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COOLING OF AN INFINITE SLAB IN A TWO-FLUID MEDIUM

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Abstract

A mixed boundary-valued problem associated with the diffusion equation, that involves the physical problem of cooling of an infinite slab in a two-fluid medium, is solved completely by using the Wiener-Hopf technique. An analytical solution is derived for the temperature distribution at the quench fronts being created by two different layers of cold fluids having different cooling abilities moving on the upper surface of the slab at constant speed. Simple expressions are derived for the values of the sputtering temperatures of the slab at the points of contact with the respective layers, assuming one layer of the fluid to be of finite extent and the other of infinite extent. The main problem is solved through a three-part Wiener-Hopf problem of a special type, and the numerical results under certain special circumstances are obtained and presented in the form of a table.

1. Introduction

The interesting physical problem considered in this paper is that of the determination of the temperature distribution in an infinite slab composed of uniform material possessing uniform thermal properties. Such a heated slab is allowed to cool down with the help of a two-fluid medium of different extents, with two different rates of cooling but moving with the same uniform speed v along one of the surfaces of the slab, producing two quench fronts (see [1]) that also propagate with the same speed. The present mixed boundary value problem for the diffusion equation is a generalisation of the problems considered in [1]–[7] where similar cooling phenomena have been investigated associated with a single fluid medium.

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The principal mathematical tool employed to solve the mixed boundary-valued problem at hand is the Wiener-Hopf technique, as described in [6] and [8], and utilised for the solution of the problems considered in [2]–[7]. Following the methods of Jones [6], the problem under consideration is reduced to that of solving a Wiener-Hopf type functional relation, known as the "three-part Wiener-Hopf problem" (see Chakrabarti [4]). This three-part Wiener-Hopf problem is solved by employing an idea similar to that of Jones (Jones [6] and Chakrabarti [4]) after reducing the problem to a system of algebraic equations involving two-infinite unknowns, whose solutions are completed in a standard manner. As in the works of Chakrabarti and Evans, we find that the sputtering temperature, i.e. the temperature of the surface of the slab at the points of contact with the two layers of fluid, can be calculated by means of simple formulae involving sine and cosine hyperbolic functions, the width of the slab, the speed v and the diffusivity k, along with the rates B_0 and B_l of cooling. Certain typical values of various parameters of the problem are considered for the purpose of numerical computation of the sputtering temperatures.

2. Formulation and reduction to the Wiener-Hopf problem

Using Cartesian co-ordinates (x, y), the boundaries of the infinite slab under consideration are represented by the lines $y = 0$ and $y = h$ respectively. Inside the material of the slab, the distribution of the steady-state temperature $u(x, y)$ satisfies the partial differential equation

$$\nabla^2 u + 2s\frac{\partial u}{\partial x} = 0, \qquad 0 < y < h, \tag{2.1}$$

with

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad s = v/2k,$$

where k is the thermal diffusivity of the material, v is the constant speed of the quench fronts $x = 0$, $x = -l$ on the surface $y = h$ of the slab, one half $(x < 0)$ of which is in contact with a two-fluid medium of different cooling abilities.

It is required to solve (2.1) under the following set of conditions:

(i) $\frac{\partial u}{\partial y} = 0$, on $y = 0$, $-\infty < x < \infty$ (insulated boundary)

(ii) $\frac{\partial u}{\partial y} = 0$, on $y = h$, $x > 0$ (insulated boundary)

(iii) $\frac{\partial u}{\partial y} + B_0 u = 0$, on $y = h$, $-l < x < 0$ ($B_0 =$ constant rate of cooling)

(iv) $\frac{\partial u}{\partial y} + B_l u = 0$, on $y = h$, $-\infty < x < -l$ ($B_l =$ constant rate of cooling in $-\infty < x < -l$)

(v) $u \to 0$ as $x \to -\infty$ (cold end at $x \to -\infty$)
(vi) $u \to 1$ as $x \to +\infty$ (constant temperature at $x \to +\infty$) and
(vii) u, $\frac{\partial u}{\partial y} \sim O(1)$ as $x \to 0$, and as $x \to -l$, on $y = h$ (bounded temperature at the quench fronts).

