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Two fixed point theorems in topological and metric spaces

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Some fixed point results are derived for mappings of contractive type in metric and topological spaces.

In the paper we shall generalize some results of Boyd and Wong [1], Ćirić [2], Massa [4], Sehgal [6], and the author [3].

DEFINITION 1. Let (X, d) be a metric space, $T : X \to X$ a mapping and $\varphi : R^+ = [0, +\infty) \to R^+$ a right continuous nondecreasing function such that $\varphi < \mathrm{id}$; that is, $\varphi(t) < t$ for all $t > 0$. For x, y in X let us denote

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(x, Ty), d(y, Tx), d(y, Ty)\}$$

and $\tilde{M}(x, y) = \max\{M(x, y), d(Tx, Ty)\}$. We shall say that the mapping T is a φ-max-contraction if

$$(\varphi\text{-max}) \qquad d(Tx, Ty) \le \varphi\big(M(x, y)\big) \quad \text{for all } x, y \text{ in } X .$$

LEMMA 1. *T is a φ-max-contraction if and only if the following condition is satisfied:*

$$(\varphi\text{-max})^{\sim} \qquad d(Tx, Ty) \le \varphi\big(\tilde{M}(x, y)\big) \quad \textit{for all } x, y \textit{ in } X .$$

Proof. Clearly, if T is a φ-max-contraction, then T satisfies the condition $(\varphi\text{-max})^{\sim}$ (because φ is nondecreasing and $M(x, y) \le \tilde{M}(x, y)$).

Let T satisfy the condition $(\varphi\text{-max})^{\sim}$. If $x, y \in X$ and $M(x, y) = \tilde{M}(x, y)$, then $(\varphi\text{-max})$ holds for these x, y . Let x, y in X be such that $M(x, y) \ne \tilde{M}(x, y)$; that is, $M(x, y) < \tilde{M}(x, y)$. Then

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$\tilde{M}(x, y) = d(Tx, Ty) > 0$, so that

$$d(Tx, Ty) \leq \varphi(\tilde{M}(x, y)) = \varphi(d(Tx, Ty)) < d(Tx, Ty) ,$$

which is a contradiction. The proof of the lemma is completed.

In what follows we shall use the following trivial (and well-known) lemma.

LEMMA 2. *If* $t_0 \in R^+$ *and* $t_{n+1} = \varphi(t_n)$ *for* $n \geq 0$ *, then* $t_n \to 0$.

DEFINITION 2. For x in X let

$$O(x, n) = \{x, Tx, \ldots, T^n x\} ,$$

and

$$O(x, \infty) = \{x, Tx, \ldots, T^n x, \ldots\} = \bigcup_{n=1}^{\infty} O(x, n) .$$

LEMMA 3. *For all* $m, n \geq 0$ *and* x *in* X *, the following inequalities hold:*

$$\text{diam } O(T^m x, n) \leq \varphi^{(m)}(\text{diam } O(x, m+n)) ; \tag{1}$$

$$\text{diam } O(T^m x, \infty) \leq \varphi^{(m)}(\text{diam } O(x, \infty)) , \tag{2}$$

provided that $\text{diam } O(x, \infty) < \infty$. (Here "diam" means "the diameter of" and $\varphi^{(m)} = \varphi \circ \varphi \circ \ldots \circ \varphi$ (m times).)

Proof. Let $1 \leq i, j \leq n+1$, $x \in X$. Then

$$\begin{aligned} d(T^i x, T^j x) &= d(T(T^{i-1}x), T(T^{j-1}x)) \leq \varphi(M(T^{i-1}x, T^{j-1}x)) \\ &= \varphi(\max\{d(T^{i-1}x, T^{j-1}x), d(T^{i-1}x, T^i x), d(T^{i-1}x, T^j x), \\ &\qquad d(T^{j-1}x, T^i x), d(T^{j-1}x, T^j x)\}) \leq \varphi(\text{diam } O(x, n+1)) , \end{aligned}$$

and hence

$$\text{diam } O(Tx, n) \leq \varphi(\text{diam } O(x, n+1)) . \tag{3}$$

From (3) one obtains easily (by mathematical induction) the inequality (1).

