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ABOUT THE # FUNCTION

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ABSTRACT. The use of decreasing rearrangement formulas, and particularly that of the weak N inequality, is illustrated by deriving from $E_\tau |f-f(\tau-)| \le E_\tau u$ (where f_t is some stochastic process and τ arbitrary stopping time) the estimate $\|f\| \le \text{Const}\,\|u\|$ in the class of structureless norms with finite dual Hardy bound.

The basic estimate is

$$f^{**}(x) - f^{**}(y) \le \frac{y}{x}\, u^{**}(y).$$

1. Ever since the paper of John and Nirenberg [6], there has been some concern regarding martingales which satisfy a condition of type $f^{\#} = \sup_n E_n |f-f_n| \in L_p$. The real variable case (and the dyadic martingale case for that matter) is discussed in [2]; showing that, if $f^{\#} \in L_p$ then $f \in L_p$.

We are looking for a similar result in the general case of cadlag processes. The notations will be explained in §2 and §3.

DEFINITION. A cadlag process $\underline{f}$ satisfies the condition # if:

(i) It is closed, adapted and $\underline{f}(-\infty) = 0$. (The latter is a normalization condition.)

(ii) There exists some $u \ge 0$, $u \in L_*$ such that, for any stopping time τ, we have

$$E_{\tau+} |f - \underline{f}(\tau-)| \le E_{\tau+} u \quad \text{on} \quad \{\tau < \infty\}. \tag{1}$$

The problem is to derive estimates about the limit function f of a cadlag process $\underline{f}$ satisfying the condition #, provided we have control over u. The quantitative (Banach space norm) relation is the following:

PROPOSITION (1). *Let Λ be a structureless norm with finite dual Hardy bound β. If f satisfies (#) for some u, with $\Lambda(u) < \infty$, then*

$$\Lambda(f) \le c\Lambda(u) \tag{2}$$

where $c \le (e+1)\beta$.

The martingale variant of the above is:

PROPOSITION (2). *Let $\underline{f}(t+) = E_t f$ be a martingale satisfying the conditions of Proposition* (1). *Then*

$$\Lambda(f) \le e\beta\Lambda(u). \tag{3}$$

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As a particular case of Proposition (2) we have

PROPOSITION (3). *If the martingale $f_n = E_n f$ satisfies for some $u \in L_p^+$ the condition*

$$E_n |f - f_n| \le E_n u \qquad \forall n \in Z \tag{4}$$

then $f \in L_p$ and

$$\|f\|_p \le ep \|u\|_p. \tag{5}$$

Estimates for the case of L_p-spaces are long known. In [3], A. Garsia proves that, with the conditions of Proposition (3) we have

$$\|Mf\|_p \le 3epq \|u\|_p \tag{6}$$

His proof also works for the continuous time case. To get the estimate (5) one can work through a B. Davis-type decomposition of L_q and then a duality argument. This is done in [7] where, due to the long detour, the resulting constant is large. It is illuminating to realize that, if one desires an inequality involving f and u then by repeating Garsia's initial proof but switching to the maximal function a little bit later than is done there, one obtains Proposition (3). On the other hand, to prove Propositions (1) and (2), we have to be more careful. As usually happens, all these norm evaluations are consequences of a distribution-function relation.

LEMMA (4). *If f satisfies the # condition, its average decreasing rearrangement satisfies*

$$f^{**}(x) - M^*(y) \le \frac{y}{x} u^{**}(y). \tag{7}$$

We postpone the proof of the lemma till the end of the paper. The consequence of (7) is the following qualitative characterization of the control over f.

PROPOSITION (5). *If f is a cadlag process (martingale) satisfying #, then the average decreasing rearrangement of f satisfies*

$$f^{**}(x) - f^{**}(y) \le \left(\frac{y}{x} + 1\right) u^{**}(y) \tag{8}$$

$$\left(\textit{respectively} \quad f^{**}(x) - f^{**}(y) \le \frac{y}{x} u^{**}(y)\right). \tag{9}$$

Proof. Since for any t, $|f_t| \le E_t(|f| + u)$ we have from the maximal theorem $M^* \le f^{**} + u^{**}$; or simply $M^* \le f^{**}$ in the martingale case.

It is worthwhile to realize that in case of potentials (i.e. when $f_n \uparrow f$), Lemma (4) is the Neveu inequality.

Proposition (2) follows from Proposition (5) via the weak N-inequality: Let

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g^*, $h^* \in V$. If $g^*(x) - g^*(y) \leq (y/x)h^*(y)$ for $x \leq y$, then $g^*(x) \leq eBh^*(x)$. (For proof see [4].)

To conclude (3), one applies some basic properties of Λ-norm, given in §2.

To prove (2), we have to simply modify the weak N-inequality. Then (8) will imply $f^{**} \leq (e+1)Bu^{**}$, which proves our claim (2), with $c = (e+1)\beta$.