In order to reduce the above mixed boundary-value problem to a Wiener-Hopf problem, we follow Jones's method [6], as described in [7] and [5].

Setting

$$u(x, y) = 1 - \varphi(x, y)e^{-sx}, \tag{2.2}$$

(2.1) transforms to the form

$$(\nabla^2 - s^2)\varphi(x, y) = 0, \tag{2.3}$$

and the conditions (i) to (vii) transform to the following new conditions:

(i)′ $\frac{\partial \varphi}{\partial y} = 0$, on $y = 0$ $(-\infty < x < \infty)$
(ii)′ $\frac{\partial \phi}{\partial y} = 0$, on $y = h$, $x > 0$
(iii)′ $\frac{\partial \varphi}{\partial y} + B_0\varphi = B_0 e^{sx}$, on $y = h$, $-l \le x \le 0$
(iv)′ $\frac{\partial \phi}{\partial y} + B_l\varphi = B_l e^{sx}$, on $y = h$, $-\infty < x < -l$
(v)′ $\varphi \sim O(e^{sx})$, as $x \to -\infty$
(vi)′ $\varphi \sim O(e^{-sx})$, as $x \to +\infty$
(vii)′ φ and $\frac{\partial \varphi}{\partial y} \sim O(1)$ as $x \to 0$ and as $x \to -l$, on $y = h$, for uniqueness of the solution.

While the above conditions (i)′ to (v)′ and (vii)′ are derivable from the conditions (i) to (v) and (vii) by using the transformation (2.2), the derivation of the condition (vi)′ requires the following attention.

The transformation (2.2) produces the partial differential equation (2.3) for the function φ, which possesses, for $x > 0$, the general solution in the form

$$\varphi(x, y) = \sum_n A_n e^{-(s^2+\beta_n^2)^{1/2}x}(\cos\beta_n y) + Ae^{sx} + Be^{-sx}, \tag{$*$}$$

which satisfies the conditions (i)′ and (ii)′, with A_n, A, B as arbitrary constants and the constants β_n satisfying the equation

$$\sin\beta_n h = 0. \tag{$**$}$$

Then, with the form $(*)$ of φ, (vi) will be satisfied, in conjunction with (2.2), only if $A = 0$, and, then (vi)′ will have to be used as a replacement of (vi) under the transformation employed here.

We then define the following Fourier transforms:

$$\Phi(\alpha, y) \equiv \Phi^+(\alpha, y) + \Phi_1(\alpha, y) + \Phi^-(\alpha, y) = \int_{-\infty}^{\infty} \varphi(x, y)e^{i\alpha x}dx, \tag{2.4}$$

with

$$\Phi^+(\alpha, y) = \int_0^{\infty} \varphi(x, y)e^{i\alpha x}dx,$$
$$\Phi_1(\alpha, y) = \int_{-l}^{0} \varphi(x, y)e^{i\alpha x}dx, \tag{2.5}$$
$$\Phi^-(\alpha, y) = \int_{-\infty}^{-l} \varphi(x, y)e^{i\alpha x}dx,$$

and observe, as in [2] and [3], by using the conditions (v)$'$, (vi)$'$ and (vii)$'$, that the functions Φ^+ and Φ'^+ are analytic in the region $\mathrm{Im}(\alpha) > -s$ and Φ^- and Φ'^- are analytic in region $\mathrm{Im}(\alpha) < s$ whilst Φ_1 represents an entire function of α, in the complex α-plane, so that $\Phi(\alpha, y)$ and $\Phi'(\alpha, y)$ are analytic functions of α in the strip $|\mathrm{Im}(\alpha)| < s$.

Using the Fourier transform to the partial differential equation (2.3) and solving the resulting ordinary differential equation satisfying the condition (i)$'$, we obtain

$$\Phi(\alpha, y) = A(\alpha)\cosh \gamma y, \tag{2.6}$$

where $\gamma = (s^2 + \alpha^2)^{1/2}$.