But the inequality (1) implies:

$$\text{diam } O(T^m x, n) \leq \varphi^{(m)}(\text{diam } O(x, n+m)) \leq \varphi^{(m)}(\text{diam } O(x, \infty))$$

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if $\operatorname{diam} O(x, \infty) < +\infty$. As $\operatorname{diam} O\left(T^m x, n\right) \to \operatorname{diam} O\left(T^m x, \infty\right)$ whenever $n \to +\infty$ (because the sequence of sets $\left\{O\left(T^m x, n\right)\right\}_{n=1}^{\infty}$ is nondecreasing), (2) follows.

THEOREM 1. *Let* (X, d) *and* φ *be as in Definition* 1 *and let* $T : X \to X$ *be a* φ*-max-contraction. Then:*

(i) $\left\{T^n x\right\}_{n=1}^{\infty}$ *is a Cauchy sequence for each* x *in* X *with bounded* T*-orbit, that is, with* $\operatorname{diam} O(x, \infty) < +\infty$ *;*

(ii) if $(\mathrm{id}-\varphi)^{-1}[0, a]$ *is bounded, where* $a = d(x, Tx)$ *, then* $\operatorname{diam} O(x, \infty) < +\infty$ *;*

(iii) if $x \in X$ *,* $\operatorname{diam} O(x, \infty)$ *is finite and the closure of* $O(x, \infty)$ *is complete (this is so if* X *is complete), then the sequence* $\left\{T^n x\right\}_{n=1}^{\infty}$ *converges to a fixed point* u *of* T *and* $d\left(T^n x, u\right) \le \varphi^{(n)}\left(\operatorname{diam} O(x, \infty)\right)$ *for all* $n \ge 1$ *;*

(iv) T *has at most one fixed point in* X *.*

Proof. *(i)* Let $t_0 = \operatorname{diam} O(x, \infty)$ and $t_{n+1} = \varphi\left(t_n\right)$ for $n \ge 0$. By Lemma 2, $\lim t_n = 0$. As $\operatorname{diam} O\left(T^n x, \infty\right) \le \varphi^{(n)}\left(t_0\right) = t_n$ by Lemma 3 (2), we see that the sequence $\left\{T^n x\right\}_{n=1}^{\infty}$ is Cauchy.

(ii) Let $n \ge 1$. Then, by Lemma 3 (1), there exists an integer k such that $1 \le k \le n$ and $d\left(x, T^k x\right) = \operatorname{diam} O(x, n)$. Then

$$\operatorname{diam} O(x, n) = d\left(x, T^k x\right) \le d(x, Tx) + d\left(Tx, T^k x\right) \le$$
$$\le d(x, Tx) + \operatorname{diam} O(Tx, n-1) \le d(x, Tx) + \varphi\left(\operatorname{diam} O(x, n)\right) ,$$

by Lemma 3 (1). Hence we have that $(\mathrm{id}-\varphi)\left(\operatorname{diam} O(x, n)\right) \le d(x, Tx)$, that is, $\operatorname{diam} O(x, n) \le \sup(\mathrm{id}-\varphi)^{-1}[0, d(x, Tx)]$, so that the sequence $\{\operatorname{diam} O(x, n)\}_{n=1}^{\infty}$ is bounded by a number r. Then $\operatorname{diam} O(x, \infty) = \lim \operatorname{diam} O(x, n) \le r$.

