On the Number of Ways of Colouring a Map.

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It is well known that any map of n regions on a sphere may be coloured in five or fewer colours.[1] The purpose of the present note is to prove the following

THEOREM. *If* $P_n(\lambda)$ *denotes the number of ways of colouring any ma; of n regions on the sphere in* λ (*or fewer*) *colours, then*

$$P_n(\lambda) \geqslant \lambda(\lambda-1)(\lambda-2)(\lambda-3)^{n-3} \qquad (n \geqslant 3,\ \lambda \neq 4). \tag{1}$$

This inequality obviously holds for $\lambda = 1, 2, 3$ so that we may confine attention to the case $\lambda > 4$. Furthermore it holds for $n = 3, 4$ since the first region may be coloured in λ ways, the second in at least $\lambda - 1$ ways, the third in at least $\lambda - 2$ ways, and the fourth, if there be one, in at least $\lambda - 3$ ways.

In the case $\lambda = 4$, not dealt with in the theorem, the inequality (1) obtains if any only if every map on the sphere can be coloured in not more than four colours, that is according as the so-called "four colour theorem" is or is not true: in fact if (1) holds for $n \geqslant 3$, $\lambda = 4$, every map can be coloured in four colours in at least one way of course; and if every map (with $n \geqslant 3$ regions) can be coloured in four colours, then from one colouring can be obtained $4 . 3 . 2$ or 24 colourings by permutation of the colours, so that $P_n(4) \geqslant 24$ as required; of course we may always assume that at least three distinct colours are used since we can always put an unused colour on any region.

In our proof of (1), we assume that (1) holds for any map containing 3, 4, ... $n-1$ regions, $n - 1 \geqslant 4$, and then show that it

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[1] First proved by P. J. Heawood in a paper, Map-Colour Theorem, *Quarterly Journal of Mathematics*, 24 (1889-90), 332-339. For the bibliography of the related "four colour theorem" with references to the important earlier papers of Cayley, Tait, F. Guthrie in this journal and elsewhere, the reader may be referred to A. Errera, Du coloriage des cartes et de quelques questions d'analysis situs, Thesis, University of Brussels, 1921 (Paris and Brussels, 1921), and C. N. Reynolds, On the Problem of Colouring Maps in Four Colours, II, *Annals of Mathematics*, vol. **28**, second series, 477-492.

holds for any map of n regions. All maps M which we consider will therefore contain n regions. The theorem follows by induction if we take $n = 5, 6, \ldots$.

The structure of any map is completely given, in so far as colouring it is concerned, by naming each pair of regions α, β which have a boundary line in common. Such regions α, β must be of different colours in any proper colouring of the map.

Call those maps of type I in which there is no region which either is multiply connected or touches itself at some vertex. Consider any map M not of type I. If there is a region α of M which is multiply connected, form a new map M_1 by drawing together two boundary lines, one on each side of this region till they touch at a

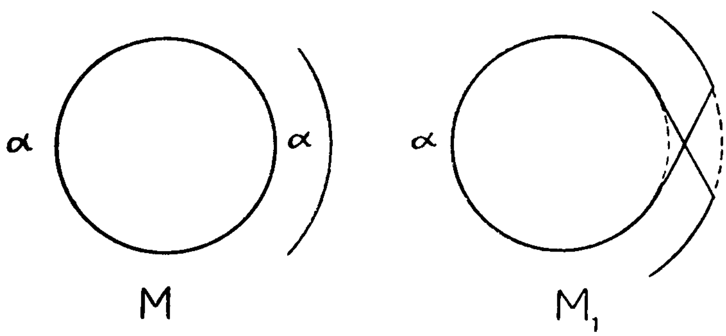

Fig. 1

point, forming a new vertex (see fig. 1). Continue this process until a map N still of n regions but with none of its regions multiply connected, is obtained. Since the pairs of regions having a common boundary are the same for N as for M, any colouring of N furnishes a colouring for M. Hence we need only prove that (1) holds for maps N in which there are no multiply connected regions.