Applying a Fourier transform to the boundary conditions (ii)$'$, (iii)$'$ and (iv)$'$, along with the use of (2.6), we obtain the following relations:

$$\Phi'(\alpha, h) \equiv \Phi'_1(\alpha, h) + \Phi'^-(\alpha, h) = A(\alpha)\gamma \sin \gamma h, \tag{2.7}$$

$$\Phi'_1(\alpha, h) + B_0\Phi_1(\alpha, h) = B_0 \cdot \frac{1 - e^{-(s+i\alpha)l}}{s + i\alpha}, \tag{2.8}$$

$$\Phi'^-(\alpha, h) + B_l\Phi^-(\alpha, h) = B_l \cdot \frac{e^{-(s+i\alpha)l}}{s + i\alpha}, \tag{2.9}$$

and

$$A(\alpha)[\gamma \sinh \gamma h + B_0 \cosh \gamma h] = \left(1 - \frac{B_0}{B_l}\right)\Phi'^-(\alpha, h) + B_0\Phi^+(\alpha, h) + \frac{B_0}{s + i\alpha}, \tag{2.10}$$

(using (2.8) and (2.9)). Eliminating $A(\alpha)$ from (2.7) and (2.10), we obtain the following three-part Wiener-Hopf functional relation for the determination of the three unknown functions $\Phi^+(\alpha, h)$, $\Phi'^-(\alpha, h)$ and $\Phi'_1(\alpha, h)$:

$$B_0\Phi^+(\alpha, h) - K_1(\alpha)\Phi'_1(\alpha, h) - (B_0/B_l)K_2(\alpha)\Phi'^-(\alpha, h) = iB_0/(\alpha - is), \tag{2.11}$$

where

$$K_1(\alpha) = 1 + (B_0/\gamma)\coth \gamma h, \quad K_2(\alpha) = 1 + (B_l/\gamma)\coth \gamma h, \tag{2.12}$$

so that

$$K_1(\alpha) = \left(1 - \frac{B_0}{B_l}\right) + \frac{B_0}{B_l} K_2(\alpha). \tag{2.13}$$

It is worth noting, at this point, that in the special case when $l = 0$, $\Phi_1'(\alpha, h) = 0$, $K_2(\alpha) = K_1(\alpha) = 1 + (B_0/\gamma)\coth\gamma h$, and we can easily restore the same results as obtained by Levine [7] and Chakrabarti [3].

3. Solution and sputtering temperatures

In order to solve the three-part Wiener-Hopf problem as given by (2.11) we proceed in a manner as described next.

We factorise $K_1(\alpha)$ in the form

$$K_1(\alpha) = K_1^+(\alpha) K_1^-(\alpha), \tag{3.1}$$

where $K_1^+(\alpha)$ and $K_1^-(\alpha)$ are analytic in the overlapping halfplanes $\mathrm{Im}(\alpha) > \tau_-$ and $\mathrm{Im}(\alpha) < \tau_+$ respectively, the details of which will be presented below, and rewrite (2.11) as

$$\frac{\Phi^+(\alpha, h)}{K_1^+(\alpha)} - \frac{1}{B_0} K_1^-(\alpha)\Phi_1'(\alpha, h) - \frac{1}{B_0} K_1^-(\alpha)\Phi'^-(\alpha, h) + \frac{1}{B_0}\left(1 - \frac{B_0}{B_l}\right)\frac{\Phi'^-(\alpha, h)}{K_1^+(\alpha)} = \frac{i}{K_1^+(\alpha)} \cdot \frac{1}{(\alpha - is)}. \tag{3.2}$$