(iii) By *(i)*, $\{T^n x\}_{n=1}^{\infty}$ is a Cauchy sequence contained in the complete subset $\overline{\theta(x, \infty)}$ of X, so that $T^n x \to u$ for some u in X. Suppose that $d(u, Tu) > 0$. Then $M(T^n x, u) = \max\{d(T^n x, Tu), d(u, Tu)\}$ for sufficiently large n. It is easy to see that φ is upper semicontinuous. As $M(T^n x, u) \to d(u, Tu)$, we have

$$d(u, Tu) = \lim d(T^{n+1}x, Tu) \le \limsup \varphi(M(T^n x, u)) \le \le \varphi(d(u, Tu)) < d(u, Tu) ,$$

a contradiction. Hence $d(u, Tu) = 0$; that is, $u = Tu$. Furthermore, we have $d(T^n x, u) \le \operatorname{diam} O(T^n x, \infty) \le \varphi^{(n)}(\operatorname{diam} O(x, \infty))$ by Lemma 3 (2).

(iv) Let u, v be two distinct fixed points of T. Then $d(u, v) = d(Tu, Tv) \le \varphi(M(u, v)) = \varphi(d(u, v)) < d(u, v)$, a contradiction. Hence T has at most one fixed point in X.

REMARK 1. Let us note that the condition (φ-max) is equivalent to the following one: there are nonnegative real functions $a, b, c, \bar{d}, e$ on $X \times X$ with $a + b + c + \bar{d} + e = 1$ such that for all x, y the following inequality holds:

$$d(Tx, Ty) \le \varphi\big(a(x, y)d(x, y)+b(x, y)d(x, Tx)+c(x, y)d(x, Ty)+ +\bar{d}(x, y)d(y, Tx)+e(x, y)d(y, Ty)\big)$$

The proof is trivial.

LEMMA 4. *Let X be a compact (separated) space, $d : X \times X \to R^+$ a function and $T : X \to X$ a mapping. Suppose that the function $f(x) = d(x, Tx)$ is lower semicontinuous on X and that the following condition is satisfied:*

(4) *for each x in X with $x \ne Tx$ there exists a positive integer $k(x)$ such that*

$$d\big(T^{k(x)}x, T^{k(x)+1}x\big) < d(x, Tx) .$$

Then T has a fixed point in X.

Proof. Let z in X be such that $f(z) = \min f(X)$. Suppose that $z \ne Tz$. Then, by (4),

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$$f\left(T^{k(z)}z\right) = d\left(T^{k(z)}z, T^{k(z)+1}z\right) < d(z, Tz) = f(z) = \min f(X) ,$$

a contradiction. Hence we have $z = Tz$. (Moreover, we have proved that the fixed point set of T contains the set $f^{-1}(\min f(X))$.)

REMARK 2. Consider the following two conditions:

(5) $d\left(Tx, T^2x\right) < d(x, Tx)$ if $x \neq Tx$;

(6) $d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ if $y \neq x \neq Tx$.

It is easy to see that (6) implies (5), and that (5) implies (4). The function $f(x)$ is lower semicontinuous on X if, for example, T is continuous and d is lower semicontinuous.

DEFINITION 3. Let X be a topological space, $\mathcal{M}$ a family of subsets of X such that M in $\mathcal{M}$ and x in X implies that $\overline{M}$, $M \cup \{x\}$, and $M \setminus \{x\}$ are in $\mathcal{M}$. Let μ be a mapping of $\mathcal{M}$ into a set such that $\mu(\overline{M}) = \mu(M) = \mu(M \cup \{x\})$ for each M from $\mathcal{M}$ and $x \in X$. (We shall call μ a pseudomeasure of noncompactness. Compare with the definition of a measure of noncompactness in Sadovskiĭ [5].) A mapping $T : X \to X$ is said to be μ-densifying if M, TM in $\mathcal{M}$ and $\mu(TM) = \mu(M)$ implies M relatively compact.