NOTE. When $u \in L_\infty$ (i.e. $f \in BMO$), our basic estimate proves the exponential growth condition, but this is not the concern here.

2. At this point is is essential to review the definition of a structureless norm. (Rearrangement invariant norm in the terminology of Boyd [1] and Luxemburg [8].)

Let V_0 be the set of positive, decreasing, right-continuous functions f^* defined on $(0, \infty)$ and V the subcone of elements satisfying $\int_0^1 f^*\, dt < \infty$, $\lim_{t\to\infty} f^*(t) = 0$.

An L-functional Λ is a positive sublinear map $\Lambda: V \to \bar{R}^+$ satisfying:

(i) For $t < \infty$, $\Lambda(\chi_t) < \infty$, where $\chi_t = (0, t)$.

(ii) If $f_n^* \uparrow f^* \in V$, then $\Lambda(f_n^*) \uparrow \Lambda(f^*)$.

(iii) If $f^{**} \leq g^{**}$ for f^*, $g^* \in V$ then $\Lambda(f^*) \leq \Lambda(g^*)$ where $f^{**}(x) = Af^*(x) = (1/x)\int_0^x f^*(t)\, dt$.

Call A the Hardy operator. Let B be the formal dual of A; then $Bf^*(x) = \int_x^\infty f^*(t)\, dt/t$. The Hardy bound α and dual Hardy bound β for Λ (with range $1 \leq \alpha$, $\beta \leq \infty$) are the "operator norms" of A and B respectively. Explicitly $\beta = \sup\{\Lambda(Bf^*); \Lambda(f^*) \leq 1\}$. For example, if $\Lambda(f^*) = (\int_0^\infty f^{*^p}\, d\mu)^{1/p}$, $1 \leq p \leq \infty$, then $\alpha = p$ and $\beta = q = p/(p-1)$.

For the general setting we start with a σ-algebra A and a measure μ, where the zero-sets are discarded. (i.e. if $\mu(A \cap E) = \phi$ for all $E \in A$ with $\mu(E) < \infty$, then $A = \phi$). Let F be the field of elements of finite measure. We use the same name (event) and same notation for an element of F and its indicator function. If we were to have started with an initial state-space, then the events would be equivalence classes (with respect to μ) of sets of finite measure.

We will use totally F-measurable functions f with values in a given separable Banach space H. For such a function f we define f^*, the decreasing rearrangement of f, as the element in V which satisfies $\mu\{|f| > \lambda\} = m\{f^* > \lambda\}$ for all $\lambda \geq 0$.

DEFINITION. A structureless space of H-valued functions is a linear subspace of measurable H-valued functions, organized as a Banach space under the norm $\|f\| = \Lambda(f^*)$, where Λ is some given L-functional. When it is convenient, we will also use the notations $\|f\| = \|f\|_\Lambda = \Lambda(f)$. As an example $L_* = \{f; \|f\|_* = \int_0^1 f^*(t)\, dt < \infty\}$.

3. We are now ready to explain the martingale and stochastic process aspects of the problem.

In the measure space (F, μ) we are given a stochastic base; an increasing, right-continuous family $\{F_t, -\infty < t < \infty\}$ of fields with UF_t dense in the basic field F. One covers the discrete-time situation by choosing $F_t = F_n$ for $n \le t < n+1$. We only consider adapted (to (F_t)) cadlag processes $\underline{f}$. Cadlag stands for "continue à droite avec limites à gauche". Intuitively, these are processes for which, at each finite time $\underline{t}$, there exists a right-hand limit $\underline{f}(t+)$ and a left-hand limit $\underline{f}(t-)$, and we may agree to take any of the above two as the value of $\underline{f}$ at $\underline{t}$.

For a precise definition, we use the one given in [5], which is more convenient for our purpose than Meyer's [9] definition.

Let D_n designate the rationals with nominator 2^n and $D = U_n D_n$. A collection of random variables $(f_t, t \in D)$ is cadlag data if for each $\varepsilon > 0$ there exists an increasing sequence $(\tau_n, n \in Z)$ of random times with $\lim_{n\to\infty} \tau_n = \infty$ and $\lim_{n\to-\infty} \tau_n = -\infty$ satisfying

$$\operatorname{osc}(n)\sup\{|f_s - f_t|;\ \tau_n < s < t < \tau_{n+1}\} \le \varepsilon, \qquad \forall n.$$

Given a random time $\tau < \infty$ put

$$\tau^{(n)} = \inf\{t \in D_n,\ t > \tau\}$$

and

$$f(\tau^+) = \lim_{n\to\infty} \sum_{t \in D_n} f_t\{\tau^{(n)} = t\}.$$

Analogously, for $\tau > -\infty$ we compute $\underline{f}(\tau -)$.