Now,

$$K_1(\alpha) = \frac{\gamma \sinh\gamma h + B_0 \cosh\gamma h}{\gamma \sinh \gamma h} = A \prod_{n=1}^{\infty} \left(\frac{\alpha^2 + \alpha_n^2}{\alpha^2 + \beta_n^2}\right), \text{ (say)} \quad (\alpha_n, \beta_n > 0) \tag{3.3}$$

where $\pm i\alpha_n$, $\pm i\beta_n$ are the zeroes of $\gamma\sinh\gamma h + B_0\cosh\gamma h$ and $\gamma \sinh \gamma h$ respectively, and A is a known constant. Moreover, we write

$$K^+(\alpha) = A^{1/2} \prod_{n=1}^{\infty} [(\alpha + i\alpha_n)/(\alpha + i\beta_n)],$$

which is free from zeroes and poles in the upper half-plane

$$\tau = \mathrm{Im}(\alpha) > \max(-\alpha_1, -\beta_1) = \tau_- \text{ (say)}$$

and

$$K_1^-(\alpha) = A^{1/2} \prod_{n=1}^{\infty} [(\alpha - i\alpha_n)/(\alpha - i\beta_n)],$$

which is free from zeroes and poles in the lower half-plane

$$\tau = \mathrm{Im}(\alpha) < \min(\alpha_1, \beta_1) = \tau_+ \text{ (say)}.$$

Let us now observe that for $\tau_- < c < c' < \tau_+$,

$$\frac{\Phi'(\alpha, h)}{K_1^+(\alpha)} = X_+ + X_- = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{\Phi'^-(\omega, h)}{K_1^+(\omega)} \frac{d\omega}{(\omega-\alpha)} - \frac{1}{2\pi i}\int_{-\infty+ic'}^{\infty+ic'} \frac{\Phi'^-(\omega, h)}{K_1^+(\omega)} \frac{d\omega}{(\omega-\alpha)}, \qquad (3.4)$$

where X_+ is analytic in $\tau > c$ and X_- is analytic in $\tau < c'$ (see Jones [6] and Noble [8]). Then from (3.2), using (3.4), we obtain

$$\frac{\Phi^+(\alpha, h)}{K_1^+(\alpha)} + \left(\frac{1}{B_0} - \frac{1}{B_l}\right) \cdot \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic} \frac{\Phi'^-(\omega, h)}{K_1^+(\omega)} \frac{d\omega}{(\omega-\alpha)} = \frac{i}{(\alpha - is)}\left[\frac{1}{k_1^+(\alpha)} - \frac{1}{K_1^+(is)}\right], \qquad (3.5)$$

which has been derived by separating the + and − functions and utilising Liouville's theorem in a manner similar to that employed by Jones [6] and other workers of the Wiener-Hopf technique (see Noble [8], also).

In order to employ Liouville's theorem in the present context, we have to bear in mind the behaviour of φ and $\frac{\partial\varphi}{\partial y}$ as $x \to 0$ and as $x \to -l$, as given by the condition (vii)$'$. Then, when the definitions (2.5) of the functions Φ^+, Φ_1 and Φ_- are made use of, along with the Abelian theorems (see Noble [7], for example) on Fourier transforms, we find that

(a) $\Phi_+(\alpha, h)$, $\Phi'_+(\alpha, h) \sim O(\frac{1}{\alpha})$, as $\mathrm{Im}\,\alpha \to +\infty$,

(b) $\Phi_1(\alpha, h)$, $\Phi'_1(\alpha, h)$ are integral functions of α with the property that

$$\Phi_1(\alpha, h) \text{ and } \Phi'_1(\alpha, h) \sim \begin{cases} e^{-i\alpha l}, & \text{as } \mathrm{Im}\,\alpha \to +\infty \\ O(\frac{1}{\alpha}), & \text{as } \mathrm{Im}\,\alpha \to -\infty \end{cases}$$

(c) $\Phi^-(\alpha, h)$, $\Phi'^-(\alpha, h) \sim O(e^{-i\alpha l}/\alpha)$, as $\mathrm{Im}\,\alpha \to -\infty$.

