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S_p TRANSFORM AND UNIFORM CONVERGENCE OF LAURENT AND POWER SERIES

BY

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ABSTRACT. If the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n z^{-n} \quad (|z| > 1)$$

is transformed to

$$f(z) = \sum_{n=0}^{\infty} \frac{\alpha_n p^n}{(1-p)^n}\left(\frac{1}{p} - z\right)^n \quad \left(\left|z - \frac{1}{p}\right| < \frac{1}{p} - 1, 0 < p < 1\right),$$

it is shown that convergence of the former at $z = 1$ implies the uniform convergence of the latter on a symmetric arc of $|z - 1/p| = 1/p - 1$ not containing $z = 1$ and that the uniform convergence of the former over a symmetric arc of $|z| = 1$ containing $z = 1$ implies uniform convergence of the latter on the entire circle $|z - 1/p| = 1/p - 1$.

1. **Introduction**. Let $f(z)$ be defined by the series

$$f(z) = \sum_{n=0}^{\infty} a_n z^{-n} \tag{1.1}$$

which is assumed to converge outside the closed disc $|z| \le 1$. We can write

$$f(z) = \sum_{n=0}^{\infty} \frac{\alpha_n p^n}{(1-p)^n}\left(\frac{1}{p} - z\right)^n \quad \text{for } \left|z - \frac{1}{p}\right| < \frac{1}{p} - 1, \tag{1.2}$$

where

$$\alpha_n = (1-p)^n \sum_{k=0}^{\infty} \binom{n+k-1}{n} p^k a_k, \; 0 < p(\text{fixed}) < 1, \; \binom{-1}{0} = 1$$
$$(n = 0, 1, \ldots).$$

We note that if S_p is the Meyer-König-Vermes matrix defined by

$$(S_p)_{nk} = (1-p)^n \binom{n+k-1}{n} p^k$$

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then

$$\boldsymbol{\alpha} = s_p \boldsymbol{a},$$

where $\boldsymbol{\alpha} = \{\alpha_0, \alpha_1, \dots\}$ and $\boldsymbol{a} = \{a_0, a_1, \dots\}$.

In this paper we show that an assumption of convergence of (1.1) at the single point $z = 1$ implies the uniform convergence of (1.2) on a symmetric arc of $|z - 1/p| = 1/p - 1$ not containing 1 and that the uniform convergence of (1.1) over a symmetric arc of $|z| = 1$ containing 1 implies uniform convergence of (1.2) on the entire circle $|z - 1/p| = 1/p - 1$.

The present work is motivated by the treatment of the Taylor transform as applied to a power series given by Jakimovski and Meyer-König [2].

2. **Results**. More explicitly we prove the following two results.

THEOREM 1. *Assume that* $\Sigma_0^\infty a_n$ *is convergent and let* ψ_0 *be a given real number* $(0 < \psi_0 < \pi)$. *Then the power series expansion* (1.2) *of the function* $f(z)$ *in* (1.1) *is uniformly convergent for* $z = 1/p - (1/p - 1)e^{i\psi}(\psi_0 \leq \psi \leq 2\pi - \psi_0)$.

THEOREM 2. *Assume that there exists a real number* $\varphi_0(0 < \varphi_0 < \pi)$ *such that the Laurent series (1.1) is uniformly convergent for* $z = e^{i\varphi}(-\varphi_0 \leq \varphi \leq \varphi_0)$. *Then the power series (1.2) is uniformly convergent on the circle* $|z - 1/p| = 1/p - 1$.

Of these two results, Theorem 1 can be deduced from a generalization of Fatou's theorem (see [5], p. 93) after transforming (1.2) by

$$\omega = \frac{1 - pz}{1 - p},$$

observing that $|\omega| < 1$ when $|z - 1/p| < 1/p - 1$. We however prove Theorem 1 directly using the same tools to prove Theorem 2 too.

