ON THE DENSITY OF THE INVERTIBLE GROUP IN C^*-ALGEBRAS

by GUYAN ROBERTSON

(Received 9th January 1975)

1.

In what follows the term C^*-algebra will mean a complex C^*-algebra with identity. We denote the identity element by 1. We shall also use the notation and terminology of Dixmier (3) without comment.

Let A be a C^*-algebra. It is well known that the group $G(A)$ of invertible elements of A is " large ", in the sense that its subgroup $U(A)$ of unitary elements actually spans the algebra. Our first result shows that $G(A)$ is in fact always dense in A in the weak (Banach space) topology. The situation is more complicated when we look at the norm topology on A. For example, if M is a von Neumann algebra then Choda (2) has shown that M has dense invertible group if and only if M is finite. Now suppose C is an abelian C^*-algebra, with maximal ideal space X. It is easily seen from (8, Theorem VII 4) that C has dense invertible group if and only if the topological covering dimension of X is less than 2. We use this to give a necessary condition for a homogeneous C^*-algebra to have dense invertible group.

We conclude in Section 4 with some miscellaneous results and remarks.

2.

Our result for the weak topology is based on a Proposition of Dixmier and Maréchal (4), which states that in a von Neumann algebra the invertible group is always dense in the strong operator topology.

Proposition 1. *Let A be a C^*-algebra. Then $G(A)$ is weakly dense in A.*

Proof. Let A act on the Hilbert space H in its universal representation. Then the strong operator closure A^- of A is the enveloping von Neumann algebra of A.

Let a be an element of the closed unit ball A_1 of A, and let V be a strong neighbourhood of a in A^-. By (4), $G(A^-)$ is strongly dense in A^-. Hence there exists an element x in $V \cap G(A^-)$. Now x has polar decomposition $x = vh$, where v is a unitary in A^-, and h is a positive element in A_1^-. By the Kaplansky density theorem, the positive part of A_1 is strongly dense in the positive part of A_1^-. By the Glimm-Kadison density theorem (6, Theorem 2), $U(A)$ is strongly dense in $U(A^-)$. Also multiplication is a strongly continuous map on $U(A^-) \times (A^-)_+$. Hence there exists $u \in U(A)$ and $h' \in A_+$ such that

$uh' \in V$. By the functional calculus (**3**, 1.5), h' may be approximated in norm by a positive invertible element $k \in A$, so that $uk \in V$. Now $uk \in G(A)$. Hence $G(A)$ is strong operator dense in A. Hence $G(A)$ is ultraweakly dense in A. However, since A is acting in its universal representation, the ultraweak topology, when restricted to A, coincides with the weak (Banach space) topology of A. This proves the result.

It is very easy to give an example of a general Banach algebra for which the above conclusion does not hold. For let B be the disc algebra. Then B consists of those continuous complex valued functions on the space Σ of complex numbers of modulus $\leqq 1$, which are analytic in the interior of Σ. Let $f \in B$ have a zero at some point in the interior of Σ, and suppose that f is not identically zero. We claim that f does not lie in the weak closure of $G(B)$.

For suppose (f_γ) is a net in $G(B)$ which converges weakly to f. By the principle of uniform boundedness the set $(\| f_\gamma \|)$ is bounded. Also $f_\gamma \to f$ pointwise on Σ. By (**1**, p. 171, Theorem 9) the family (f_γ) is normal. Hence, by (**1**, p. 171, Corollary), since each f_γ is never zero, a limit function of the set (f_γ) is either never zero or is identically zero. But this contradicts our choice of f, and hence our claim is proved.

We note that in the above example the norm and the weak closures of $G(B)$ in fact coincide. For if a function f in B is either identically zero or has all its zeros contained in the boundary of Σ then it clearly lies in the norm closure of $G(B)$.

We turn now to the more difficult question of the norm density of the invertible group in C^*-algebras. We noted earlier that the answer is known completely in the case of abelian C^*-algebras. Now the simplest class of non-commutative C^*-algebras is that of homogeneous C^*-algebras. Recall that a C^*-algebra A is called homogeneous of degree n (n a positive integer) if all its irreducible representations are of degree n. We are able to provide an answer to our question in the case of some such algebras.