These infinity behaviours (a), (b) and (c) of the various complex functions of α ultimately force the separated + and − functions in (3.2) obtained after using the split as given by (3.4), to be identically equal to zero when Liouville's theorem is also made use of.

We then evaluate the integral in (3.5), by closing the contour in the lower half-plane, and use the residues at the poles, to get

$$\frac{\Phi^+(\alpha, h)}{K_1^+(\alpha)} + \left(\frac{1}{B_0} - \frac{1}{B_l}\right) A^{-1/2} \sum_{j=1}^{\infty} \frac{\Phi'^-(-i\alpha_j, h)(-i\alpha_j + i\beta_j)}{(-i\alpha_j - \alpha)} \prod_{\substack{n=1 \\ n\neq j}}^{\infty}{}' \left(\frac{-i\alpha_j + i\beta_n}{-i\alpha_j + i\alpha_n}\right)$$
$$= \frac{i}{(\alpha - is)}\left[\frac{1}{K_1^+(\alpha)} - \frac{1}{K_1^+(is)}\right], \tag{3.6}$$

where α lies in the upper half-plane $\tau > c$.

We again rewrite (2.11) as

$$\Phi_1'(\alpha, h) = \frac{B_0}{K_1(\alpha)}\left[\Phi^+(\alpha, h) - \frac{i}{\alpha - is}\right] - \frac{B_0}{B_l} K(\alpha)\Phi'^-(\alpha, h), \tag{3.7}$$

where

$$\frac{K_2(\alpha)}{K_1(\alpha)} = K(\alpha) = \frac{\gamma \sinh \gamma h + B_l \cosh \gamma h}{\gamma \sinh \gamma h + B_0 \cosh \gamma h}, \tag{3.8}$$

and use the factorisation $K(\alpha) = K^+(\alpha)K^-(\alpha)$, where

$$K^{\pm}(\alpha) = B^{1/2} \prod_{n=1}^{\infty} [(\alpha \pm i\alpha_n')/(\alpha \pm i\alpha_n)], \tag{3.9}$$

$\pm i\alpha_n$ and $\pm i\alpha_n'$ being the zeroes of $\gamma \sinh \gamma h + B_{0,l} \cosh \gamma h$ respectively, and B being a known constant, different from the constant A, appearing earlier.

We then divide both sides of (3.7) by $K^+(\alpha)e^{-i\alpha l}$, bearing in mind the behaviour (b) of $\Phi_1(\alpha, h)$ and $\Phi_1'(\alpha, h)$ as $\alpha \to \infty$, to obtain the relation:

$$\frac{B_0 e^{i\alpha l}}{K_1(\alpha)K^+(\alpha)}\left[\Phi^+(\alpha, h) - \frac{i}{\alpha - is}\right] - \frac{B_0}{B_l} e^{i\alpha l} K^-(\alpha)\Phi'^-(\alpha, h)$$
$$= \frac{\Phi_1'(\alpha, h)e^{i\alpha l}}{K^+(\alpha)}. \tag{3.10}$$

Splitting the first term in (3.10) into the sum $Y_+ + Y_-$, where Y_+ is analytic in $\tau > d$ and Y_- is analytic in $\tau > d'$ $(\tau < d < d' < \tau_+)$ in the usual manner, and using Liouville's theorem as before, we obtain that

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$$-\frac{B_0}{B_l}e^{i\alpha l}K^-(\alpha)\Phi'^-(\alpha, h) = \frac{B_0}{AB^{1/2}}\cdot\frac{1}{2\pi i}\int_{-\infty+id'}^{\infty+id'}\prod_{n=1}^{\infty}\left[\frac{(\omega^2+\beta_n^2)e^{i\omega l}}{(\omega-i\alpha_n)(\omega_i+\alpha_n')}\right] \cdot\left[\Phi^+(\omega, h)-\frac{i}{\omega-is}\right]\frac{d\omega}{(\omega-\alpha)}, \tag{3.11}$$

when (3.3) is also made use of.