3. **Auxiliary results**. To describe the procedure we construct a matrix A which transforms the partial sums of (1.1) into the partial sums of (1.2). In this context we assume only that $\boldsymbol{\alpha} = S_p \boldsymbol{a}$ exists noting that a necessary and sufficient condition therefor (see [3], p. 272) is

$$(3.1) \qquad a_k = O\left(\frac{1}{k^n p^k}\right) \qquad \text{for fixed } n = 0, 1, \dots \text{ as } k \to \infty.$$

Let u and v denote points of the circles $|z| = 1$ and $|z - 1/p| = 1/p - 1$, respectively. We shall use the notation

$$(3.2) \qquad u = e^{i\varphi}(0 \leq \varphi < 2\pi),$$

$$(3.3) \qquad \beta = e^{i\psi}(0 \leq \psi < 2\pi),$$

$$(3.4) \qquad v = \frac{1}{p} - \left(\frac{1}{p} - 1\right)\beta; \qquad \text{i.e. } \beta = \frac{1 - pv}{1 - p},$$

$$
(3.5)\qquad
\begin{aligned}
t_n &= a_0 + \frac{a_1}{u} + \ldots + \frac{a_n}{u^n}\\
\gamma_n &= \alpha_0 + \alpha_1\beta + \ldots + \alpha_n\beta^n
\end{aligned}
$$

and, for $n = 0, 1, \ldots$ (n fixed),

$$
(3.6)\qquad \eta_k \equiv \eta_k(n) = (pu)^k \sum_{m=0}^{n} \binom{m+k-1}{m} (1-p)^m \beta^m \qquad (k = 0, 1, \ldots).
$$

In the first instance, because of (3.1),

$$
|t_k \eta_k| \le (|a_0| + \ldots + |a_k|)|\eta_k| \le \frac{M}{k^{n+1}} \sum_{m=0}^{n} \binom{m+k-1}{m} (1-p)^m \qquad (k = 1, 2, \ldots)
$$

for a suitable constant M. Consequently

$$
(3.7)\qquad t_k\eta_k \to 0 \text{ as } k \to \infty.
$$

Using this fact we can write

$$
\begin{aligned}
\gamma_n &= \alpha_0 + \alpha_1\beta + \ldots + \alpha_n\beta^n\\
&= \sum_{m=0}^{n} \beta^m(1-p)^m \sum_{k=0}^{\infty} \binom{m+k-1}{m} p^k a_k\\
&= \sum_{k=0}^{\infty} (t_k - t_{k-1})\eta_k\\
&= \sum_{k=0}^{\infty} (\eta_k - \eta_{k+1}) t_k\\
&= \sum_{k=0}^{\infty} a_{nk} t_k,
\end{aligned}
$$

where

$$
a_{nk} = (pu)^k \sum_{m=0}^{n} (1-p)^m \beta^m \left[\binom{m+k-1}{m} - pu \binom{m+k}{m} \right].
$$

Rewriting

$$
(3.8)\qquad
\begin{aligned}
a_{nk} = {} & (1-pu)(pu)^k \binom{n+k}{n} (1-pv)^n + (pu)^k p(v-u)\\
& \times \sum_{m=0}^{n-1} \binom{m+k}{m} (1-pv)^m \qquad (n, k = 0, 1, \ldots).
\end{aligned}
$$

This proves that

$$
(3.9)\qquad \boldsymbol{\gamma} = A\boldsymbol{t}
$$

where $A = A(p, u, v)$ is the matrix with the elements a_{nk}. If $u = v = 1$, then (3.9) reduces to the well-known relation

$$\sigma = Fs,$$

where

$$A(p,1,1) = F = \left(\frac{1-p}{p}(S_p)_{nk+1}\right)$$

is a regular sequence to sequence matrix (see [4], p. 558).

In the following theorem we establish the convergence preserving nature of the matrix A.

THEOREM 3. *The matrix* $A = A(p,u,v)$ *defines a sequence to sequence convergence preserving transformation for each triple* (p,u,v) *with*

$$0 < p < 1,\ |u| = 1,\ v = \frac{1}{p} - \left(\frac{1}{p} - 1\right)\beta\ (|\beta| = 1,\ \beta \neq 1).$$

To prove this we need the following lemma.