Let A be an n-homogeneous C^*-algebra (for some positive integer n). We recall some notation from (**3**, 3.5). Let $\mathrm{Irr}_n(A)$ denote the set of irreducible representations of A on an n-dimensional Hilbert space H_n. Equip $\mathrm{Irr}_n(A)$ with the topology of simple strong convergence on A. i.e. $\pi_\gamma \to \pi$ in $\mathrm{Irr}_n(A)$ means $\pi_\gamma(a)\beta \to \pi(a)\beta$ for all $a \in A$, $\beta \in H_n$. $\hat{A}$ denotes the set of unitary equivalence classes of irreducible representations of A, with the Jacobson topology. We have a canonical map $\mathrm{Irr}_n(A) \to \hat{A}$, which is a quotient map for the respective topologies (**3**, 3.5.8). Let det denote the determinant on the matrix algebra $L(H_n)$ of all operators on H_n. Given $x \in A$, the map $\pi \to \pi(x)$ is continuous on $\mathrm{Irr}_n(A)$. Hence so also is the map $\pi \to \det \pi(x)$. This map is also constant on unitary equivalence classes of elements of $\mathrm{Irr}_n(A)$, and so, by passing to the quotient, defines a continuous complex-valued function ∂_x on $\hat{A}$.

For completeness we give the definition of covering dimension for topological spaces.

Definition (**8**, p. 9). Let X be a normal topological space. We say that X has dimension $\leqq n$ (n a positive integer) if every finite open covering of X has a finite open refinement of order $\leqq n+1$ (i.e. each point of X is contained in at most $n+1$ sets of this refinement). We write this as $\dim X \leqq n$.

We can now prove

Proposition 2. *Let A be a C^*-algebra which is homogeneous of finite degree and has uniformly dense invertible group. Then* $\dim \hat{A} \leqq 1$.

Proof. Note firstly that we are able to apply the dimension theory of (**8**) to $\hat{A}$ because $\hat{A}$ is a compact Hausdorff space (**3**, 3.1.8 and 3.6.4). By (**8**, Theorem VII 4), it suffices to show that the invertible elements are dense in $C(\hat{A})$ (the sup norm algebra of continuous complex-valued functions on $\hat{A}$).

Let $f \in C(\hat{A})$, and let $\varepsilon > 0$. By a special case of the Dauns-Hofmann theorem (**3**, 10.5.6), there exists $a \in A$ such that for all $\pi \in \hat{A}$,

$$\pi(a) = f(\pi)\pi(1).$$

Now, by continuity of det and (**3**, 1.3.7), there exists $\delta > 0$ such that, if $x \in A$ and $\| x-a \| < \delta$ then

$$|\det \pi(x) - \det \pi(a)| < \varepsilon \quad (\pi \in \mathrm{Irr}_n(A))$$

Since $G(A)$ is dense in A, there exists $x \in G(A)$ with $\| x-a \| < \delta$. Then, for each $\pi \in \hat{A}$,

$$|\partial_x(\pi) - f(\pi)| = |\det \pi(x) - \det \pi(a)| < \varepsilon.$$

Since x is invertible, $\pi(x)$ is invertible for each $\pi \in \hat{A}$, and so ∂_x is non-zero on $\hat{A}$. Also, as we have already remarked, $\partial_x \in C(\hat{A})$. This completes the proof.

Examples of homogeneous C^*-algebras are those of the form

$$A = C(X) \otimes M_n,$$

where X is a compact Hausdorff space and M_n is the full $n \times n$ matrix algebra. A routine, but tedious, argument, using induction on n, shows that the converse of Proposition 2 holds for such algebras. A more general result is proved in (**11**), using the structure theory of (**5**). However, we do not know whether the full converse of Proposition 2 is true or not.

4.

We now give a useful characterisation of invertible elements in C^*-algebras which have dense invertible groups.

Proposition 5. *Let A be a C^*-algebra with $G(A)$ dense in A. Let $x \in A$. The following are equivalent:*

(1) $x \in G(A)$;

(2) $f(x^*x) > 0$ *for each* $f \in P(A)$;

(3) $f(x^*x) > 0$ *for each* $f \in E(A)$;

(4) $\pi(x)$ *is invertible for each* $\pi \in \hat{A}$.

https://doi.org/10.1017/S001309150001066X Published online by Cambridge University Press

Proof. Obviously (1) implies (4) and (3) implies (2). Also, for any C^*-algebra A, (1) implies (3). For if $x \in G(A)$, then by (3, 2.1.2), for each $f \in E(A)$,

$$1 = |f(x^{-1}x)| < f(x^*x)f(x^{-1}x^{-1*}).$$

(2) implies (1): Suppose x is a singular element of A. Then x lies in the boundary of $G(A)$ and hence is a two-sided topological divisor of zero. Hence x is not left invertible, and so Ax is a proper closed left ideal of A containing x. Hence x is contained in a maximal left ideal L of A. By (3, 2.9.5), L is the left kernel of some pure state f of A, and so $f(x^*x) = 0$.