We then evaluate the integral in (3.11), by closing the contour in the upper half-plane, and use the residues at the poles, to get

$$\begin{aligned}
&-\frac{B_0}{B_l}e^{i\alpha l}K^-(\alpha)\Phi'^-(\alpha, h)\\
&\quad=\frac{B_0}{AB^{1/2}}\left[\sum_{j=1}^{\infty}\frac{\Phi^+(i\alpha_j, h)e^{-\alpha_j l(\beta_j^2-\alpha_j^2)}}{(i\alpha_j-\alpha)(i\alpha_j+i\alpha_j')}\right.\\
&\qquad\cdot\prod_{\substack{n=1\\n\neq j}}^{\infty}{}'[(\beta_n^2-\alpha_j^2)/(i\alpha_j-i\alpha_n)(i\alpha_j+i\alpha_n')]\\
&\qquad+\frac{i}{(i\alpha_j-is)(i\alpha_j-\alpha)}\sum_{j=1}^{\infty}\frac{(\beta_j^2-\alpha_j^2)e^{-\alpha_j l}}{(i\alpha_j+i\alpha_j')}\\
&\qquad\times\prod_{\substack{n=1\\n\neq j}}^{\infty}{}'[(\beta_n^2-\alpha_j^2)/(i\alpha_j-i\alpha_j)(i\alpha_j+i\alpha_n')]\\
&\qquad\left.-\frac{ie^{-sl}}{(is-\alpha)}\prod_{n=1}^{\infty}\left[\frac{(\beta_n^2-s^2)}{(is-i\alpha_n)(is+i\alpha_n')}\right]\right],
\end{aligned} \tag{3.12}$$

where α lies in the lower half-plane $\tau < d'$.

Using $\alpha = i\alpha_r$ in (3.6) and $\alpha = -i\alpha_r$ in (3.12), we obtain the following two-infinite system of algebraic equations for the two-infinite unknowns $\Phi^+(i\alpha_r, h) \equiv x_r$ and $\Phi'^-(-i\alpha_r, h) \equiv y_r$, respectively:

$$\left.\begin{aligned} a_r x_r + \textstyle\sum_{j=1}^{\infty} b_{r_j} y_j = f_r,\\ c_r y_r + \textstyle\sum_{j=1}^{\alpha'} d_{r_j} x_j = g_r,\end{aligned}\right] \quad (r = 0, 1, 2, \ldots) \tag{3.13}$$

where

$$a_r = \prod_{n=1}^{\infty}\left(\frac{\alpha_r+\beta_n}{\alpha_r+\alpha_n}\right), \qquad c_r = e^{\alpha_r l}\cdot\prod_{n=1}^{\infty}\left(\frac{\alpha_r+\alpha'_n}{\alpha_r+\alpha_n}\right),$$

$$b_{r_j} = -\left(\frac{1}{B_0}-\frac{1}{B_l}\right)\left(\frac{\beta_j-\alpha_j}{\alpha_r+\alpha_j}\right)\cdot\prod_{\substack{n=1\\ n\neq j}}^{\infty}{}'\left(\frac{\beta_n-\alpha_j}{\alpha_n-\alpha_j}\right),$$

$$d_{r_j} = -\frac{B_l}{AB}\cdot\left[\frac{(\beta_j^2-\alpha_j^2)e^{-\alpha_j l}}{(\alpha_j+\alpha'_j)(\alpha_j+\alpha_r)}\right]\cdot\prod_{\substack{n=1\\ n\neq j}}^{\infty}{}'\left[\frac{(\beta_n^2-\alpha_j^2)}{(\alpha_j+\alpha'_n)(\alpha_j-\alpha_n)}\right],$$

$$f_r = -\frac{1}{(s-\alpha_r)}\cdot\left[\prod_{n=1}^{\infty}\left\{\left(\frac{\alpha_r+\beta_n}{\alpha_r+\alpha_n}\right)-\left(\frac{s+\beta_n}{s+\alpha_n}\right)\right\}\right],$$