Definition. A cadlag process $\underline{f}$ is the collection of random variables $f(\tau+)$, $f(\tau-)$ defined for each random time τ with $-\infty < \tau < \infty$, such that there exists cadlag data f_t which furnishes the values $f(\tau \pm)$.

In our case (adapted processes) we start with adapted data and only consider stopping times. Good examples of cadlag processes are martingales and increasing processes.

A process is called closed if there exists $f_{(-\infty)} = \lim_{t\to-\infty} \underline{f}_t$ and $f_{(\infty)} = \lim_{t\to\infty} \underline{f}_t$. We denote $f_{(\infty)} = f$.

We denote by E_τ the conditional expectation operator corresponding to the field $F_{\tau+}$, where τ is a stopping time.

We need the martingale maximal theorem in its general form: Let $\underline{w}_t$ be an adapted process, $M_t = \sup_{s<t} |\underline{w}_s|$ and $M = \lim_{t\to\infty} M_t$. If there exists some $w \in L_*^+$ such that $|\underline{w}_t| \le E_t w$ for all t, then $M^* \le w^{**}$. For proposition (3) we only need the formulation $\|M\|_p \le q\, \|f\|_p$.

Proof of Proposition (3). First we realize that the L_1-estimate (for all cases) follows by letting $\tau \to -\infty$. One gets $\|f\|_1 \le \|u\|_1$. Hence we can suppose $1 < p < \infty$.

For $\xi>0$ we define the stopping time τ as $\tau=\inf\{n, M_n>\xi\}$. Then $\{\tau<\infty\}=\{M>\xi\}\geq\{|f|>\xi\}$ and $\{\tau=n\}=\{M_n>\xi\geq M_{n-1}\}\leq\{|f_n|>\xi\geq|f_{n-1}|\}\in F_n$.

$$I_1=\int\{|f|>\xi\}(|f|-\xi)\,d\mu\leq\sum_n\int\{|f|>\xi\}\{\tau=n\}(|f|-|f_{n-1}|)\,d\mu$$

$$\leq\sum_n\int\{\tau=n\}\,|f-f_{n-1}|\,d\mu=\sum_n\int\{\tau=n\}E_n\,|f-f_{n-1}|\,d\mu$$

$$\leq\sum_n\int\{\tau=n\}u\,d\mu=\int\{\tau<\infty\}u\,d\mu=\int\{M>\xi\}u\,d\mu=I_2$$

We note that, if $\xi=f^*(x)$ then $I_1=x(f^{**}(x)-f^*(x))$, but I_2 is not of a right (rearrangment type) form. Integrate both sides against $p(p-1)\lambda^{p-2}$. Write $J_i=\int_0^\infty p(p-1)\lambda^{p-2}I_i\,d\lambda$. Then $J_1=\int|f|^p\,d\mu$ and $J_2=p\int M^{p-1}u\,d\mu<p\,\|M\|_p^{p-1}\,\|u\|_p\leq p\cdot q^{p-1}\,|f\|_p^{p-1}\,\|u\|_p$. Since $q^{p-1}\leq e$ we obtain $\|f\|_p\leq pe\,\|u\|_p$. Q.E.D.

We included this proof to accentuate the force of relation (9).

Proof of Lemma (4). Given $\xi>0$ define the stopping time τ as $\tau=\inf\{t; M_t>\xi\}$. Then we have $\{|f|>\xi\}\leq\{M>\xi\}=\{\tau<\infty\}$, $\{\tau=\infty\}=\{M\leq\xi\}\leq\{|f|\leq\xi\}$ and $|f(\tau-)|\leq\xi$. Hence, for any $A\in F$

$$I_0=\int A(|f|-\xi)\,d\mu\leq\int A\{\tau=\infty\}(|f|-\xi)\,d\mu+\int A\{\tau<\infty\}(|f|-|f(\tau-)|\,d\mu$$

$$\leq\int A\{|f|\geq\xi\}\{\tau=\infty\}(|f|-\xi)\,d\mu+\int A\{\tau<\infty\}\,|f-f(\tau-)|\,d\mu.$$

Since $\{\tau=\infty\}\{|f|\geq\xi\}=0$, we have

$$I_0\leq\int\{\tau<\infty\}\,|f-f(\tau-)|\,d\mu=\int\{\tau<\infty\}E_\tau\,|f_\tau-f(\tau-)|\,d\mu$$

$$\leq\int\{\tau<\infty\}E_\tau u\,d\mu=\int\{M>\xi\}u\,d\mu.$$

Let $\xi=M^*(y)$ and $\mu(A)=x$, then

$$I_0=\int A(|f|-\xi)\,d\mu\leq\int_0^y u\,d\mu\leq yu^{**}(y).$$

Since A was arbitrarily chosen with $\mu(A)=x$, we have

$$I_0=\int_0^x(f^*(t)-M^*(y))\,dt\leq yu^{**}(y)$$

or $x(f^{**}(x)-M^*(y))\leq yu^{**}(y)$. Q.E.D.

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