Now this map N without multiply connected regions may have one or more regions β which touch themselves at a vertex ν (fig. 2).

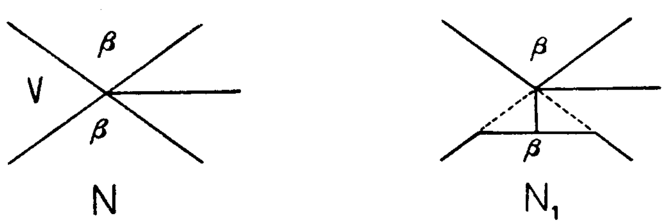

Fig. 2

Take one end of such a region and draw it away, putting in a new boundary line. We thus have a new map N_1 in which a new pair of regions, namely those on each side of this end of β have a boundary line in common. Since β touched itself at ν before, these two regions are distinct. Continue this process till there are no more regions touching themselves at a vertex. The map N' thus formed is therefore of Type I. But every pair of regions which had a common boundary in M have still a common boundary in N'. Hence every possible colouring of N' is a possible colouring of M. Also M and N' contain the same number of regions. Therefore if (1) holds for N', it holds for M. We have thus still to consider only maps M of regions of type I.

Call those maps of type II which are of type I, while all vertices are triple. Consider any map M of type I but not of type II. Then at least four distinct regions abut at some vertex ν. Draw one of them, α, away from ν, putting in a new boundary line between the regions on each side of α at ν, thus forming a new triple vertex (fig. 3)

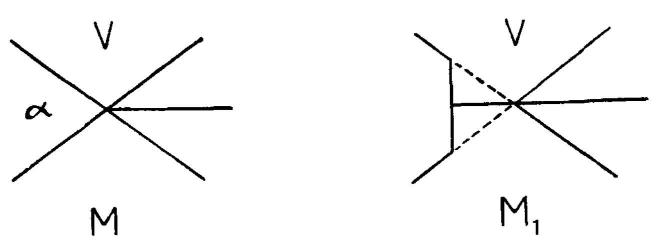

Fig. 3

and reducing the number of regions abutting at ν by one. Continue until all the vertices are triple, forming the map N. As every pair of regions with a common boundary in M have still a common boundary in N, every colouring of N is a colouring of M. Thus we see that if (1) holds for every map of n regions of type II, it holds for every map of n regions of type I, and therefore for every map of n regions.

Call those maps of type III which are of type II and in which no pair of regions form a multiply connected region. Such a map contains no 2-sided region of course. Consider a map M of type II but not of type III, containing then at least one pair of simply connected regions α, β touching each other along $k > 2$ distinct boundaries (fig. 4).

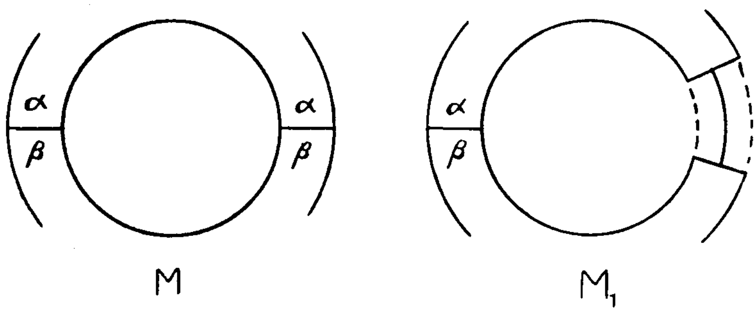

Fig. 4

Select one of these boundaries as l (see figure) which is of course terminated by two triple vertices. Open up the map along this boundary and put a new boundary line across the "canal" so formed. Evidently the new map constructed is still of type II, and all regions previously in contact are still in contact, although one contact of two regions previously not in contact has been introduced. Continue this process until a map N is obtained in which no two regions have more than one boundary line in common. Evidently N is of type III, and if (1) holds for N it will hold also for M. Hence we may restrict attention to maps of type III.