THEOREM 2. *Let X be a (separated) topological space, $d : X \times X \to R^+$ a function and $T : X \to X$ a continuous μ-densifying mapping such that the function $f(x) = d(x, Tx)$ is lower semicontinuous on each compact subset of X (the last condition is satisfied if, for example, d is lower semicontinuous on each compact subset of $X \times X$). Suppose that $O(x^*, \infty)$ is in $\mathcal{M}$ for some $x^* \in X$, and that the condition (4) of Lemma 4 is satisfied. Then T has a fixed point in X .*

Proof. Let K be the closure of $O(x^*, \infty)$. As

$$T\left(O(x^*, \infty)\right) \cup \{x^*\} = O(x^*, \infty)$$

and T is continuous, we have $TK \cup \{x^*\} = K$. From the last equality and the inclusion $O(x^*, \infty) \in \mathcal{M}$ it follows that K and TK are in $\mathcal{M}$. But $\mu(TK) = \mu(TK \cup \{x\}) = \mu(K)$, so that K is relatively compact. As K is closed, it is compact. Hence T is a self-mapping of the compact space K into itself and satisfies all conditions of Lemma 4. We conclude that T has a fixed point in $K \subset X$.

DEFINITION 4. Let us remember some notions. Let (X, d) be a metric space and for any subset M of X let us define:

$$Q_\alpha(M) = \{a > 0 : M \text{ has a finite } a\text{-covering}\} ,$$

$$Q_\chi(M) = \{a > 0 : M \text{ can be covered by finitely many } a\text{-balls}\} ,$$

$$Q_J(M) = \{a > 0 : M \text{ contains no infinite } a\text{-discrete set}\} ,$$

$$Q_{\chi_i}(M) = \{a > 0 : M \text{ can be covered by finitely many } a\text{-balls centered in } M\} ,$$

and $\mu(M) = \inf Q_\mu(M)$ for $\mu = \alpha, \chi, J, \chi_i$, respectively. The set functions α, χ, J, χ_i are called the Kuratowski's, Hausdorff, Istrătescu's, and inner Hausdorff measure of noncompactness, respectively (of the metric space (X, d)). We shall say that a mapping $T : X \to X$ densifies with respect to μ, where $\mu = \alpha, \chi, J$, or χ_i, if $\mu(TM) < \mu(M)$ for each non-precompact subset M of X.

COROLLARY. *Let (X, d) be a complete metric space, $T : X \to X$ a continuous mapping which densifies with respect to the Kuratowski's, Hausdorff, Istrătescu's, or inner Hausdorff measure of non-compactness of the space (X, d). If $O(x^*, \infty)$ is bounded for some x^* in X and the condition* (4) *of Lemma* 4 *is satisfied, then T has a fixed point in X.*

Proof. Let M be the family of all bounded subsets of X. It is easy to see that T is μ-densifying (where $\mu = \alpha, \chi, J$, or χ_i), and Theorem 2 may be applied. The corollary is proved.

References

[1] D.W. Boyd and J.S.W. Wong, "On nonlinear contractions", *Proc. Amer. Math. Soc.* 20 (1969), 458-464.

[2] Lj.B. Ćirić, "A gneeralization of Banach's contraction principle", *Proc. Amer. Math. Soc.* 45 (1974), 267-273.

[3] Josef Daneš, "Some fixed point theorems in metric and Banach spaces", *Comment. Math. Univ. Carolinae* 12 (1971), 37-51.

[4] Silvio Massa, "Generalized contractions in metric spaces", *Boll. Un. Mat. Ital.* **10** (1974), 689-694.

[5] Б.Н. Садовский [B.N. Sadovskii], "Предельно компактные и уплотняющие операторы" [Limit-compact and condensing operators], *Uspehi Mat. Nauk* **27** no. 1 (1972), 81-146; *Russian Math. Surveys* **27**, no. 1 (1972), 85-155.

[6] V.M. Sehgal, "On fixed and periodic point for a class of mappings", *J. London Math. Soc.* (2) **5** (1972), 571-576.

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