(4) implies (1): Suppose x is a singular element of A, and let L be as above. Then the canonical representation π of A on the Hilbert space A/L is irreducible and $\pi(x)$ is not invertible, since $x \in L$.

As an example of the way in which the above result may be applied we have the following

Corollary 6. *Let A be a liminal C^*-algebra (with identity) and let $x \in A$. Then conditions* (1) *to* (4) *of Proposition* 5 *are equivalent.*

Proof. By Proposition 5, we need only imbed A in a C^*-algebra B with dense invertible group, such that every pure state of B restricts to a pure state of A.

Let π be the reduced atomic representation of A (a direct sum of irreducible representations of A, taking exactly one representation from each unitary equivalence class). Then π is a faithful representation of A and the strong closure B of $\pi(A)$ is a finite von Neumann algebra, and hence has dense invertible group (2, Theorem 5). It is easy to see that each pure state of B restricts to a pure state of A.

Finally we return to the case of abelian C^*-algebras. Let A be an abelian C^*-algebra with maximal ideal space X. We may write $A = C(X)$. As we noted previously (10), it is shown by Peck (9) that when A is separable (i.e. X is metric), the following are equivalent.

(1) $A_1 = \text{co } U(A)$.

(2) $\dim X \leqq 1$ (i.e. A has dense invertible group).

In the non-separable case the proof that (2) implies (1) in (9) goes over without change using the results in (8) corresponding to those in (7). We now give a simplification of Peck's argument, which also proves that (1) implies (2) in the non-separable case.

Suppose $A = C(X)$ satisfies (1). In order to show that $\dim X \leqq 1$, we need only show that for every closed subset F of X and continuous map g from F into the unit circle $S = \{e^{i\beta}\colon 0 \leqq \beta \leqq 2\pi\}$ there exists a continuous extension of g to the whole of X (8, Theorem VII, 5). Take such g and F. By the Tietze extension theorem g extends to an element $\bar{g}$ of $C(X)$ with $\|\bar{g}\| \leqq 1$. By hypothesis $\bar{g}$ can be expressed as a convex combination of unitaries in $C(X)$, i.e. there exist positive scalars β_k and maps u_k from X into S with

$$\bar{g} = \sum_{k=1}^{n} \beta_k u_k, \quad \sum_{k=1}^{n} \beta_k = 1.$$

Now for $x \in F$, $|g(x)| = 1$ and

$$g(x) = \bar{g}(x) = \sum_{k=1}^{n} \beta_k u_k(x).$$

Hence $g(x) = u_k(x)$ $(1 \leqq k \leqq n)$. Thus any u_k, say u_1, will give the required extension.

REFERENCES

(1) L. AHLFORS, *Complex analysis* (McGraw-Hill, 1952).

(2) H. CHODA, An extremal property of the polar decomposition in von Neumann algebras, *Proc. Japan Acad.* **46** (1970), 341-344.

(3) J. DIXMIER, *Les C*-algèbres et leurs representations* (Gauthier-Villars, 1969).

(4) J. DIXMIER and O. MARÉCHAL, Vecteurs totalisateurs d'une algèbre de von Neumann, *Commun. Math. Phys.* **22** (1971), 44-50.

(5) J. M. G. FELL, The structure of algebras of operator fields, *Acta Math.* **106** (1961), 233-280.

(6) J. GLIMM and R. V. KADISON, Unitary operators in C^*-algebras, *Pac. J. Math.* **10** (1960), 547-556.

(7) W. HUREWICZ and H. WALLMAN, *Dimension theory* (Princeton University Press, 1948).

(8) J. NAGATA, *Modern dimension theory* (North Holland, 1965).

(9) N. T. PECK, Representation of functions in $C(X)$ by means of extreme points, *Proc. Amer. Math. Soc.* **18** (1967), 133-135.

(10) A. G. ROBERTSON, A note on the unit ball in C^*-algebras, *Bull. London Math. Soc.* **6** (1974), 333-335.

(11) A. G. ROBERTSON, Ph.D. Thesis (Newcastle University, 1974).

MATHEMATICAL INSTITUTE
20 CHAMBERS STREET
EDINBURGH EH1 1HZ