Call those maps of type IV which are of type III and in which no three regions form a multiply connected region. Such a map contains no 3-sided region of course. Consider a map of type III but not of type IV, in which some three regions α_1, α_2, α_3 are then multiply connected. Of course each of the three regions touches the two others along one and only one boundary line. Thus, the three regions form a ring which separates M into two parts K, L distinct from α_1, α_2, α_3 (fig. 5). Now form two new maps N_1 and N_2,

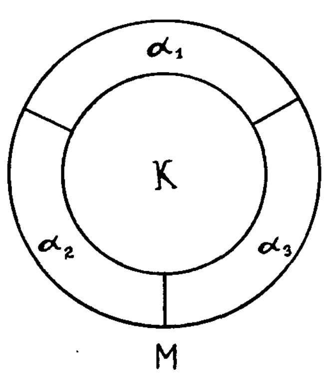

Fig 5

of at least four regions each, by shrinking these parts to a point. If the number of regions in N_1 and N_2 be denoted by n_1 and n_2 respectively, we have

$$n_1 + n_2 = n + 3.$$

Since the number of regions in N_1 or in N_2 is less than n, we have

$$P_{n_i}(\lambda) \geqslant \lambda(\lambda-1)(\lambda-2)(\lambda-3)^{n_i-3} \qquad (i = 1, 2)$$

in which $P_{n_i}(\lambda)$ stands for the number of ways of colouring N_i $(i=1, 2)$ in λ colours.

In order to colour M, we take any one of the $\lambda(\lambda-1)(\lambda-2)$ choices of colours for $\alpha_1, \alpha_2, \alpha_3$, corresponding to each of which we have $P_{n_1}(\lambda)/[\lambda(\lambda-1)(\lambda-2)]$ choices for N_1 and $P_{n_2}(\lambda)/[\lambda(\lambda-1)(\lambda-2)]$ choices for N_2. Thus for M we have

$$P_n(\lambda) = \lambda(\lambda-1)(\lambda-2)\frac{P_{n_1}(\lambda)P_{n_2}(\lambda)}{[\lambda(\lambda-1)(\lambda-2)]^2}$$
$$\geqslant \lambda(\lambda-1)(\lambda-2)(\lambda-3)^{n-3}.$$

Hence we may assume that M is of type IV.

Call those maps of type V which are of type IV and which contain no 4-sided region. Consider a map M of type IV but not of type V, in which the 4-sided region β has a common boundary with $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ taken in cyclical order. Then α_i has a common boundary with α_{i+1} (putting $\alpha_5 = \alpha_1$), since all the vertices in M are triple. Furthermore, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ have only these boundaries in common with one another; for if α_i and α_j are not adjacent cyclically and yet touch, then the three regions α_i, α_j and β would form a multiply connected region.

Colour the map M', obtained by reducing β to a point, in all possible ways. These we may divide into four sets, depending on whether the colours of $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are essentially of the types (a, b, a, b), (a, b, a, c), (a, b, c, b), or (a, b, c, d). Call the number of colourings of the first type, p_1, of the second, p_2, of the third, p_3, and of the fourth, p_4. With each of these colourings of M' we have $\lambda - i$ colours for β, where i is the number of colours employed in the ring about β. Thus we have for M

$$(2) \qquad P_n(\lambda) = (\lambda-2)p_1 + (\lambda-3)(p_2+p_3) + (\lambda-4)p_4.$$

Form now the four maps A, B, C, D obtained respectively by letting α_1, β, and α_3 coalesce, letting α_1, β coalesce, letting α_2, β, α_4 coalesce, and letting α_2, β coalesce (fig. 6).