$$g_r = -\frac{B_l}{AB}\cdot\left[\sum_{j=1}^{\infty}\frac{(\beta_j^2-\alpha_j^2)e^{-\alpha_j l}}{(\alpha_j+\alpha'_j)(\alpha_j+\alpha_r)(s-\alpha_j)}\cdot\prod_{\substack{n=1\\ n\neq j}}^{\infty}{}'\left(\frac{(\beta_n^2-\alpha_j^2)}{(\alpha_j-\alpha_n)(\alpha_j+\alpha'_n)}\right)\right.$$
$$\left. +\frac{e^{-sl}}{(s+\alpha_r)}\cdot\prod_{n=1}^{\infty}\left(\frac{(\beta_n^2-s^2)}{(s-\alpha_n)(s+\alpha'_n)}\right)\right]. \tag{3.14}$$

The solution of the system of equations (3.13) will determine the unknown constants x_r, y_r for all $r = 0, 1, 2, \ldots$. It is interesting to note that the Wiener-Hopf functional relation (2.11) containing the three unknowns $\Phi^+(\alpha, h)$, $\Phi'_1(\alpha, h)$ and $\Phi'^-(\alpha, h)$, is tackled in such an elegant way that the determination of only two unknown functions $\Phi^+(\alpha, h)$ and $\Phi'^-(\alpha, h)$ will solve the present problem completely.

The temperature distribution can finally be determined, by using (3.6) and (3.12). The sputtering temperature at the quench front $x = 0$, i.e., at the points of contact with the first fluid stratum (or the first quench front), is obtained from the relation (see Chakrabarti [3])

$$u_0(h) \equiv 1-\varphi(0, h) = \lim_{|\alpha|\to\infty}[1+i\alpha\Phi^+(\alpha, h)]$$
$$= \frac{1}{K_1^+(is)} - \left(\frac{1}{B_0}-\frac{1}{B_l}\right)\sum_{j=1}^{\infty} y_j(\beta_j-\alpha_j)\prod_{\substack{n=1\\ n\neq j}}^{\infty}{}'\left(\frac{\beta_n-\alpha_j}{\alpha_n-\alpha_j}\right), \tag{3.15}$$

(using (3.6)).

In the special case when $l = 0$, the expression for the sputtering temperature at the quench front $x = 0$ reduces to the same form as obtained by Levine [7] and subsequently by Chakrabarti [3].

The sputtering temperature at the second quench front $x = -l$ i.e., at the points of contact with the second fluid stratum, is given by (2.2)

$$u(-l, h) \equiv 1 - \varphi(-l, h)e^{sl}, \tag{3.16}$$

where

$$\varphi(-l, h) = \lim_{|\alpha|\to\infty} [i\alpha e^{i\alpha l}\Phi^{-}(\alpha, h)].$$

Using (3.12) and (2.9), we have

$$\Phi^{-}(\alpha, h) = \frac{e^{-(s+i\alpha)l}}{s+i\alpha} - \frac{1}{B_l}\Phi'^{-}(\alpha, h).$$

So,

$$\lim_{|\alpha|\to\infty} [i\alpha e^{i\alpha l}\Phi^{-}(\alpha, h)] = \lim_{|\alpha|\to\infty}\left[\frac{i\alpha e^{-sl}}{s+i\alpha} - \frac{1}{B_l}\Phi'^{-}(\alpha, h)i\alpha e^{i\alpha l}\right]$$

$$= e^{-sl} - \frac{1}{AB}\left[\sum_{j=1}^{\infty} x_j \cdot \frac{(\beta_j^2 - \alpha_j^2)e^{-\alpha_j l}}{(\alpha_j + \alpha_j')} {\prod_{\substack{n=1\\ n\neq j}}^{\infty}}' \left(\frac{(\beta_n^2 - \alpha_j^2)}{(\alpha_j + \alpha_n')(\alpha_j - \alpha_n)}\right)\right.$$
$$- \sum_{j=1}^{\infty} \frac{e^{-\alpha_j l}}{(\alpha_j - s)}\left(\frac{\beta_j^2 - \alpha_j^2}{\alpha_j + \alpha_j'}\right) {\prod_{\substack{n=1\\ n\neq j}}^{\infty}}' \left(\frac{(\beta_n^2 - \alpha_j^2)}{(\alpha_j + \alpha_n')(\alpha_j - \alpha_n)}\right)$$
$$\left. - e^{-sl}\prod_{n=1}^{\infty}\left(\frac{(\beta_n^2 - s^2)}{(s-\alpha_n)(s+\alpha_n')}\right)\right].$$