LEMMA 4. *Let the real number* p $(0 < p < 1)$ *and the complex number*

$$v = \frac{1}{p} - \left(\frac{1}{p} - 1\right)\beta \quad \text{with} \quad \beta = e^{i\psi}\ (0 < \psi < 2\pi)$$

be given. Then

$$\omega_n = \sum_{k=0}^{\infty} p^{k+1}\left|\sum_{m=0}^{n-1}\binom{m+k}{m}(1-p)^m\beta^m\right| \leq \frac{1}{|v|-1} + \frac{4}{|v-1|} \qquad (n = 1,2,\ldots).$$

PROOF. We first show that the series which defines ω_n is convergent. Since $|(1-p)\beta| = 1 - p$ we have

$$\omega_n \leq \sum_{m=0}^{n-1}(1-p)^m p \sum_{k=0}^{\infty}\binom{m+k}{m}p^k = \frac{np}{1-p}.$$

Let $\mu = \dfrac{p}{1-p}$ so that $0 < \mu < \infty$. We write

$$\omega_n = \sum_{k=0}^{\infty} p^{k+1}\left|\sum_{m=0}^{n-1}\binom{m+k}{m}(1-p)^m\beta^m\right| = T_1 + T_2 \qquad (n = 1,2,\ldots)$$

with

$$T_1 = \sum_{k\geq\mu n}, \qquad T_2 = \sum_{k<\mu n}.$$

Applying Abel's inequality to the innersum of T_1 we get

$$T_1 \leq \frac{2(1-p)^{n-1}}{|1-\beta|}\sum_{k\geq\mu n} p^{k+1}\binom{n+k-1}{k}$$

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$$\leq \frac{2p(1-p)^{n-1}}{|1-\beta|} \sum_{k=0}^{\infty} \binom{n+k-1}{k} p^k$$

$$= \frac{2p}{(1-p)|1-\beta|} = \frac{2}{|\nu - 1|}.$$

For T_2 we have

$$T_2 = \sum_{k<\mu n} p^{k+1} \left| \sum_{m=0}^{\infty} \binom{m+k}{m} (1-p)^m \beta^m - \sum_{m=n}^{\infty} \binom{m+k}{m} (1-p)^m \beta^m \right| \leq T_2' + T_2'',$$

where

$$T_2' = \sum_{k<\mu n} p^{k+1} \left| \sum_{m=0}^{\infty} \binom{m+k}{m} (1-p)^m \beta^m \right| \leq \sum_{k=0}^{\infty} \frac{1}{|\nu|^{k+1}} = \frac{1}{|\nu| - 1},$$

and

$$T_2'' = \sum_{k<\mu n} p^{k+1} \left| \sum_{m=n}^{\infty} \binom{m+k}{m} (1-p)^m \beta^m \right|.$$

Again applying Abel's inequality to the inner sum of T_2'' we get

$$T_2'' \leq \frac{2(1-p)^n}{|1-\beta|} \sum_{k<\mu n} \binom{n+k}{k} p^{k+1} \leq \frac{2(1-p)^n}{|1-\beta|} p \sum_{k=0}^{\infty} \binom{n+k}{k} p^k$$

$$= \frac{2p}{(1-p)|1-\beta|} = \frac{2}{|\nu - 1|}.$$

Hence

$$\omega_n \leq T_1 + T_2' + T_2'' \leq \frac{1}{|\nu| - 1} + \frac{4}{|\nu - 1|},$$

and the lemma is proved.

PROOF OF THEOREM 3. It is enough to show that the matrix A satisfies the well known necessary and sufficient conditions for a matrix to be conservative (see e.g. [1]).

$$\lim_{n\to\infty} a_{nk} = \left(1 - \frac{u}{\nu}\right)\left(\frac{u}{\nu}\right)^k \qquad (k = 0, 1, \ldots).$$

If $\boldsymbol{a} = \{1, 0, 0, \ldots\}$, then $\boldsymbol{\alpha} = \{1, 0, 0, \ldots\}$ and $\boldsymbol{t} = \boldsymbol{\gamma} = \{1, 1, \ldots\}$ so that (3.9) gives

$$\sum_{k=0}^{\infty} a_{nk} = 1 \qquad (n = 0, 1, 2, \ldots).$$

Now,

$$\sum_{k=0}^{\infty} |a_{nk}| \leq S_1 + S_2,$$

where

$$S_1 \leq |1 - pu|(1-p)^n \sum_{k=0}^{\infty} \binom{n+k}{k} p^k = \frac{|1 - pu|}{1 - p}$$

