \partial U \cap \overline{\Omega}_0 .$$

(One achieves this by taking $\psi_0 \le 1$ everywhere on $\partial\Omega_0$ and $\psi_0 \le -L$ on $\partial\Omega_0 \sim \{x : |x_1| > 3\delta/4\}$, where L is a sufficiently large constant.)

We now introduce coordinates y_1, y_2, y_3 for $\mathbb{R}^3$ as described in §1.

Then (2.5) implies that (1.11) holds with Σ_0 in place of Σ and with $\mu = \frac{1}{2}$. Hence, by the discussion of the latter part of §1, we can find a function $\tilde{w} = \tilde{w}(y_1, y_2) = w + \alpha y_2^{k+2}\phi(y_1/y_2)$ with $M(\tilde{w}) \le 0$ on $W = \{(y_1, y_2) : 0 < y_2 < \delta, |y_1|/y_2 < 1, |y_1| < \delta\}$ and with graph $\tilde{w}$ corresponding, in the (x_1, x_2, x_3) coordinates, to graph w, where $w \in C^2(U \cap \Omega_0)$ with $U \cap \partial\Omega_0 = \{x \in \Sigma_0 : |x_1| < \delta\}$. From construction w has the properties:

$$\frac{\partial w}{\partial \eta_0} = \infty \quad \Big(\text{in the sense that} \lim_{\substack{\xi \to x \\ \xi \in U \cap \Omega_0}} Dw(\xi)\cdot\eta_0(x) = \infty \text{ for each } x \in \Sigma_0 \cap U\Big),$$

$$0 < w \le \delta \text{ on } U \cap \Omega_0 , \quad w = \delta \text{ on } \partial U \cap \Omega_0 ,$$

$$M(w) \ge 0 \text{ on } U .$$

Hence we have $M(-w) \le 0$, $\partial(-w)/\partial\eta_0 = -\infty$, $-\delta \le -w < 0$ on $U \cap \Omega_0$, $-w = -\delta$ on $\partial U \cap \Omega_0$. Since, as described above, we can arrange $u_0 < -1 < -w$ on $\partial U \cap \Omega_0$, we then deduce from a standard comparison principle that $u_0 \le -w$ on $\overline{U}$. In particular we deduce $u_0 \le 0$ on $U \cap \partial\Omega_0$.

If we now let u_1 be the solution of (0.1) with Ω_1 in place of Ω and with ψ_1 in place of ψ, where $\psi_1 \le 1$ everywhere and $\psi_1 = \psi_0$ on $\partial\Omega_0 \cap \partial\Omega_1$. Then, since by the maximum principle $u_1 \le 1 \le \psi_0$ on $\Sigma_0 \cap \Omega_1$, we deduce $u_1 \le u_0$ on Ω_0. On the other hand it is standard, since $H_1 \equiv 0$ on V_q, that $u_1 \equiv \psi_1$ on V_q. Combining these facts we have, for all q large enough to ensure that $|\xi_q| < \delta/2$ and $|\zeta_q| < \delta/2$, $u_1(\xi_q) = 1$ and $u_1(\zeta_q) \le u_0(\zeta_q) \le 0$. Thus u_1 is not continuous at 0.

Thus we have established the following theorem.

THEOREM 2. *Let* Ω_0, Ω_1 *be any* $C^{k+1,1}$ *domains as described above*

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(in (2.4), (2.5), (2.6)) *and let* u_1 *be the solution of* (0.1) *with* Ω_1 *in place of* Ω *and with* ψ_1 *in place of* ψ . (ψ_1 *as described above.*) *Then* $\lim_{\substack{x\to 0\\ x\in\Omega_1}} u_1(x)$ *does not exist. In fact the trace of* u_1 *on* $\partial\Omega_1$ *has no limit at* $x = 0$.

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