The possible types of colourings of A are those possible for M' in which α_1 and α_3 are of the same colour, and only those. These types are (a, b, a, b) and (a, b, a, c). Thus if we call p_A the number of ways of colouring A we have

$$p_A = p_1 + p_2.$$

The possible colourings of B are those of M' in which α_1 and α_3 are of different colour, *i.e.* (a, b, c, b) and (a, b, c, d), so that we have

$$p_B = p_3 + p_4.$$

Similarly we find

$$p_C = p_1 + p_3,$$

$$p_D = p_2 + p_4.$$

These equations give

$$p_A + p_B + p_C + p_D = 2(p_1 + p_2 + p_3 + p_4),$$
$$p_A + p_C - p_B - p_D = 2(p_1 - p_4).$$

Hence from equation (2) we find

$$\begin{aligned} P_n(\lambda) &= (\lambda - 3)(p_1 + p_2 + p_3 + p_4) + (p_1 - p_4) \\ &= \frac{(\lambda - 3)}{2}(p_A + p_B + p_C + p_D) + \tfrac{1}{2}(p_A + p_C - p_B - p_D) \\ &= \frac{(\lambda - 2)}{2}(p_A + p_C) + \frac{(\lambda - 4)}{2}(p_B + p_D). \end{aligned}$$

Now the maps A, C each contain $n - 2$ regions, while the maps B, D each contain $n - 1$ regions, and, as $n > 4$, the number of regions in each of these maps is $\geqslant 3$. Therefore we have

$$p_A \geqslant \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-5},$$
$$p_B \geqslant \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-4},$$

with similar inequalities for p_C and p_D.

Hence we obtain from the above equation for $P_n(\lambda)$,

$$\begin{aligned} P_n(\lambda) &\geqslant (\lambda - 2)[\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-5}] \\ &\qquad + (\lambda - 4)[\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-4}] \\ &\geqslant \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-5}(\lambda^2 - 6\lambda + 10) \\ &> \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-3}. \end{aligned}$$

Hence we may assume that M is of type V.

Now as is well known every map on the sphere containing only triple vertices and no multiply connected or 2-, 3-, 4-sided regions, and therefore every map of type V, contains at least twelve 5-sided regions.[1]

Take any map M of type V, the only type we have left to consider, in which some 5-sided region β touches in cyclic order $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$. Evidently none of these regions α_i touch each other except in this cyclic order; for if α_i and α_j did so, the three regions α_i, α_j, and β would form a multiply connected region. Colour the partial map M' obtained by letting β shrink to a point, in all possible ways. These we may divide into eleven types, in which the colours of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are essentially of the types;

$$(a, b, c, b, c), \ (c, a, b, c, b), \ (b, c, a, b, c), \ (c, b, c, a, b),$$
$$(b, c, b, c, a); \ (a, b, a, c, d), \ (d, a, b, a, c), \ (c, d, a, b, a),$$
$$(a, c, d, a, b), \ (b, a, c, d, a); \ (a, b, c, d, e).$$

Let us call the number of colourings of these respective types $p_1, \ldots p_5, q_1, \ldots q_5$ and r. We have obviously for M

$$(3) \qquad P_n(\lambda) = (\lambda - 3) \sum_i p_i + (\lambda - 4) \sum_i q_i + (\lambda - 5) r.$$

Consider the ten maps $A_1, \ldots A_5, B_1, \ldots B_5$ formed as follows: For A_1 we let coalesce in M $\alpha_1, \beta, \alpha_3$; for A_2, we let coalesce $\alpha_2, \beta, \alpha_4$, etc.; for B_1 we let coalesce α_1, β; for B_2 we let coalesce α_2, β, etc. (see fig. 6).