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and

$$S_2 \leq |v-u| \sum_{k=0}^{\infty} p^{k+1} \left| \sum_{m=0}^{n-1} \binom{m+k}{m} (1-p)^m \beta^m \right|.$$

By Lemma 4 we get

$$S_2 \leq \frac{|v-u|}{|v|-1} + 4\frac{|v-u|}{|v-1|}$$

and hence we have

$$\sum_{k=0}^{\infty} |a_{nk}| \leq \frac{|1-pu|}{1-p} + \frac{|v-u|}{|v|-1} + 4\frac{|v-u|}{|v-1|} \qquad (n = 0, 1 \ldots).$$

Hence Theorem 3.

Since $A(p, u, v)$ is convergence preserving $\boldsymbol{\gamma} = A\boldsymbol{t}$ is convergent. Also

$$\begin{aligned}\lim_{n\to\infty} \gamma_n &= \left(1 - \frac{u}{v}\right) \sum_{k=0}^{\infty} t_k \left(\frac{u}{v}\right)^k + \left(\sum_{m=0}^{\infty} a_m u^{-m}\right)\left[1 - \left(1 - \frac{u}{v}\right) \sum_{k=0}^{\infty} \left(\frac{u}{v}\right)^k\right] \\ &= \sum_{m=0}^{\infty} a_m u^{-m} \left(\frac{u}{v}\right)^m \\ &= \sum_{m=0}^{\infty} a_m v^{-m}.\end{aligned}$$

4. Proof of Theorems 1 and 2. Let us assume that $\Sigma_0^\infty a_n = s$. By Theorem 3 the matrix $A(p, 1, v)$ is convergence preserving for $0 < p < 1$,

$$v = \frac{1}{p} - \left(\frac{1}{p} - 1\right)\beta \qquad (|\beta| = 1,\ \beta \neq 1).$$

Therefore $\Sigma_0^\infty \alpha_n \beta^n$ converges pointwise on the whole circle $|\beta| = 1$. But this convergence is not uniform on the whole circle $|\beta| = 1$. To prove this we put

$$z_n = \frac{1}{p} - \left(\frac{1}{p} - 1\right) e^{i\pi/n+1}.$$

Now

$$\left|1 - \frac{1}{z_n}\right| \sum_{k=0}^{\infty} \frac{1}{|z_n|^k} = \frac{|z_n - 1|}{|z_n| - 1} \to \infty \text{ as } n \to \infty.$$

So there exists a sequence $\boldsymbol{s} = \{s_0, s_1, \ldots, s_n, \ldots\}$ with the properties $s_n \to 0$ and $(1 - 1/z_n) \Sigma_{k=0}^\infty s_k z_n^{-k}$ not bounded. Define

$$f(z) = \left(1 - \frac{1}{z}\right) \sum_{k=0}^{\infty} s_k z^{-k} \quad \text{for } |z| > 1.$$

Then

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$$f(z) = \sum_{k=0}^{\infty} a_k z^{-k},\ a_k = s_k - s_{k-1} \qquad (k = 0, 1, \ldots),\ s_{-1} = 0.$$

Now $f(1) = \Sigma_{k=0}^{\infty} a_k = 0$. If $\boldsymbol{\alpha} = S_p \boldsymbol{a}$, then we have

$$(1.2) \qquad f(z) = \sum_{k=0}^{\infty} \frac{\alpha_k p^k}{(1-p)^k}\left(\frac{1}{p} - z\right)^k \qquad \text{for } \left|z - \frac{1}{p}\right| \leq \frac{1}{p} - 1.$$

If this series converges uniformly for $|z - 1/p| = 1/p - 1$, then it would be uniformly convergent for $|z - 1/p| \leq 1/p - 1$ and $f(z)$ would be continuous on the disc $|z - 1/p| \leq 1/p - 1$ which contradicts the fact that $\{f(z_n)\}$ is not bounded.

In other words (1.2) does not converge uniformly on the entirety of its circle of convergence when (1.1) converges for $z = 1$. However Theorem 1 holds.

