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BOOK REVIEWS

ROMANOWSKA, A. B. and SMITH, J. D. H., *Modal theory: an algebraic approach to order, geometry and convexity* (Research and Exposition in Mathematics 9, Heldermann Verlag, Berlin 1985) xii + 158 pp.

This research monograph introduces the new algebraic discipline concerned with modes and modals. This is in fact the study of a particular kind of universal algebras. A universal algebra (A,Ω) is idempotent if each element is idempotent for each operation (i.e. $x\dots x\omega)=x$ for all $x\in A$, $\omega\in\Omega$) and it is entropic if for each $\omega\in\Omega$, ω of type n, then $\omega:(A^n,\Omega)\to(A,\Omega)$ is an Ω-homomorphism, where (A^n,Ω) is the Ω-algebra direct product of n copies of A. Thus entropy can be thought of as a kind of distributivity. Modes are universal algebras which are idempotent and entropic. A modal $(D,+,\Omega)$ is an algebra such that $(D,+)$ is a join semilattice, (D,Ω) is a mode and the operations Ω distribute over $(D,\ +)$. Thus a modal is like a semilattice module over a mode.

The authors provide a background of universal algebra, then introduce varieties of modes and modals in the first three chapters. The last three chapters deal with applications to numbers, lattices and a general structure theory. The natural context for this theory is that of ordered sets and convexity, hence the title. In that context, a general algebraic theory is developed with some interesting structural results and applications. For those who are interested in an abstract approach to convexity and order from an algebraic point of view, this volume will be of interest. But of course, it is a specialised area, though one that could develop into a useful discipline. The book is produced from camera-ready typescript and the standard of production is quite good.

J. D. P. MELDRUM

BRYANT, V., *Metric spaces: iteration and application* (Cambridge University Press, 1985) 105 pp., cloth £15.00, paper £4.95.

This excellent little book contains an introduction to metric spaces based on the idea of obtaining roots of equations by iteration. The author's aim has been "to provide a book which can be read and enjoyed by a wide range of students" not only specialist mathematicians but also others, such as engineers, for whom an overdose of ε's and δ's would soon prove fatal.

The first chapter provides the motivation for all that follows. Basic ideas are introduced in a natural way and illustrated by means of examples. The following chapters then develop the theory required to discuss the introductory ideas rigorously. After the definition of a metric space and a good supply of examples for later use, convergence of sequences, closed sets, completeness and (sequential) compactness are treated in turn. (Open sets and compactness via open covers are mentioned briefly towards the end of the book.) The climax is the Contraction Mapping Theorem, with applications to linear algebra, differential equations and the Implicit Function Theorem. As a postlude, the final chapter shows how ideas from classical analysis such as continuity and the Intermediate Value Theorem can be related to metric spaces.

The presentation is lucid and the author's enjoyment of the subject and interest in his reader's well-being shine brightly. Exercises are liberally sprinkled throughout the text. There is the odd misprint or infelicity of style but these are insignificant. The author has succeeded admirably in his aim and the book can be warmly recommended. It should woo many an engineer, while

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mathematicians will find it a useful preparation for grappling with the more traditional approach in other books.

ADAM C. MCBRIDE

KAHANE, J.-P., *Some random series of functions* (Cambridge Studies in Advanced Mathematics 5, Cambridge University Press, 2nd ed., 1985) 305 pp., £30.00.

By the time the invitation to review this book arrived I had already bought my own copy to displace the first edition [D.C. Heath, Lexington 1968] from my desk. Harmonic analysts will not have waited to read reviews before ensuring that, at the very least, their department libraries acquire a copy.

What sort of problem does Kahane consider? Everyone knows that if we play at heads and tails repeatedly for stakes $a_0, a_1, \ldots, a_N \in \mathbb{R}$ then (provided certain regularity conditions are observed) my winnings $\sum_{j=0}^{N} \pm a_j$ will be approximately normally distributed with mean 0 and variance $\sum_{j=0}^{N} a_j^2$. Thus, if I choose signs $\pm$ at random, then, with high probability,

$$\left|\sum_{j=0}^{N} \pm a_j\right| \text{ is of comparable size to } \left(\sum_{j=0}^{N} a_j^2\right)^{1/2}.$$

It is not difficult to extend this result to two dimensions and obtain for $a_0, a_1, \ldots, a_N \in \mathbb{C}$ (subject to certain regularity conditions) that, with high probability,

$$\left|\sum_{j=0}^{N} \pm a_j\right| \text{ is of comparable size to } \left(\sum_{j=0}^{N} |a_j|^2\right)^{1/2}.$$

From this it is not hard to deduce that if $a_0, a_1, \ldots \in \mathbb{C}$ then, with high probability,

$$\sup_N \left|\sum_{j=0}^{N} \pm a_j\right| \text{ is of comparable size to } \left(\sum_{j=0}^{\infty} |a_j|^2\right)^{1/2}$$

and so, if $\sum_{j=0}^{\infty} |b_j|^2 = \infty$, then, with probability 1,

$$\sup_N \left|\sum_{j=0}^{N} \pm b_j r^j\right| \to \infty \quad \text{as } r \to 1-.$$

It follows that if $\sum |c_j|^2 = \infty$ and $|\omega| = 1$, then, with probability 1,

$$\sup_N \left|\sum_{j=0}^{N} \pm c_j \omega^j r^j\right| \to \infty \quad \text{as } r \to 1-.$$

But (thanks to the ideas of Borel, Lebesgue and Kolmogorov) we can state that if each of a countable collection of events occurs with probability 1 then they all happen together with probability 1. Thus if $\omega_1, \omega_2, \ldots$ form a countable dense subset of the circle $|z| = 1$ we know that, with probability 1,

$$\sup_N \left|\sum_{j=0}^{N} \pm c_j \omega_k^j r^j\right| \to \infty \quad \text{as } r \to 1- \text{ for each of } j = 1, 2, \ldots.$$

We have, in fact, proved the following theorem:

Theorem A. *If* $c_0, c_1, \ldots \in \mathbb{C}$, $\sum_{j=0}^{\infty} |c_j|^2 = \infty$ and $\limsup |c_n|^{1/n} = 1$ then, with probability 1, the circle of convergence $|z| = 1$ is a natural boundary for the Taylor series $\sum_{j=0}^{\infty} \pm c_j z^j$.