We finally derive from (3.16) that

$$u(-l, h) = \frac{1}{AB}\left[\sum_{j=1}^{\infty} x_j \cdot \frac{(\beta_j^2 - \alpha_j^2)e^{(s-\alpha_j)l}}{(\alpha_j + \alpha_j')} {\prod_{\substack{n=1\\ n\neq j}}^{\infty}}' \left(\frac{(\beta_n^2 - \alpha_j^2)}{(\alpha_j + \alpha_n')(\alpha_j - \alpha_n)}\right)\right.$$
$$- \sum_{j=1}^{\infty} \frac{(\beta_j^2 - \alpha_j^2)e^{(s-\alpha_j)l}}{(\alpha_j - s)(\alpha_j + \alpha_j')} \cdot {\prod_{\substack{n=1\\ n\neq j}}^{\infty}}' \left(\frac{(\beta_n^2 - \alpha_j^2)}{(\alpha_j + \alpha_n')(\alpha_j - \alpha_n)}\right)$$
$$\left. - \prod_{n=1}^{\infty}\left(\frac{(\beta_n^2 - s^2)}{(s-\alpha_n)(s+\alpha_n')}\right)\right]. \tag{3.17}$$

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4. Numerical results

The numerical results for the sputtering temperatures $u_0(h)$ and $u(-l, h)$ at the points of contact to the first and second fluid strata, that is at the first and second quench fronts respectively for suitable choices of λ, μ, s and l, where $\lambda = B_0 h$, $\mu = B_l h$, are tabulated below taking $A \cdot B = 1$. The equations (3.15) and (3.17) have been utilised for the numerical evaluation of the sputtering temperatures.

A study of the table of the values of the sputtering temperatures in Set I, shows that for fixed μ, s and $l (s < l)$, $u_0(h)$ and $u(-l, h)$ increase with the increase in values of λ up to $\lambda = 0.4$ and then decrease with the increasing values of λ.

A study of the table of the values of the sputtering temperatures in Set II, shows that for fixed μ, s and $l (s > l)$, $u_0(h)$ and $u(-l, h)$ increase with the increase in values of λ.

TABLE: Calculation of sputtering temperatures $u_0(h)$ and $u(-l, h)$

Set I:

$\mu = 0.02$, $s = 0.01$, $l = 0.5$	$\lambda = 0.04$	$\lambda = 0.06$	$\lambda = 0.08$	$\lambda = 0.2$
$u_0(h) =$	0.3139361	0.4141611	0.4691014	0.5717536
$u(-l, h) =$	0.1900475	0.2509164	0.285951	0.344757

$\mu = 0.02$, $s = 0.01$, $l = 0.5$	$\lambda = 0.4$	$\lambda = 0.6$	$\lambda = 0.8$	$\lambda = 1$
$u_0(h) =$	0.5888806	0.5771541	0.5593662	0.540358
$u(-l, h) =$	0.3335399	0.3073943	0.2821618	0.2598951

Set II:

$\mu = 0.02$, $s = 0.05$, $l = 0.02$	$\lambda = 0.04$	$\lambda = 0.06$	$\lambda = 0.08$	$\lambda = 0.2$	$\lambda = 0.4$
$u_0(h) =$	0.5459645	0.6465706	0.7023641	0.8428629	0.9646779
$u(-l, h) =$	0.4238668	0.4971518	0.5421164	0.6531928	0.7129149

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