DIRECT PROOF OF THEOREM 1. Let $\Sigma_0^{\infty} a_n = s$,

$$v = \frac{1}{p} - \left(\frac{1}{p} - 1\right)\beta \text{ with } \beta = e^{i\psi}\ (0 < \psi < 2\pi),$$

$$s_n = a_0 + a_1 + \ldots + a_n$$

and

$$\gamma_n = \alpha_0 + \alpha_1\beta + \ldots + \alpha_n\beta^n \text{ with } \boldsymbol{\alpha} = S_p\boldsymbol{a} \qquad (n = 0, 1, ..).$$

Then

$$\boldsymbol{\gamma} = B\,\boldsymbol{s},$$

where $B = A(p, 1, v)$. The matrix B has the column limits

$$b_k = \left(1 - \frac{1}{v}\right)\left(\frac{1}{v}\right)^k \qquad (k = 0, 1, \ldots)$$

and row sum

$$\sum_{k=0}^{\infty} b_{nk} = 1 \qquad (n = 0, 1, \ldots).$$

Since $\Sigma_0^{\infty} b_k = 1$, we have

$$\sum_{k=0}^{\infty} (b_{nk} - b_k) = 0 \qquad (n = 0, 1, \ldots)$$

and

$$\gamma_n - \sum_{k=0}^{\infty} b_k s_k = \sum_{k=0}^{\infty} (b_{nk} - b_k)(s_k - s) \qquad (n = 0, 1, \ldots).$$

Given $\epsilon > 0$, there exist a $K > 0$ and a natural number $m = m(\epsilon)$ such that

$$|s_k - s| < K \text{ for all } k, \qquad |s_k - s| < \epsilon \text{ for } k > m.$$

This yields the estimate

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$$\left|\gamma_n - \sum_{k=0}^{\infty} b_k s_k\right| \leq K \sum_{k=0}^{m} |b_{nk} - b_k| + \epsilon \sum_{k=m+1}^{\infty} |b_{nk} - b_k|.$$

Now $\lim_{n\to\infty} b_{nk} = b_k$ implies that there exists a natural number $N = N(\epsilon)$ such that

$$|b_{nk} - b_k| < \epsilon \text{ for } n > N \qquad \text{and } k = 0, 1, \ldots, m.$$

Thus

$$\left|\gamma_n - \sum_{k=0}^{\infty} b_k s_k\right| \leq Km\epsilon + \epsilon\left(\sum_{k=0}^{\infty} |b_{nk}| + \sum_{k=0}^{\infty} |b_k|\right) \leq \epsilon\left(Km + 5 + 2\frac{|\nu - 1|}{|\nu| - 1}\right).$$

The factor multiplying ϵ is less than a constant independent of ν but depending on ψ_0 under the restriction $\psi_0 \leq \psi \leq 2\pi - \psi_0$. Theorem 1 is proved.

PROOF OF THEOREM 2. It is enough to prove this theorem for small values of p and φ_0; so we assume in addition that

$$0 < p < \frac{1}{2}, \qquad 0 < \varphi_0 < \frac{\pi}{4}.$$

There are uniquely defined numbers ν_0 and ψ_0 $(3\pi/2 < \psi_0 < 2\pi)$ such that

$$\nu_0 = \frac{1}{p} - \left(\frac{1}{p} - 1\right)\beta_0 \text{ with } \beta_0 = e^{i\psi_0}, \qquad \nu_o = |\nu_0| u_0 \text{ with } u_0 = e^{i\varphi_0}.$$

We put

(4.1) $$f(u) = \sum_{n=0}^{\infty} a_n u^{-n} \text{ with } u = e^{i\varphi} \qquad (0 < |\varphi| \leq \varphi_0)$$

(4.2) $$\nu = \frac{1}{p} - \left(\frac{1}{p} - 1\right)\beta \text{ with } \beta = e^{i\psi}(2\pi - \psi_0 > \psi > \psi_0), \qquad \beta \neq 1$$

(4.3) $$t_n(u) = \sum_{k=0}^{n} a_k u^{-k}, \qquad \gamma_n(\nu) = \sum_{k=0}^{n} \alpha_k \beta^k.$$