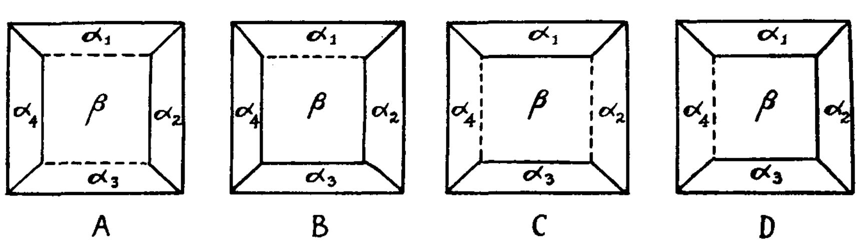

Fig. 6

[1] This is an immediate consequence of the Euler formula applied to such a map. *Cf.*, for instance, my paper, The Reducibility of Maps, *American Journal of Mathematics*, **35** (1915), 115-128.

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We see, similarly to the case of a 4-sided region,

$$p_{A_1} = p_4 + p_5 + q_1$$
$$p_{A_2} = p_5 + p_1 + q_2$$
$$\dots\dots\dots\dots\dots$$
$$p_{B_1} = p_1 + q_2 + q_3 + q_5 + r$$
$$p_{B_2} = p_2 + q_3 + q_4 + q_1 + r$$
$$\dots\dots\dots\dots\dots$$

Summing, we find

$$\sum_i p_{A_i} = 2\sum_i p_i + \sum_i q_i$$
$$\sum_i p_{B_i} = \sum_i p_i + 3\sum_i q_i + 5r$$

from which equations follow

$$2\sum_i p_{A_i} + \sum_i p_{B_i} = 5\left(\sum_i p_i + \sum_i q_i + r\right),$$
$$3\sum_i p_{A_i} - \sum_i p_{B_i} = 5\left(\sum_i p_i - r\right).$$

Equation (3) gives then

$$P_n(\lambda) = (\lambda - 4)\ \left(\sum_i p_i + \sum_i q_i + r\right) + \sum_i p_i - r$$
$$= \frac{(\lambda - 4)}{5}\left(2\sum_i p_{A_i} + \sum_i p_{B_i}\right) + \frac{1}{5}\left(3\sum_i p_{A_i} - \sum_i p_{B_i}\right)$$
$$= \frac{2\lambda - 5}{5}\sum_i p_{A_i} + \frac{\lambda - 5}{5}\sum p_{B_i}\,.$$

As each map A_i obviously contains $n - 2 \geqslant 10$ regions, and each map B^i contains $n - 1 \geqslant 11$ regions, we have

$$p_{A_i} \geqslant \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-5}$$
$$p_{B_i} \geqslant \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-4}$$

for $i = 1, 2, 3, 4, 5$. Therefore, we conclude

$$P_n(\lambda) \geqslant \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-5}[(2\lambda - 5) + (\lambda - 3)(\lambda - 5)]$$
$$\geqslant \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-5}[\lambda^2 - 6\lambda + 10]$$
$$> \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-3}.$$

Thus (1) holds for every map M whatsoever of n regions, and hence, by induction, for every map. Note that only in this last reduction did we need to restrict λ not to be 4.

It is deserving of remark that the inequality (1) *of the theorem is the best equality of its type, i.e.* for every pair of numbers $\lambda > 4$ and $n \geqslant 3$ there is a map M in which

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^{n-3}.$$

For take a region α_1; let α_2 touch α_1 along a boundary line; let α_3 touch α_1 and α_2; let α_4 touch in order α_1, α_2, α_3; let α_5 touch in order α_1, α_2, α_4, and continue thus, using n regions. We may then colour in succession α_1 in λ ways, α_2 in $\lambda - 1$ ways, α_3 in $\lambda - 2$ ways, α_4 in $\lambda - 3$ ways, α_5 in $\lambda - 3$ ways, etc. Hence $P_n(\lambda)$ has the stated value when M is of this special type.

In conclusion we may observe that the method of the paper can be applied to maps on multiply connected surfaces of genus $p > 0$ as well as to maps on surfaces of genus zero like the sphere.