By hypothesis the series in (4.1) converges uniformly with respect to u in the closed interval $-\varphi_0 \leq \varphi \leq \varphi_0$. We show that

$$\sum_{n=0}^{\infty} \alpha_n \beta^n = \lim_{n\to\infty} \gamma_n$$

exists uniformly for the values of β and ν specified in (4.2). For the values u specified in (4.1) we have

(4.4) $$\boldsymbol{\gamma}(\nu) = A(p, u, \nu)\boldsymbol{t}(u).$$

Connecting now u and ν by the relation $u = \nu/|\nu|$ and putting

$$C = A(p, \nu/|\nu|, \nu)$$

(4.4) reduces to

$$\gamma(v) = Ct\left(\frac{v}{|v|}\right).$$

The column limits of the matrix C are

$$c_k = \left(1 - \frac{1}{|v|}\right)\left(\frac{1}{|v|}\right)^k \qquad (k = 0, 1, \ldots).$$

The row sum of C equals 1 and $\Sigma_{k=0}^{\infty} c_k = 1$. So we have

$$\sum_{k=0}^{\infty} (c_{nk} - c_k) = 0. \tag{4.5}$$

Now,

$$\begin{aligned} f(v) &= \sum_{k=0}^{\infty} a_k v^{-k} \\ &= \sum_{k=0}^{\infty} \left[t_k\left(\frac{v}{|v|}\right) - t_{k-1}\left(\frac{v}{|v|}\right)\right]\frac{1}{|v|^k} \\ &= \left(1 - \frac{1}{|v|}\right) \sum_{k=0}^{\infty} \left(\frac{1}{|v|}\right)^k t_k\left(\frac{v}{|v|}\right) \\ &= \sum_{k=0}^{\infty} c_k t_k\left(\frac{v}{|v|}\right). \end{aligned}$$

From this and (4.5) it follows that

$$\gamma_n(v) - f(v) = \sum_{k=0}^{\infty} (c_{nk} - c_k)\left[t_k\left(\frac{v}{|v|}\right) - f\left(\frac{v}{|v|}\right)\right].$$

Given $\epsilon > 0$, there exist a constant $K > 0$ and a natural number $m = m(\epsilon)$ such that

$$\left|t_k\left(\frac{v}{|v|}\right) - f\left(\frac{v}{|v|}\right)\right| \le K \qquad \text{for all } k$$

$$\left|t_k\left(\frac{v}{|v|}\right) - f\left(\frac{v}{|v|}\right)\right| < \epsilon \qquad \text{for } k > m,$$

where these inequalities are true uniformly for all v under consideration. This yields

$$\left|\gamma_n(v) - f(v)\right| \le K \sum_{k=0}^{m} |c_{nk} - c_k| + \epsilon \sum_{k=m+1}^{\infty} |c_{nk} - c_k|.$$

Since $c_{nk} \to c_k$ as $n \to \infty$, we have a natural number $N = N(\epsilon)$ such that

$$|c_{nk} - c_k| < \epsilon \text{ for } n > N \qquad (k = 0, 1, \ldots, m).$$

Thus, for $n > N$,

$$\begin{aligned} |\gamma_n(v) - f(v)| &\le \epsilon K m + \epsilon\left(\sum_{k=0}^{\infty} |c_{nk}| + \sum_{k=0}^{\infty} |c_k|\right) \\ &= \epsilon\left(Km + 1 + \sum_{k=0}^{\infty} |c_{nk}|\right). \end{aligned}$$

Since $|1 - pu| \leq 1 + p$, $|v - u| = |v| - 1$ and $|v - 1| \geq |v| - 1$ we have, as in proof of Theorem 3,

$$\sum_{k=0}^{\infty} |c_{nk}| \leq 5 + \frac{1+p}{1-p} \qquad (n = 0, 1, \ldots).$$

Hence

$$|\gamma_n(v) - f(v)| \leq \epsilon\left(6 + Km + \frac{1+p}{1-p}\right) \qquad \text{for } n > N.$$

Combining with Theorem 1 and the convergence of $\Sigma_0^\infty \alpha_n$ the proof of Theorem 2 is now complete.

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